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NOTE ON NEW HOMOTOPY PERTURBATION METHOD FOR SOLVING NON-LINEAR INTEGRAL EQUATIONS

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Abstract. In this paper, exact solution for the second kind of nonlinear integral equations are presented. An application of modified new homotopy perturbation method is applied to solve the second kind of non-linear integral equations such that Voltrra and Fredholm integral equations. The results reveal that the modified new homotopy perturbation method is very effective and simple and gives the exact solution. Also the comparison of the results of applying this method with those of applying the homotopy perturbation method reveals the effectiveness and convenience of the new technique.

Keywords: homotopy perturbation method; integral equations; Voltrra integral equations; Fredholm integral equations.

2010 AMS Subject Classification: 45G10.

1. Introduction

In recent years, a large amount of literature developed concerning the modified decomposition method introduced by Wazwaz [8] by applying it to a large size of applications in applied mathematics. A new perturbation method called homotopy perturbation method (HPM) [1-4], was proposed by He and systematical description which is, in fact, coupling of the traditional perturbation method and homotopy in topology. Until recently, the application of the HPM [2] in non-linear problems has been developed by scientists and engineers, because this method is the most effective and convenient ones for both weakly and strongly non-linear equations, also Tarig

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M. Elzaki in [6,7] combine integral transform and homotopy perturbation method (HPM) to solve nonlinear equations.

In this article we obtain a new modification of HPM to solve the second kind of nonlinear integral equations such that Voltrra and Fredholm integral equation.

Definition 1.1

For $\in$ we have, $P_n(\mathrm{x}) = \in P_m(\mathrm{x})$ where, $\in$ and x are dimensionless, $P_n(\mathrm{x}), P_m(\mathrm{x})$ are polynomials in x of orders n, m respectively, and $n \geq m$.

When $\in = 0$, the problem reduces to, $P_n(\mathrm{x}) = 0$, which is called the reduced or unperturbed equation.

Consider a nonlinear equation in the form, $Lu + Nu = 0$

Where L and N , are linear and nonlinear operators respectively. A homotopy can be constructed in the form,

$$Lu + p\left(Nu - Lu\right) = 0$$

The non-linear Fredhom integral equations are given by,

$$u(\mathrm{x}) = \mathrm{f(x)} + \int_0^1 k\,(\mathrm{x,y})\left\{R\,(\mathrm{u(y)}) + N\,(\mathrm{u(y)})\right\} dy \tag{1}$$

And the non-linear Voltrra integral equations are given by,

$$u(\mathrm{x}) = \mathrm{f(x)} + \int_0^x k\,(\mathrm{x,y})\left\{R\,(\mathrm{u(y)}) + N\,(\mathrm{u(y)})\right\} dy \tag{2}$$

$u(\mathrm{x})$ is an unknown function that will be determined, $\mathrm{k(x,y)}$, is the kernel of the integral equation, $\mathrm{f(x)}$ is an analytical function, $\mathrm{R(u)}, N(\mathrm{u})$ are linear and non-linear factions of u respectively.

2. The New Modified HPM

The new modified homotopy perturbation method depends mainly on splitting the function $\mathrm{f(x)}$ into two parts.

To illustrate the new modified homotopy perturbation method we consider the equation (1), as.

$$L(u) = u(\mathrm{x}) - \mathrm{f}_1(\mathrm{x}) - \mathrm{f}_2(\mathrm{x}) - \int_0^1 k\,(\mathrm{x,y})\left\{R(\mathrm{u(y)}) + N\,(\mathrm{u(y)})\right\} dy = 0 \tag{3}$$

Where that: $f(\mathrm{x}) = f_1(\mathrm{x}) + f_2(\mathrm{x}),$ as a possible remedy, we can define $H(\mathrm{u,p})$ by,

$$\begin{aligned} &H(\mathrm{u,p}) = (1-\mathrm{p})\,\mathrm{F(u)} + pL(\mathrm{u}), \\ &H(\mathrm{u},0) = \mathrm{F(u)}\ , \quad H(\mathrm{u},1) = L(\mathrm{u}), \end{aligned} \tag{4}$$

Then we have the following two ways:

(i) We choose, $\mathrm{F(u)} = u(\mathrm{x}) - f_1(\mathrm{x}),$ such that,

$$\int_0^1 k(\mathrm{x,y})\{R(\mathrm{u(y)}) + N(\mathrm{u(y)})\}dy + f_2(\mathrm{x}) = 0$$

(ii) We choose, $\mathrm{F(u)} = u(\mathrm{x}) - f_2(\mathrm{x}),$ such that,

$$\int_0^1 k(\mathrm{x,y})\{R(\mathrm{u(y)}) + N(\mathrm{u(y)})\}dy + f_1(\mathrm{x}) = 0$$

The HPM uses the homotopy parameter p , as expanding parameter to obtain,

$$u = u_0 + pu_1 + p^2u_2 + \ldots \tag{5}$$

When $p \to 1$, then equation (5) corresponds to the equation (4), and equation (5) becomes a solution of equation (3), i.e.

$$u = \lim_{p\to 1} u = u_0 + u_1 + u_2 + \ldots \tag{6}$$

The series (6), is convergent for most cases, and also the rate of convergence depends on $L(\mathrm{u})$.

