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ON EXISTENCE OF SOLUTIONS FOR INTEGRAL BOUNDARY VALUE PROBLEM OF DIFFERENTIAL EQUATIONS WITH FRACTIONAL ORDER $q \in (4,5]$

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Abstract. This paper studies the existence of solutions for nonlinear fractional differential equations of order $q \in (4,5]$ with integral boundary conditions. Our results are based on some fixed point theorems such as Banach, Krasnoselskii, and Leray-Schauder degree theory. The theorems are illustrated by an example.

Keywords: Existence; Fractional differential equations; Integral boundary; Fixed point theorems.

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1. Introduction

In this article, we will devote to considering the existence of solution of the integral and anti-periodic boundary value problem

$$(1)\qquad \begin{cases} {}^cD_{t_0}^q x(t) = f(t,x(t)), t \in J = [t_0,T], T > t_0, q \in (4,5] \\ x^{(k)}(t_0) - \theta_k x^{(k)}(T) = \beta_k \int_{t_0}^{T} g_k(t,x(t))dt, k = 0,1,2,3,4, \end{cases}$$

where ${}^cD_{t_0}^q$ denotes the Caputo fractional derivative of order q, f, and $g_k : J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\theta_k, \beta_k \in \mathbb{R}$ with $\theta_k \neq 1$ for each $k = 0,1,2,3,4$.

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The problem (1) can be considered as a generalization of many previous problems, for examples, but not exclusively, the anti periodic boundary value problems discussed in ([1]-[7]) with appropriate choices of θ_k and β_k.

Fractional differential equations have been widely used in many areas of science and engineering, which is due to the intensive developments of the theory of fractional calculus and the applications arising in various fields. For more details, we refer to [9],[12],[14], [15],[19],[21] and references therein. Particularly, anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes and have recently received considerable attention ([11]). Therefore, the existence of anti-periodic and integral boundary conditions for differential equations and inclusions of some orders are discussed in details (see [1]-[7],[10],[16]-[18],[20] and the references therein). The authors used some types of fixed point theorems to get sufficient conditions for the existence of a solution. Motivated by the theses works, we discuss the existence problem of (1) by using some fixed point theorems.

The interested fact for higher-order anti-periodic and integral fractional boundary value problem is the inheritance property of all characteristics of lower-order fractional anti-periodic problems. Hence, our results generalize the existing results on anti-periodic and integral fractional boundary value problems.

This paper is organized as follows. In Section 2, we introduce some preliminaries about fractional differential equations and related topics to main results. In Section 3, we discuss the main problems of existence of solutions for problem (1) by applying some well known fixed point theorems. To validate the theoretical manner, we give illustrative examples at the end of the article.

2. Preliminaries

We recall in this section some facts from fractional calculus (see [13]) and obtain a basic lemma that is essential for the results in the sequel.

The Riemann–Liouville fractional integral of order $q > 0$ is defined as

$$I_{t_0}^{q} f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(t)ds,$$

provided the integral exists. The Caputo derivative of a function x is defined as

$${}^{c}D_{t_0}^{q} x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} x^{(n)}(s)ds, n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Hereafter, we assume that x is a real valued function defined on J and has at most continuous fourth derivative, and f or y are fractional integrable functions of order q.

Lemma 2.1.([13]) *For $0 < n-1 < q < n$, we have*

$${}^{c}D_{t_0}^{q} \left(c_0 + c_1 (t-t_0) + c_2 (t-t_0)^2 + \cdots + c_{n-1} (t-t_0)^{n-1} \right) = 0,$$

where $c_k \in \mathbb{R}, k = 0,1,2,...,n-1$. Moreover

$$I_{t_0}^{q}\, {}^{c}D_{t_0}^{q} x(t) = x(t) + c_0 + c_1 (t-t_0) + c_2 (t-t_0)^2 + \cdots + c_{n-1} (t-t_0)^{n-1}.$$

Let $C(J,\mathbb{R})$ denotes the Banach space of all continuous functions endowed with the usual maximum norm. To study the existence problems of nonlinear problem (1), we need the following lemma.

