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ON AN ALGORITHM FOR EXTRACTING HEIGHT RIDGES ON 2-D IMAGES

NASSAR H. ABDEL-ALL, M. A. SOLIMAN, R. A. HUSSIEN, WADAH M. EL-NINI*

Department of Mathematics, Faculty of Science, Assiut University, Assiut, Egypt

Abstract. Ridges are one of the key feature of interest in areas such as computer vision and image processing. Even though a significant amount of research has been directed to defining and extracting ridges some fundamental challenges remain. The authors have recently shown [16] the attraction of ridge and height ridge as a generalized local maximum in 2-D Riemannian space by directly calculations. Here, we are concerned with provide an algorithm to finding the height ridges on 2-D function which given by either explicit equation or discretely as a table of values. Some examples are given and plotted. Found that the results of algorithm matching to results of our paper [16] by directly calculations.

Keywords: Ridges; Height ridges; Ridge directions.

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1. Introduction

Ridges, often described intuitively as the crests connecting mountain peaks, are one of the most sought after features in areas ranging from computer vision [7, 10] and image processing

*Corresponding author

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[4] to tensor analysis [15, 18] and combustion simulations [6]. Consequently, defining and extracting ridges from digital data has received significant attention across different communities resulting in various competing concepts and a plethora of algorithms.

Eberly et al. [4] compare several definitions of ridges, extend Haralick's 'height' definition [8] into multiple dimensions, and establish a set of desired invariants for ridge definitions. Their conclusion is that the so-called height ridges produce qualitatively superior results. This sentiment appears to be wide spread since height ridges are a commonly used ridge structure [4, 5, 12, 13, 14].

The authors in [1, 2] presented a new approach to extract ridges (local maximum) on images of 2-D functions, and in [16] presented a new approach to extract height ridges (generalized local maximum) on images of 2-D functions by directly calculations.

Here, we will presented algorithm to construct height ridges on a subpixel level by selecting an initial guess to a ridge point, searching for a nearby ridge point, then traversing the ridge curve by following its tangents (the ridge direction).

The basic concepts in linear algebra which utilized in this paper can be found in a text on matrix analysis such as [9], also basic concepts in differential geometry (local extrema and tensors) can be found in a standard calculus text such as [11, 12, 17].

The paper is organized as follows. In the second section we review an algorithm to height ridge finding on the function which given by either explicit equation or discretely as a table of values. Supsection (2.1) review the algorithm in Euclidean space and supsection (2.2) review the algorithm in Riemannian space. The third section gives experimental results.

2. Height ridge algorithm

Constructing closed form representations for ridges of high-dimensional functions and images are generally intractable. So, ridge algorithms which lend themselves to numerical computation must be used instead.

Here, we provides ridge algorithm which support the application of ridge finding where the function is given by either explicit equation or discretely as a table of values. In case a table of values it is helpful to have a closed form formula for DP which involves only explicit occurrences of the derivatives of function f and the eigenvalues and eigenvectors of $D^2 f$. In this case, splines may be used as a smooth representation of f, more detailed about splines can be found in [3].

According to the height ridge definition, a point $x \in \mathbb{R}^2$ is a 1-dimensional ridge point if $P(x) = u^t Df = 0$ and $\alpha(x) < 0$, where $\alpha(x)$ is eigenvalue of $D^2 f$ and u is the corresponding eigenvector. The ridge direction is DP which is tangent to the level curve $P = 0$.

The idea to ridges constructed is start with provide an initial guess (x_0, y_0) to a ridge point, then a search is made for a nearby ridge point (x_1, y_1) (we call this step ridge flow), finally from this point a ridge curve can be traversed if we know what the ridge direction is (we call this step ridge traversal). Both steps involve computing the gradient of P, so, to facilitate detection of zeros of P, we require $P(x)$ to be at least a C^1 function or be $f(x)$ a C^3 function. We use the tensor notation.

2.1. The algorithm in 2-D Euclidean geometry

Assume that $f \in C^3(\mathbb{R}^2, \mathbb{R})$, a smoothness condition which we will see guarantees continuity of ridge directions, except possibly at umbilics where $\alpha = \beta$ (Umbilics are places where the eigenvectors may become discontinuous even though the eigenvalues remain continuous). In this paper we will suppose that ridges lie on umbilic-free regions, that is, in regions for which $\alpha < \beta$.

The eigensystems for $D^2 f$ are given by $f_{,ij} u_j = \alpha u_i$ and $f_{,ij} v_j = \beta v_i$, the vectors u and v form an orthonormal system ($u_i u_i = 1$, $v_i v_i = 1$, $u_i v_i = 0$, and $e_{ij} u_i v_j = 1$, where e_{ij} is the permutation tensor on two symbols). Define $P = u_i f_{,i}$ and $Q = v_i f_{,i}$.

According to the height ridge definition, a point $x \in \mathbb{R}^2$ is a 1-D ridge point if $P(x) = 0$ and $\alpha(x) < 0$.