Note that: we can study equation (2) by the same method.

3. Numerical Examples

In this section we illustrate some examples of non-linear Fredholm and voltrra integral equations.

Example 1.

Consider the non-linear Fredholm integral equation,

$$u(\mathrm{x}) = \mathrm{sinhx} - 1 + \int_0^1 \left(\cosh^2 y - u^2(\mathrm{y})\right)dy \tag{7}$$

Because $\int_0^1 \left(\cosh^2 y - u^2(\mathrm{y})\right)dy = 0,$ *at* $u(\mathrm{x}) = \mathrm{sinhx}$, then we split $f(\mathrm{x})$ given by, $f(\mathrm{x}) = \mathrm{sinhx} - 1$, into two parts, namely,

$$f_1(\mathrm{x}) = \mathrm{sinhx}, \quad f_2(\mathrm{x}) = -1, \quad \Rightarrow F(\mathrm{u}) = \mathrm{u(x)} - f_1(\mathrm{x})$$

Then, $$L(u) = u(\mathrm{x}) - \mathrm{f}_1(\mathrm{x}) + \mathrm{f}_2(\mathrm{x}) - \int_0^1 \left(\cosh^2 y - u^2(\mathrm{y})\right) dy = 0.$$

By substituting $F(\mathrm{u})$ and L(u) in equation (4), and equating the terms with identical power of p, we have,

$$p^0: \quad u_0(\mathrm{x}) = \mathrm{f}_1(\mathrm{x}) = \mathrm{sinhx}$$

$$p^1: \quad u_1(\mathrm{x}) = \mathrm{f}_2(\mathrm{x}) + \int_0^1 \left(\cosh^2 y - u_0^2(\mathrm{y})\right) dy = 0$$

$$p^{k+1}: \quad u_{k+1}(\mathrm{x}) = \mathrm{f}_2(\mathrm{x}) + \int_0^1 \left(\cosh^2 y - u_k^2(\mathrm{y})\right) dy = 0$$

Then the solution of equation (7), is. $u(\mathrm{x}) = \lim_{p \to 1} u = u_0 + u_1 + u_2 + ... = \mathrm{sinhx}$

Example 2.

Consider the following Volterra integral equation,

$$u(\mathrm{x}) = 12x + x^2 - 2x^3 - \frac{x^6}{30} - 11 \sin x + 2\int_0^x \left((\mathrm{x} - \mathrm{y})^3 u(\mathrm{y})\right) dy, \tag{8}$$

Because $2\int_0^x \left((\mathrm{x} - \mathrm{y})^3 u(\mathrm{y})\right) dy = \frac{x^6}{30} - 12x + 2x^3 + 12 \sin x$, *at* $u(\mathrm{x}) = \mathrm{x}^2 + \sin x$, then we split $f(\mathrm{x})$ given by, $f(\mathrm{x}) = 12x + x^2 - 2x^3 - \frac{x^6}{30} - 11 \sin x$, into two parts, namely,

$f_1(\mathrm{x}) = x^2 + \mathrm{sinx}$, $f_2(\mathrm{x}) = 12x - 2x^3 - \frac{x^6}{30} - 12 \sin x$, such that, $f(\mathrm{x}) = f_1(\mathrm{x}) + f_2(\mathrm{x})$.

Then, $\mathrm{F}(u) = u(\mathrm{x}) - \mathrm{f}_1(\mathrm{x})$, $L(u) = u(\mathrm{x}) - \mathrm{f}_1(\mathrm{x}) - \mathrm{f}_2(\mathrm{x}) - 2\int_0^x \left((\mathrm{x} - \mathrm{y})^3 u(\mathrm{y})\right) dy = 0.$

Substituting $F(\mathrm{u})$ and L(u) in equation (4), and equating the terms with identical power of p, we get,

$$p^0: \quad u_0(\mathrm{x}) = \mathrm{f}_1(\mathrm{x}) = x^2 + \mathrm{sinx}$$

$$p^1: \quad u_1(\mathrm{x}) = \mathrm{f}_2(\mathrm{x}) + 2\int_0^x \left((\mathrm{x} - \mathrm{y})^3 u_0(\mathrm{y})\right) dy = 0$$

$$p^{k+1}: \quad u_{k+1}(\mathrm{x}) = \mathrm{f}_2(\mathrm{x}) + 2\int_0^x \left((\mathrm{x} - \mathrm{y})^3 u_k(\mathrm{y})\right) dy = 0$$

Then the exact solution of equation (8), is. $u(\mathrm{x}) = \lim_{p\to 1} u = u_0 + u_1 + u_2 + ... = x^2 + \mathrm{sinx}$

Example 3.