Lemma 2.2. *For any $y \in C(J,\mathbb{R})$, the unique solution of the boundary value problem*

$$\begin{cases} {}^{c}D_{t_0}^{q} x(t) = y(t), t \in J, 4 < q \leq 5, \\ x^{(k)}(t_0) - \theta_k x^{(k)}(T) = \beta_k \int\limits_{t_0}^{T} g_k(t)dt,\ k = 0,1,2,3,4, \end{cases} \tag{2}$$

is

$$\begin{aligned} x(t) \quad = \quad & \int\limits_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\,ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k} \lambda_k(t) \int\limits_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} y(s)\,ds \\ & + \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k} \lambda_k(t) \int\limits_{t_0}^{T} g_k(s)ds \end{aligned} \tag{3}$$

where

$$\alpha_k = \prod_{m=0}^{k} (1-\theta_m), \lambda_k(t) = \sum_{m=0}^{k} \gamma_{m,k} \begin{pmatrix} k \\ m \end{pmatrix} (t-t_0)^m (T-t_0)^{k-m}, k = 0,1,2,3,4,$$

and

$$\begin{aligned}
\gamma_{0,0} =& 1, \gamma_{1,1} = \alpha_0, \gamma_{2,2} = \alpha_1, \gamma_{3,3} = \alpha_2, , \gamma_{4,4} = \alpha_3 \\
\gamma_{0,1} =& \theta_0, \gamma_{0,2} = \theta_0(\theta_1+1), \gamma_{0,3} = \theta_0\left(\theta_1\theta_2+2\theta_1+2\theta_2+1\right), \\
\gamma_{0,4} =& \theta_0\left(\theta_1\theta_2\theta_3+3\theta_1\theta_2+5\theta_1\theta_3+3\theta_2\theta_3+3\theta_1+3\theta_3+1\right), \\
\gamma_{1,2} =& 2\alpha_0\theta_1, \gamma_{1,3} = 3\alpha_0\theta_1(\theta_2+1), \gamma_{1,4} = 4\alpha_0\theta_1\left(\theta_2\theta_3+2\theta_2+2\theta_3+1\right), \\
\gamma_{2,3} =& 3\alpha_1\theta_2, \gamma_{2,4} = 6\theta_2\alpha_1\left(\theta_3+1\right), \gamma_{3,4} = 4\theta_3\alpha_2.
\end{aligned}$$

Proof. Using Lemma 2.1, for some constants $c_0, c_1, c_2.c_3, c_4 \in \mathbb{R}$, we have

$$\begin{aligned}
x(t) &= I_{t_0}^{q} y(t) - c_0 - c_1\left(t-t_0\right) - c_2\left(t-t_0\right)^2 - c_3\left(t-t_0\right)^3 - c_4\left(t-t_0\right)^4 \\
&= \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\,ds \\
&\quad -c_0 - c_1\left(t-t_0\right) - c_2\left(t-t_0\right)^2 - c_3\left(t-t_0\right)^3 - c_4\left(t-t_0\right)^4. \qquad (4)
\end{aligned}$$

Applying the boundary conditions for problem (2) in (4), we find that

$$\begin{aligned}
c_0 =& \frac{-\theta_0}{(1-\theta_0)} \int_{t_0}^{T} \frac{(T-s)^{q-1}}{\Gamma(q)} y(s)\,ds - \frac{\theta_0\theta_1\left(T-t_0\right)}{(1-\theta_0)(1-\theta_1)} \int_{t_0}^{T} \frac{(T-s)^{q-2}}{\Gamma(q-1)} y(s)\,ds \\
& - \frac{\theta_0\theta_2(\theta_1+1)\left(T-t_0\right)^2}{2(1-\theta_0)(1-\theta_1)(1-\theta_2)} \int_{t_0}^{T} \frac{(T-s)^{q-3}}{\Gamma(q-2)} y(s)\,ds \\
& - \frac{\theta_0\theta_3\left(\theta_1\theta_2+2\theta_1+2\theta_2+1\right)\left(T-t_0\right)^3}{6(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)} \int_{t_0}^{T} \frac{(T-s)^{q-4}}{\Gamma(q-3)} y(s)\,ds \\
& - \frac{\theta_0\theta_4\left(\theta_1\theta_2\theta_3+3\theta_1\theta_2+5\theta_1\theta_3+3\theta_2\theta_3+3\theta_1+3\theta_3+1\right)\left(T-t_0\right)^4}{24(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)} \times \\
& \int_{t_0}^{T} \frac{(T-s)^{q-5}}{\Gamma(q-4)} y(s)\,ds - \frac{\beta_0}{(1-\theta_0)} \int_{t_0}^{T} g_0(s,x(s))ds \\
& - \frac{\theta_0\beta_1\left(T-t_0\right)}{(1-\theta_0)(1-\theta_1)} \int_{t_0}^{T} g_1(s,x(s))ds \\
& - \frac{\theta_0\beta_2(\theta_1+1)\left(T-t_0\right)^2}{2(1-\theta_0)(1-\theta_1)(1-\theta_2)} \int_{t_0}^{T} g_2(s,x(s))ds
\end{aligned}$$