Since u and v form an orthonormal system we can write

$$f_{,i} = P u_i + Q v_i.$$

Moreover, assuming we are in a region where u is differentiable, $u_i u_i = 1$ implies $u_i u_{i,j} = 0$. Differentiating P yields

$$P_{,k} = u_i f_{,ik} + u_{i,k} f_{,i} = \alpha u_i + Q v_i u_{i,k}.$$

Differentiating $f_{,ij} u_j = \alpha u_i$ yields

$$f_{,ij} u_{j,k} + f_{,ijk} u_j = \alpha u_{i,k} + \alpha_{,k} u_i,$$

where we have used $f_{,ij} u_j = \alpha u_i$. Contracting with v_i and using
$v_i f_{,ij} = \beta v_j$ and $u_i v_i = 0$ we obtain

$$\beta v_j u_{j,k} + f_{,ijk} v_i u_j = \alpha v_i u_{i,k}.$$

Therefore, $v_i u_{i,k} = f_{,ijk} v_i u_j / (\alpha - \beta)$. Substituting this in the previous equation for $P_{,k}$ yields

$$P_{,k} = \alpha u_k + \frac{Q}{\alpha - \beta} f_{,ijk} v_i u_j. \tag{2.1}$$

The eigenvectors and eigenvalues are computed using second derivatives of f, the quantity $Q = v_i f_{,i}$?requires first derivatives of f, and $P_{,k}$ additionally requires third derivatives of f. All calculations do not require explicit formulas for the derivatives of eigenvectors $u_{i,j}$ or $v_{i,j}$.

2.1.1. Ridge flow

Given an initial approximation $\mathscr{A}$ to a ridge point, a flow path to the ridge is determined by gradient descent. Ridge points occur as absolute minimum points for the function $P^2(x)/2$ where $\alpha(x) < 0$. The gradient descent is modeled by

$$\frac{dx_i(t)}{dt} = (-P^2(x(t))/2)_{,i} = -P(x(t)) P_{,i}(x(t)), \quad x_i(0) = \mathscr{A}_i,\, i = 1, 2. \tag{2.2}$$

The solution curve terminates at time $T > 0$ if $P(x(T)) = 0$ or if a positive local minimum is reached, in which case a different starting point should be used. The point $\mathscr{R} = x(T)$ will be used as the starting ridge point for ridge traversal.

2.1.2. Ridge traversal

Let $\mathscr{R}$ be the initial ridge point obtained by the construction in the ridge flow. If $T(x)$ is a tangent vector to the ridge, then the ridge can be traversed by solving a system of ordinary differential equations, $dx/dt = T(x)$. To determine $T(x)$, note that the ridge curve is a solution to $P(x) = 0$, so it is (part of) a level curve for P. The gradient of P is therefore normal to the ridge; a tangent to the ridge is orthogonal to the normal, so $T_i(x) = e_{ij}P_{,j}(x)$.

The system of differential equations determining the traversal is therefore

$$\frac{dx_i(t)}{dt} = \pm e_{ij}P_{,j}(x(t)), \tag{2.3}$$

such that the two traversals $x_i(0) = \mathscr{R}_i$, $i = 1,2$ are required.

2.2. The algorithm in 2-D Riemannian geometry

In Riemannian Geometry, Equation (2.1) generalizes to

$$P_{,k} = \alpha u_k + \frac{Q}{\alpha - \beta} f_{,ijk} v^i u^j, \tag{2.4}$$

which specifies the covariant derivative of P. The model for ridge flow given by equation (2.2) generalizes as follows. As a parameterized curve $x^i(t)$, the tangent to the ridge flow path is the contravariant vector dx^i/dt. The gradient $P_{,i}(x)$ is covariant, so we need its contravariant counterpart. Thus, the ridge flow is modeled by

$$\frac{dx^i}{dt} = -Pg^{ij}P_{,j} = -PP^i_{,.}, \; x^i(0) = \mathscr{A}, \; i = 1,2, \tag{2.5}$$

where $\mathscr{A}$ is an initial approximation to the ridge. If the flow terminates at time $T > 0$ where $P(x(T)) = 0$, then $\mathscr{R} = x(T)$ is used as the starting ridge point for ridge traversal. Ridge traversal given by equation (2.3) generalizes to

$$\frac{dx^i}{dt} = \pm g^{ij} e_{jk} g^{kl} P_{,l} = \pm e^{ij} P_{,j}, \quad x^i(0) = \mathscr{R}^i(0), \; i = 1,2. \tag{2.6}$$

Note that raising of indices on both covariant tensors e_{ij} and $P_{,i}$ is required to produce a contravariant vector dx^i/dt.

3. Applications for hight ridge

(1) Consider the parabolic hyperboloid surface (saddle surface) represented by the function $f(x_1,x_2) = x_1^2 - x_2^2$.

$$\mathbf{f}_{,\mathbf{i}} = \begin{pmatrix} 2x_1 \\ \\ -2x_2 \end{pmatrix}, \qquad \mathbf{f}_{,\mathbf{ij}} = \begin{pmatrix} 2 & 0 \\ \\ 0 & -2 \end{pmatrix}.$$

The eigenvalues of $f_{,ij}$ are $\alpha = -2$ and $\beta = 2$, observe that $\alpha(x_1,x_2) < 0 < \beta(x_1,x_2)$ for all (x_1,x_2). The corresponding eigenvectors are $u = (u_1,u_2) = (0,1)$ and $v = (v_1,v_2) = (1,0)$ respectively.