Consider the non-linear Fredholm integral equation,

$$u(\mathrm{x}) = \mathrm{cosx} - x + \int_0^1 x\left(\sin^2 y + u^2(\mathrm{y})\right)dy\,, \tag{9}$$

Use the same method in example 1, we assume that, $f_1(\mathrm{x}) = \mathrm{cosx}$, $f_2(\mathrm{x}) = -x$,

Because $\int_0^1 x\left(\sin^2 y + u^2(\mathrm{y})\right)dy = x\,,\quad at \quad u(\mathrm{x}) = \mathrm{cosx}$,Then we have,

$F(\mathrm{u}) = \mathrm{u(x)} - f_1(\mathrm{x})$, and $L(u) = u(\mathrm{x}) - \mathrm{f}_1(\mathrm{x}) - \mathrm{f}_2(\mathrm{x}) - \int_0^1 x\left(\sin^2 y + u^2(\mathrm{y})\right)dy$

Substituting $F(\mathrm{u})$ and L(u) in equation (4), and equating the terms with identical power of p, we get,

$$p^0: \quad u_0(\mathrm{x}) = \mathrm{f}_1(\mathrm{x}) = \mathrm{cosx}$$

$$p^1: \quad u_1(\mathrm{x}) = \mathrm{f}_2(\mathrm{x}) + \int_0^1 x\left(\sin^2 y + u_0^2(\mathrm{y})\right)dy = 0$$

$$p^{k+1}: \quad u_{k+1}(\mathrm{x}) = \mathrm{f}_2(\mathrm{x}) + \int_0^1 x\left(\sin^2 y + u_k^2(\mathrm{y})\right)dy = 0$$

Then the exact solution of equation (9), is, $u(\mathrm{x}) = \mathrm{cosx}$

Example 4.

Consider the following non-linear Volterra integral equation,

$$u(\mathrm{x}) = x - \int_0^x \left(\sinh(\mathrm{x}-\mathrm{y})\, u(\mathrm{y})\right)dy\,, \tag{10}$$

$f(\mathrm{x}) = \mathrm{x}$, then we can choose, $f_1(\mathrm{x}) = \mathrm{x} - \frac{x^3}{6}$, $f_2(\mathrm{x}) = \frac{x^3}{6}$, $\Rightarrow f(\mathrm{x}) = f_1(\mathrm{x}) + f_2(\mathrm{x})$

Because $\int_0^x \sinh(\mathrm{x}-\mathrm{y})u(\mathrm{y})dy = \frac{x^3}{6}$, $at \;\; u(\mathrm{x}) = \mathrm{x} - \frac{x^3}{6}$, then $F(\mathrm{u}) = \mathrm{u(x)} - f_1(\mathrm{x})$, and

$L(\mathrm{u}) = u(\mathrm{x}) - f_1(\mathrm{x}) - f_2(\mathrm{x}) + \int_0^x \sinh(\mathrm{x}-\mathrm{y})u(\mathrm{y})dy = 0$.

Substituting $F(\mathrm{u})$ and $\mathrm{L(u)}$ in equation (4), and equating the like power of p ,we get,

$$p^0: \quad u_0(x) = f_1(x) = x - \frac{x^3}{6}$$

$$p^1: \quad u_1(x) = f_2(x) - \int_0^x \sinh(x-y) u_0(y)\,dy = 0$$

$$p^{k+1}: \quad u_{k+1}(x) = f_2(x) - \int_0^x \sinh(x-y) u_k(y)\,dy = 0$$

Then we find the exact solution of equation (10) in the form**:** $u(x) = x - \frac{x^3}{6}$**.**

4. Conclusion

In this paper, we introduce the new modification of homotopy perturbation method, to obtain the exact solution of the second kind of non-linear integral equations such that Voltrra and Fredholm integral equations. This new method is based on the homotopy perturbation method, and has been applied directly without using linearization or any restrictive assumptions.

Conflict of Interests

The authors declare that there is no conflict of interests.

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