$$-\frac{\theta_0\beta_3\left(\theta_1\theta_2+2\theta_1+2\theta_2+1\right)(T-t_0)^3}{6(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)}\int_{t_0}^{T} g_3(s,x(s))ds$$

$$-\frac{\theta_0\beta_4\left(\theta_1\theta_2\theta_3+3\theta_1\theta_2+5\theta_1\theta_3+3\theta_2\theta_3+3\theta_1+3\theta_3+1\right)(T-t_0)^4}{24(1-\theta_0)(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)}\times$$

$$\int_{t_0}^{T} g_4(s,x(s))ds,$$

$$c_1 = -\frac{\theta_1}{(1-\theta_1)}\int_{t_0}^{T}\frac{(T-s)^{q-2}}{\Gamma(q-1)}y(s)\,ds-\frac{\theta_1\theta_2\,(T-t_0)}{(1-\theta_1)(1-\theta_2)}\int_{t_0}^{T}\frac{(T-s)^{q-3}}{\Gamma(q-2)}y(s)\,ds$$

$$-\frac{\theta_1\theta_3(\theta_2+1)\,(T-t_0)^2}{2(1-\theta_1)(1-\theta_2)(1-\theta_3)}\int_{t_0}^{T}\frac{(T-s)^{q-4}}{\Gamma(q-3)}y(s)\,ds$$

$$-\frac{\theta_1\theta_4\left(\theta_2\theta_3+2\theta_2+2\theta_3+1\right)(T-t_0)^3}{6(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T}\frac{(T-s)^{q-5}}{\Gamma(q-4)}y(s)\,ds$$

$$-\frac{\beta_1}{(1-\theta_1)}\int_{t_0}^{T} g_1(s,x(s))ds-\frac{\theta_1\beta_2\,(T-t_0)}{(1-\theta_1)(1-\theta_2)}\int_{t_0}^{T} g_2(s,x(s))ds$$

$$-\frac{\theta_1\beta_3(\theta_2+1)\,(T-t_0)^2}{2(1-\theta_1)(1-\theta_2)(1-\theta_3)}\int_{t_0}^{T} g_3(s,x(s))ds$$

$$-\frac{\theta_1\beta_4\left(\theta_2\theta_3+2\theta_2+2\theta_3+1\right)(T-t_0)^3}{6(1-\theta_1)(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T} g_4(s,x(s))ds,$$

$$c_2 = -\frac{\theta_2}{2(1-\theta_2)}\int_{t_0}^{T}\frac{(T-s)^{q-3}}{\Gamma(q-2)}y(s)\,ds-\frac{\theta_2\theta_3\,(T-t_0)}{2(1-\theta_2)(1-\theta_3)}\int_{t_0}^{T}\frac{(T-s)^{q-4}}{\Gamma(q-3)}y(s)\,ds$$

$$-\frac{\theta_2\theta_4\,(\theta_3+1)\,(T-t_0)^2}{4(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T}\frac{(T-s)^{q-5}}{\Gamma(q-4)}y(s)\,ds$$

$$-\frac{\beta_2}{2(1-\theta_2)}\int_{t_0}^{T} g_2(s,x(s))ds-\frac{\theta_2\beta_3\,(T-t_0)}{2(1-\theta_2)(1-\theta_3)}\int_{t_0}^{T} g_3(s,x(s))ds$$

$$-\frac{\theta_2\beta_4(\theta_3+1)(T-t_0)^2}{4(1-\theta_2)(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T} g_4(s,x(s))ds,$$