There are no 1-D ridge points with respect to β because $\beta = 2 > 0$ (see definition of hight ridges in [16]). But there are 1-D ridge points with respect to $\alpha = -2 < 0$, we use the above algorithm to find it

$$P = u_i f_{,i} = u_1 f_{,1} + u_2 f_{,2} = -2x_2.$$

$$Q = v_i f_{,i} = v_1 f_{,1} + v_2 f_{,2} = 2x_1.$$

$$P_{,k} = \alpha u_k + \frac{Q}{\alpha - \beta} f_{,ijk} v_i u_j$$

$$\Rightarrow P_{,k} = \alpha \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{Q}{\alpha - \beta} \begin{pmatrix} f_{,111}v_1u_1 + f_{,121}v_1u_2 + f_{,211}v_2u_1 + f_{,221}v_2u_2 \\ f_{,112}v_1u_1 + f_{,122}v_1u_2 + f_{,212}v_2u_1 + f_{,222}v_2u_2 \end{pmatrix}$$

$$\Rightarrow P_{,k} = (-2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{2x_1}{-2-2}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

We start with the initial approximation $\mathscr{A}_i = x_i(0) = (1,-1)$, as shows in FIGURE 1.

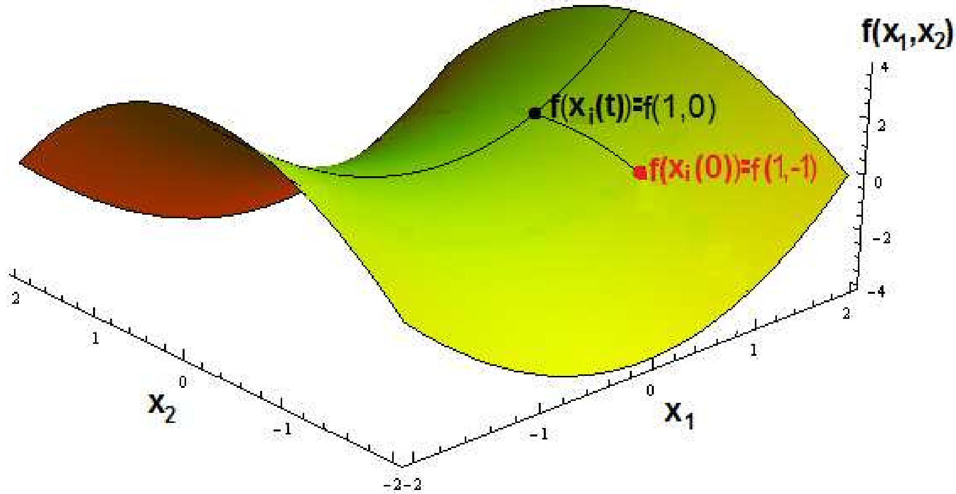

FIGURE 1. The initial ridge $f(\mathscr{A}_i) = f(x_i(0)) = f(1,-1)$ and nearby ridge point $f(\mathscr{R}_i) = f(x_i(t)) = f(1,0)$.

Now, we search for a nearby ridge point $x_i(t)$ from the initial approximation $\mathscr{A}_i = x_i(0) = (1,-1)$, using the subsection 2.1.1 (Ridge Flow), we have the two ordinary equations differential (3.1) and (3.2)

$$\frac{dx_1(t)}{dt} = -P(x(t))P_{,1}(x(t)) = -(-2x_2)(0) = 0, \quad x_1(0) = \mathscr{A}_1 = 1 \tag{3.1}$$

$$\frac{dx_2(t)}{dt} = -P(x(t))P_{,2}(x(t)) = -(-2x_2)(-2) = -4x_2, \quad x_2(0) = \mathscr{A}_2 = -1. \tag{3.2}$$

The general solution for (3.1) is given from

$$\frac{dx_1(t)}{dt} = 0 \Rightarrow x_1(t) = C_1$$

at the initial condition $x_1(0) = 1 \Rightarrow C_1 = 1$.

The special solution for (3.1) is

$$x_1(t) = 1 \tag{3.3}$$

Similarity, the general solution for (3.2) is given from

$$\frac{dx_2(t)}{dt} = -4x_2, \Rightarrow x_2(t) = C_2 e^{-4t}$$

at the initial condition $x_2(0) = -1 \Rightarrow -1 = C_2 e^0 \Rightarrow C_2 = -1$.

The special solution (3.2) is

$$x_2(t) = -e^{-4t}. \tag{3.4}$$

Now, from (3.3) and (3.4) we have

$P(x_1(t), x_2(t)) = P(1, -e^{-4t}) = -2(-e^{-4t}) = 2e^{-4t}$, therefor

$P(x_1(t), x_2(t)) \cong 0$ as along as $2e^{-4t} \cong 0$ or $t \to \infty$, which equivalent to $P(x_1(t), x_2(t)) = 0$ as along as $(x_1(t), x_2(t)) = (1, 0)$. We will use this point $\mathscr{R} = (x_1(t), x_2(t)) = (1, 0)$ as the starting ridge point for ridge traversal.