$$\begin{aligned} c_3 \;=\;& -\frac{\theta_3}{6(1-\theta_3)}\int_{t_0}^{T}\frac{(T-s)^{q-4}}{\Gamma(q-3)}y(s)\,ds-\frac{\theta_3\theta_4(T-t_0)}{6(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T}\frac{(T-s)^{q-5}}{\Gamma(q-4)}y(s)\,ds\\ &-\frac{\beta_3}{6(1-\theta_3)}\int_{t_0}^{T} g_3(s,x(s))ds-\frac{\theta_3\beta_4(T-t_0)}{6(1-\theta_3)(1-\theta_4)}\int_{t_0}^{T} g_4(s,x(s))ds, \end{aligned}$$

and

$$c_4=-\frac{\theta_4}{24(1-\theta_4)}\int_{t_0}^{T}\frac{(T-s)^{q-5}}{\Gamma(q-4)}y(s)\,ds-\frac{\beta_4}{24(1-\theta_4)}\int_{t_0}^{T} g_4(s,x(s))ds.$$

Substituting the values of c_0, c_1, c_2, c_3 and c_4 in (4), and arranging the terms into compact expression, one can obtain (3). This completes the proof.

3. Existence results

The main results of the article will be considered in this section. Before all, we state well-known fixed point theorems (see [8]) which are needed to prove the existence of solution for (1).

Theorem 3.1. *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta \in \Omega$ and let $\Psi : \overline{\Omega} \to X$ be a completely continuous operator such that $\|\Psi x\| \le \|x\|$, $x \in \partial\Omega$. Then Ψ has a fixed point in $\overline{\Omega}$.*

Theorem 3.2. *Let X be a Banach space. Assume that $\Psi : X \to X$ is completely continuous operator and the set $V = \{x \in X | x = \lambda \Psi x, 0 < \lambda < 1\}$ is bounded. Then Ψ has a fixed point in X.*

Theorem 3.3. *Let Ω be a closed convex and nonempty subset of a Banach space X. Let Φ, Θ be operators defined on Ω such that*

(i) $\Phi x + \Theta y \in \Omega$ whenever $x, y \in \Omega$;

(ii) Φ is compact and continuous;

(iii) Θ is a contraction mapping.

Then there exists $z \in \Omega$ such that $z = \Phi z + \Theta z$.

In view of Lemma 2.2, define an operator $\Psi : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ as

$$
\begin{aligned}
(\Psi x)(t) &= \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \sum_{k=0}^{4} \frac{\theta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s,x(s))ds \\
&\quad + \sum_{k=0}^{4} \frac{\beta_k}{k!\alpha_k}\lambda_k(t) \int_{t_0}^{T} g_k(s,x(s))ds. \qquad (5)
\end{aligned}
$$

Observe that problem (1) has a solution $x \in C(J,\mathbb{R})$ if and only if it satisfies the fixed point equation $\Psi x = x$. Before going on to first result, the following hypothesis is essential.

(A): Let $L_k, k = 0,1,2,3,4,5,$ be positive constants such that $|g_k(t,x(t))| \leq L_k$, and $|f(t,x(t))| \leq L_5$, for $t \in J,\ x \in C(J,\mathbb{R})$.

Lemma 3.4. *Assume that hypothesis (A) holds. Then, the operator Ψ is completely continuous.*

Proof. The continuity of g_k and f imply the continuity of the operator Ψ. By virtue of (5), we have

$$
\begin{aligned}
|(\Psi x)(t)| &\leq \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,x(s))| ds \\
&\quad + \sum_{k=0}^{4} \frac{|\theta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |f(s,x(s))| ds \\
&\quad + \sum_{k=0}^{4} \frac{|\beta_k|}{k!\,|\alpha_k|} |\lambda_k(t)| \int_{t_0}^{T} |g_k(s,x(s))| ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{L_5(t-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{|\lambda_k(t)|}{k!\,|\alpha_k|} \left(\frac{L_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + L_k\,|\beta_k|\,(T-t_0) \right) \\
&\leq \max_{t\in J} \left(\frac{L_5(t-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{|\lambda_k(t)|}{k!\,|\alpha_k|} \left(\frac{L_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + L_k\,|\beta_k|\,(T-t_0) \right) \right) \\
&= L
\end{aligned}
$$