The tangent vector to the ridge is

$$T_i(x) = e_{ij}P_{,j} = \begin{pmatrix} e_{1j}P_{,j} \\ e_{2j}P_{,j} \end{pmatrix} = \begin{pmatrix} e_{11}P_{,1} + e_{12}P_{,2} \\ e_{21}P_{,1} + e_{22}P_{,2} \end{pmatrix} = \begin{pmatrix} P_{,2}(x) \\ -P_{,1}(x) \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

using the subsection 2.1.2 (Ridge Traversal), we have the two ordinary equations differential (3.5) and (3.6)

$$\frac{dx_1(t)}{dt} = \pm T_1(x) = \pm e_{1j}P_{,j}(x(t)) = \pm(-2) = \mp 2, \quad x_1(0) = \mathscr{R}_1 = 1 \tag{3.5}$$

$$\frac{dx_2(t)}{dt} = \pm T_2(x) = \pm e_{2j}P_{,j}(x(t)) = 0, \quad x_2(0) = \mathscr{R}_2 = 0 \tag{3.6}$$

the general solution for (3.5) is given from

$$\frac{dx_1(t)}{dt} = \mp 2 \Rightarrow x_1(t) = \mp 2t + C_1$$

at the initial condition $x_1(0) = 1 \Rightarrow C_1 = 1$.

The special solution for (3.5) is

$$x_1(t) = \mp 2t + 1 \tag{3.7}$$

Similarity, the general solution for (3.6) is given from

$$\frac{dx_2(t)}{dt} = 0 \Rightarrow x_2(t) = C_2$$

at the initial condition $x_2(0) = 0 \Rightarrow C_2 = 0$.

The special solution for (3.6) is

$$x_2(t) = 0. \tag{3.8}$$

The 1-D ridge points with respect to $\alpha = -2$ are

$(x_1(t), x_2(t)) = (\mp 2t+1, 0)$ for all t (note that $P(\mp 2t+1, 0) = 0$ for all t), as shows in FIGURE 2.

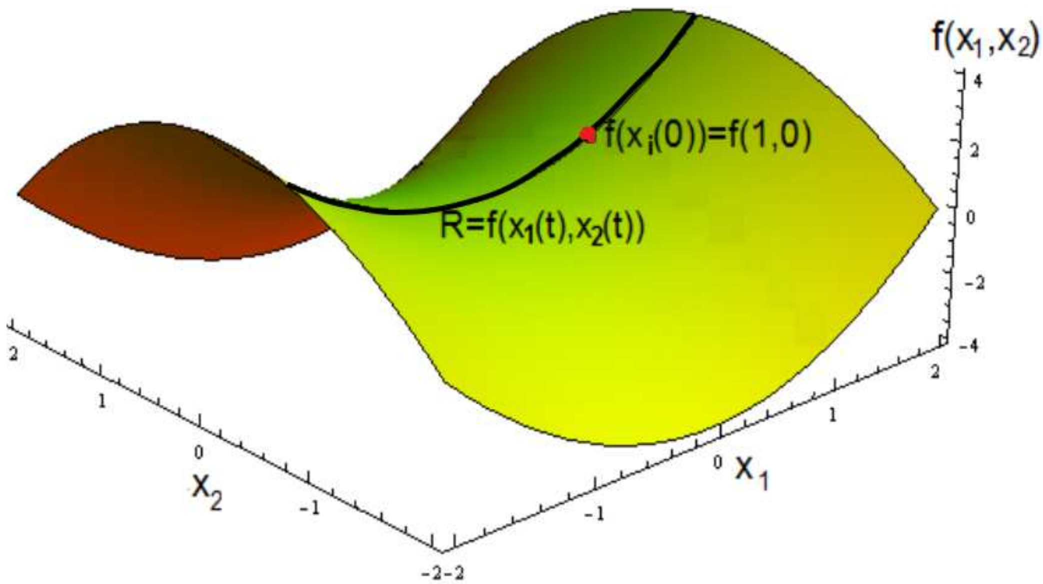

FIGURE 2. The 1-D ridge points with respect to $\alpha = -2$ on the image of $f(x_1, x_2) = x_1^2 - x_2^2$.

We note that results of algorithm in above example it matching to results in our paper [16] by directly calculations with the same example.

(2) Consider the surface represented by the function $f(x_1, x_2) = -x_2^2 + e^{-x_1^2}$. We have

$$\mathbf{f}_{,\mathbf{i}} = \begin{pmatrix} -2x_1 e^{-x_1^2} \\ \\ -2x_2 \end{pmatrix}, \qquad \mathbf{f}_{,\mathbf{ij}} = \begin{pmatrix} (4x_1^2-2)e^{-x_1^2} & 0 \\ \\ 0 & -2 \end{pmatrix}.$$

The eigenvalues of $f_{,ij}$ are $\alpha = -2$ and $\beta = (4x_1^2-2)e^{-x_1^2}$, observe that $\alpha(x_1, x_2) < 0$ for all (x_1, x_2), while $\beta(x_1, x_2) < 0$ at $x_1 \in (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and for all x_2. The corresponding eigenvectors are $u = (u_1, u_2) = (0, 1)$ and $v = (v_1, v_2) = (1, 0)$ respectively.

• There are 1-D ridge points with respect to $\alpha = -2 < 0$, we use the previous algorithm to find it

$$P = u_i f_{,i} = u_1 f_{,1} + u_2 f_{,2} = -2x_2.$$

$$Q = v_i f_{,i} = v_1 f_{,1} + v_2 f_{,2} = -2x_1 e^{-x_1^2}.$$

$$P_{,k} = \alpha u_k + \frac{Q}{\alpha - \beta} f_{,ijk} v_i u_j$$

$$\Rightarrow P_{,k} = \alpha \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{Q}{\alpha - \beta} \begin{pmatrix} f_{,111}v_1u_1 + f_{,121}v_1u_2 + f_{,211}v_2u_1 + f_{,221}v_2u_2 \\ f_{,112}v_1u_1 + f_{,122}v_1u_2 + f_{,212}v_2u_1 + f_{,222}v_2u_2 \end{pmatrix}$$

$$\Rightarrow P_{,k} = (-2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{-2x_1 e^{-x_1^2}}{-2 - (4x_1^2 - 2)e^{-x_1^2}} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

We start with the initial approximation $\mathscr{A}_i = x_i(0) = (-1, \frac{1}{2})$, as shows in FIGURE 3.

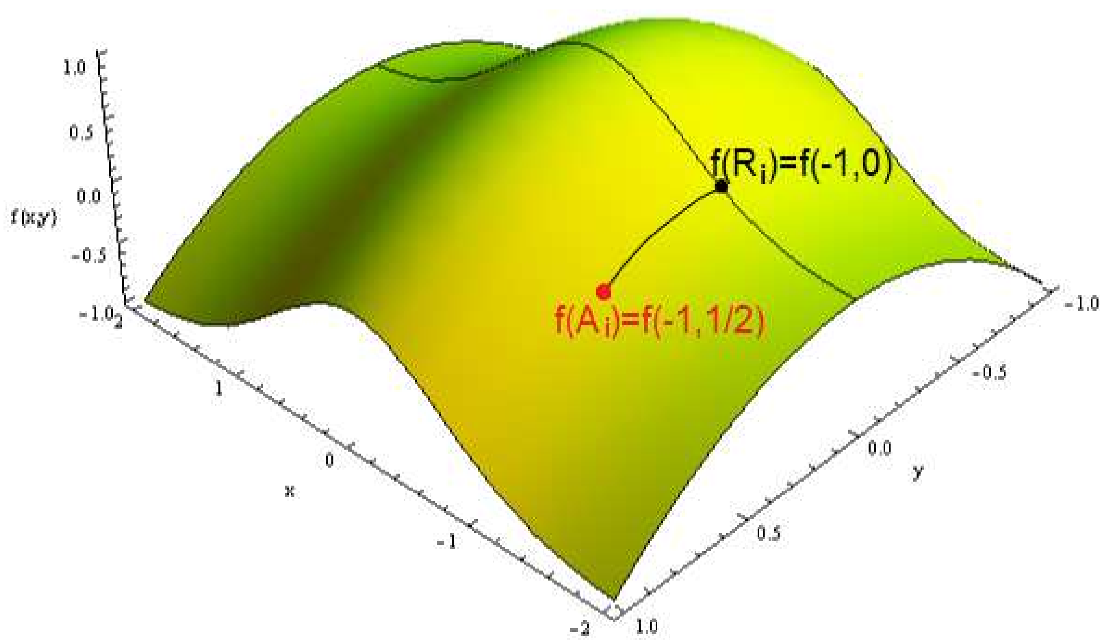

FIGURE 3. The initial ridge $f(\mathscr{A}_i) = f(x_i(0)) = f(-1, \frac{1}{2})$ and nearby ridge point $f(\mathscr{R}_i) = f(x_i(t)) = f(-1, 0)$.

Now we search for a nearby ridge point $x_i(t)$ from the initial approximation

$\mathscr{A}_i = x_i(0) = (-1, \frac{1}{2})$, using the subsection 2.1.1 (Ridge Flow), we have the two ordinary equations differential (3.9) and (3.10)

$$\frac{dx_1(t)}{dt} = -P(x(t))P_{,1}(x(t)) = -(-2x_2)(0) = 0, \quad x_1(0) = \mathscr{A}_1 = -1 \tag{3.9}$$

$$\frac{dx_2(t)}{dt} = -P(x(t))P_{,2}(x(t)) = -(-2x_2)(-2) = -4x_2, \quad x_2(0) = \mathscr{A}_2 = \frac{1}{2}. \tag{3.10}$$

The general solution for (3.9) is given from

$$\frac{dx_1(t)}{dt} = 0 \Rightarrow x_1(t) = C_1$$

at the initial condition $x_1(0) = -1 \Rightarrow C_1 = -1$.

The special solution for (3.9) is

$$x_1(t) = -1 \tag{3.11}$$

Similarity, the general solution for (3.10) is given from

$$\frac{dx_2(t)}{dt} = -4x_2, \Rightarrow x_2(t) = C_2 e^{-4t}$$

at the initial condition $x_2(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = C_2 e^0 \Rightarrow C_2 = \frac{1}{2}$.

The special solution for (3.10) is

$$x_2(t) = \frac{1}{2} e^{-4t}. \tag{3.12}$$

Now, from (3.11) and (3.12) we have

$P(x_1(t), x_2(t)) = P(1, \frac{1}{2}e^{-4t}) = -2(\frac{1}{2}e^{-4t}) = -e^{-4t}$, therefore

$P(x_1(t), x_2(t)) \cong 0$ as along as $-e^{-4t} \cong 0$ or $t \to \infty$, which equivalent to $P(x_1(t), x_2(t)) = 0$ as along as $(x_1(t), x_2(t)) = (-1, 0)$. We will use this point $\mathscr{R} = (x_1(t), x_2(t)) = (-1, 0)$ as the starting ridge point for ridge traversal.

The tangent vector to the ridge is

$$T_i(x) = e_{ij}P_{,j} = \begin{pmatrix} e_{1j}P_{,j} \\ e_{2j}P_{,j} \end{pmatrix} = \begin{pmatrix} e_{11}P_{,1} + e_{12}P_{,2} \\ e_{21}P_{,1} + e_{22}P_{,2} \end{pmatrix} = \begin{pmatrix} P_{,2}(x) \\ -P_{,1}(x) \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

using the subsection 2.1.2 (Ridge Traversal), we have the two ordinary equations differential (3.13) and (3.14)

$$\frac{dx_1(t)}{dt} = \pm T_1(x) = \pm e_{1j}P_{,j}(x(t)) = \pm(-2) = \mp 2, \quad x_1(0) = \mathscr{R}_1 = -1 \tag{3.13}$$

$$\frac{dx_2(t)}{dt} = \pm T_2(x) = \pm e_{2j}P_{,j}(x(t)) = 0, \quad x_2(0) = \mathscr{R}_2 = 0 \tag{3.14}$$

The general solution for (3.13) is given from

$$\frac{dx_1(t)}{dt} = \mp 2 \Rightarrow x_1(t) = \mp 2t + C_1$$

at the initial condition $x_1(0) = -1 \Rightarrow C_1 = -1$.

The special solution for (3.13) is

$$x_1(t) = \mp 2t - 1 \tag{3.15}$$

Similarity, the general solution for (3.14) give from

$$\frac{dx_2(t)}{dt} = 0 \Rightarrow x_2(t) = C_2$$

at the initial condition $x_2(0) = 0 \Rightarrow C_2 = 0$.

The special solution for (3.14) is

$$x_2(t) = 0. \tag{3.16}$$

The 1-D ridge points with respect to $\alpha = -2 < 0$ are $(x_1(t), x_2(t)) = (\mp 2t - 1, 0)$ for all t (note that $P(\mp 2t - 1, 0) = 0$ for all t), as shows in FIGURE 4.

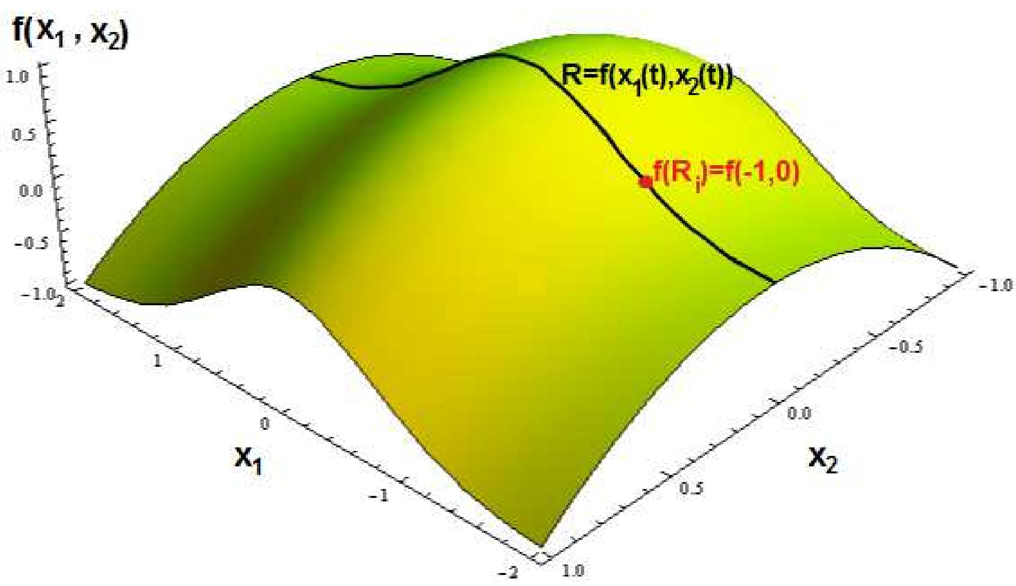

FIGURE 4. The 1-D ridge points with respect to $\alpha = -2$ on the image of $f(x_1, x_2) = -x_2^2 + e^{-x_1^2}$

• There are 1-D ridge points with respect to β, note that $\beta = (4x_1^2 - 2)e^{-x_1^2} < 0$ as along as $x_1 \in (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, therefore

$$Q = v_i f_{,i} = v_1 f_{,1} + v_2 f_{,2} = -2x_1 e^{-x_1^2}.$$

$$P = u_i f_{,i} = u_1 f_{,1} + u_2 f_{,2} = -2x_2.$$

$$Q_{,k} = \beta v_k + \frac{P}{\beta - \alpha} f_{,ijk} u_i v_j$$

$$\Rightarrow Q_{,k} = \beta \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \frac{P}{\beta - \alpha} \begin{pmatrix} f_{,111}u_1v_1 + f_{,121}u_1v_2 + f_{,211}u_2v_1 + f_{,221}u_2v_2 \\ f_{,112}u_1v_1 + f_{,122}u_1v_2 + f_{,212}u_2v_1 + f_{,222}u_2v_2 \end{pmatrix}$$

$$\Rightarrow Q_{,k} = (4x_1^2 - 2)e^{-x_1^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{-2x_2}{(4x_1^2 - 2)e^{-x_1^2} - 2} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (4x_1^2 - 2)e^{-x_1^2} \\ 0 \end{pmatrix}.$$

We start with the initial approximation $\mathscr{A}_i = x_i(0) = (-1, \frac{1}{2})$, as shows in FIGURE 5.

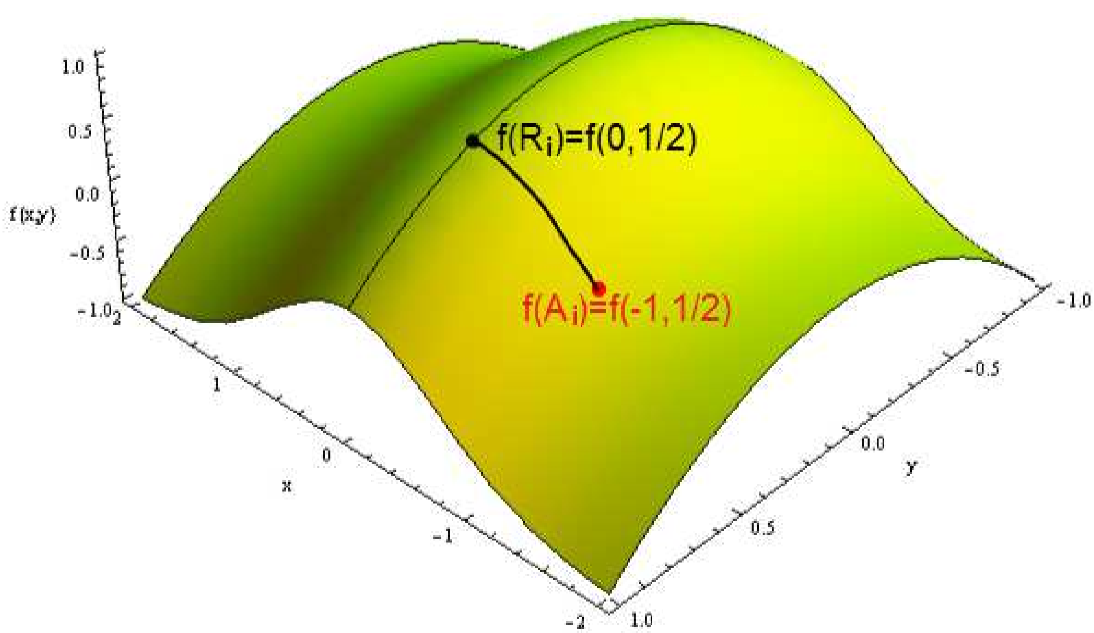

FIGURE 5. The initial ridge $f(\mathscr{A}_i) = f(x_i(0)) = f(-1, \frac{1}{2})$ and nearby ridge point $f(\mathscr{R}_i) = f(x_i(t)) = f(0, \frac{1}{2})$.

Now, we search for a nearby ridge point $x_i(t)$ from the initial approximation

$\mathscr{A}_i = x_i(0) = (-1, \frac{1}{2})$, using the subsection 2.1.1 (Ridge Flow), we have the two ordinary equations differential (3.17) and (3.18)

$$\frac{dx_1(t)}{dt} = -Q(x(t))Q_{,1}(x(t)) = -(-2x_1e^{-x_1^2})(4x_1^2-2)e^{-x_1^2} = 4x_1(2x_1^2-1)e^{-2x_1^2}, x_1(0) = -1 \tag{3.17}$$

$$\frac{dx_2(t)}{dt} = -Q(x(t))Q_{,2}(x(t)) = -(-2x_1e^{-x_1^2})(0) = 0, \quad x_2(0) = \frac{1}{2}. \tag{3.18}$$

The general solution for (3.17) is given from

$$\frac{dx_1(t)}{dt} = 4x_1(2x_1^2-1)e^{-2x_1^2} \quad or \quad \frac{dx_1(t)}{4x_1(2x_1^2-1)e^{-2x_1^2}} = dt$$

$$\Rightarrow \phi(x_1) = t + C_1, \quad \phi(x_1) = \int \frac{dx_1(t)}{4x_1(2x_1^2-1)e^{-2x_1^2}}$$

at the initial condition $x_1(0) = -1 \Rightarrow C_1 = \phi(-1)$.

The special solution for (3.17) is

$$\phi(x_1) = t + \phi(-1) \tag{3.19}$$

Similarity, the general solution for (3.18) is given from

$$\frac{dx_2(t)}{dt} = 0, \Rightarrow x_2(t) = C_2$$

at the initial condition $x_2(0) = \frac{1}{2} \Rightarrow C_2 = \frac{1}{2}$.

$\therefore$ The special solution for (3.18) is

$$x_2(t) = \frac{1}{2}. \tag{3.20}$$

Now, from (3.19) , (3.20) we have $Q(x_1(t), x_2(t)) = 0$ as along as $(x_1(t), x_2(t)) = (0, \frac{1}{2})$. We will use this point $\mathscr{R} = (x_1(t), x_2(t)) = (0, \frac{1}{2})$ as the starting ridge point for ridge traversal.

The tangent vector to the ridge is

$$T_i(x) = e_{ij}Q_{,j} = \begin{pmatrix} e_{1j}Q_{,j} \\ e_{2j}Q_{,j} \end{pmatrix} = \begin{pmatrix} e_{11}Q_{,1} + e_{12}Q_{,2} \\ e_{21}Q_{,1} + e_{22}Q_{,2} \end{pmatrix} = \begin{pmatrix} Q_{,2}(x) \\ -Q_{,1}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -2(2x_1^2-1)e^{-x_1^2} \end{pmatrix}$$

using the subsection 2.1.2 (Ridge Traversal), we have the two ordinary equations differential (3.21) and (3.22)

$$\frac{dx_1(t)}{dt} = \pm T_1(x) = 0, \quad x_1(0) = \mathscr{R}_1 = 0 \tag{3.21}$$

$$\frac{dx_2(t)}{dt} = \pm T_2(x) = \pm(2-4x_1^2)e^{-x_1^2}, \quad x_2(0) = \mathscr{R}_2 = \frac{1}{2} \tag{3.22}$$

The general solution for (3.21) is given from

$$\frac{dx_1(t)}{dt} = 0 \Rightarrow x_1(t) = C_1$$

at the initial condition $x_1(0) = 0 \Rightarrow C_1 = 0$.

The special solution for (3.21) is

$$x_1(t) = 0 \tag{3.23}$$

Similarity, the general solution for (3.22) is given from

$$\frac{dx_2(t)}{dt} = \pm(2-4x_1^2)e^{-x_1^2}$$

$$\Rightarrow x_2(t) = \pm(2-4x_1^2)e^{-x_1^2}t + C_2$$

but from (3.23) $x_1(t) = 0$, for all t, then

$$\frac{dx_2(t)}{dt} = \pm(2-0)e^{0}t + C_2 = \pm 2t + C_2$$

at the initial condition $x_2(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = 0 + C_2 \Rightarrow C_2 = \frac{1}{2}$.

The special solution for (3.22) is

$$x_2(t) = \pm 2t + \frac{1}{2}. \tag{3.24}$$

The 1-D ridge points with respect to $\beta = (4x_1^2 - 2)e^{-x_1^2}$ are $(x_1(t), x_2(t)) = (0, \pm 2t + \frac{1}{2})$ for all t (note that $P(0, \pm 2t + \frac{1}{2}) = 0$ for all t), as shows in FIGURE 6.

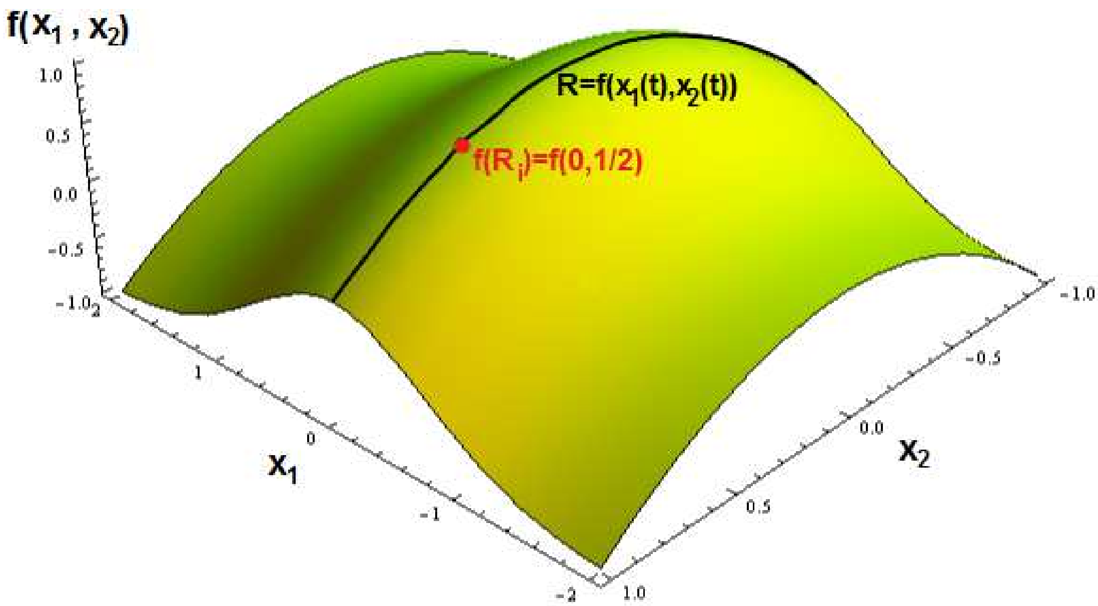

FIGURE 6. The 1-D ridge points with respect to $\beta = (4x_1^2 - 2)e^{-x_1^2}$ on the image of $f(x_1,x_2) = -x_2^2 + e^{-x_1^2}$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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