which implies that $\|\Psi x\| \leq L$. Furthermore,

$$
\left| (\Psi x)^{'}(t) \right|
$$

$$
\begin{aligned}
&\leq \int_{t_0}^{t} \frac{(t-s)^{q-2}}{\Gamma(q-1)} |f(s,x(s))| ds \\
&\quad + \sum_{k=0}^{4} \frac{|\theta_k|}{k! |\alpha_k|} \left|\lambda_k^{'}(t)\right| \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} |f(s,x(s))| ds \\
&\quad + \sum_{k=0}^{4} \frac{|\beta_k|}{k! |\alpha_k|} \left|\lambda_k^{'}(t)\right| \int_{t_0}^{T} |g_k(s,x(s))| ds \\
&\leq \frac{L_5 (t-t_0)^{q-1}}{\Gamma(q)} + \sum_{k=0}^{4} \frac{\left|\lambda_k^{'}(t)\right|}{k! |\alpha_k|} \left(\frac{L_5 |\theta_k| (T-t_0)^{q-k}}{\Gamma(q-k+1)} + L_k |\beta_k| (T-t_0) \right) \\
&\leq \max_{t\in J} \left(\frac{L_5 (t-t_0)^{q}}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{\left|\lambda_k^{'}(t)\right|}{k! |\alpha_k|} \left(\frac{L_5 |\theta_k| (T-t_0)^{q-k}}{\Gamma(q-k+1)} + L_k |\beta_k| (T-t_0) \right) \right) \\
&= L^{'}
\end{aligned}
$$

and this implies that $\left\|(\Psi x)^{'}\right\| \leq L$. Hence, for $t_1,\ t_2 \in J$, we have

$$
|(\Psi x)(t_2) - (\Psi x)(t_1)| \leq \int_{t_1}^{t_2} \left|(\Psi x)^{'}(s)\right| ds \leq L^{'}(t_2 - t_1).
$$

This implies the equicontinuity of Ψ on J. Thus, by the Arzela-Ascoli theorem, the operator Ψ is completely continuous.

To establish the first existence result based on the fixed point Theorem 3.1, we need the following assumption.

(B): Let σ, τ and ρ be positive constants such that

$$
\begin{cases}
\tau = \max_{t\in J} \left(\frac{(t-t_0)^q}{\Gamma(q+1)} + \sum\limits_{k=0}^{4} \frac{|\lambda_k(t)|}{k! |\alpha_k|} \left(\frac{|\theta_k| (T-t_0)^{q-k}}{\Gamma(q-k+1)} + |\beta_k| (T-t_0) \right) \right), \sigma\tau < 1, \\
|f(t,x(t))| \leq \sigma |x(t)|, \\
|g_k(t,x)| \leq \sigma |x(t)|, k = 0,1,2,3,4,
\end{cases}
$$

for $|x(t)| < \rho, t \in J$.

Notice that assumption (A) can be followed by assumption (B).

Theorem 3.5. *Assume that hypothesis (B) holds. Then, the problem (1) has at least one solution.*

Proof. Define a bounded nonempty open subset $\Omega = \{x \in C(J,\mathbb{R}) : \|x\| < \rho\}$. Then, by Lemma 3.4, the operator $\Psi : \overline{\Omega} \to C(J,\mathbb{R})$ is completely continuous and satisfying

$$|(\Psi x)(t)| \le \sigma \left(\frac{(t-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{|\lambda_k(t)|}{k!\,|\alpha_k|} \left(\frac{|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + |\beta_k|\,(T-t_0) \right) \right) \|x\| . \tag{6}$$

In accordance with hypothesis (B), Equation (6) imply

$$\|\Psi x\| \le \|x\|,\ x \in \partial\Omega.$$

Hence, by Theorem 3.1, Ψ has at least one fixed point which is a solution of the problem (1). This finishes the proof.

Remark 3.6. *If alternatively, assuming that* $\lim_{x\to 0} \frac{f(t,x)}{x} = \lim_{x\to 0} \frac{g_k(t,x)}{x} = 0$, *and* $\varepsilon \max_{t\in J} \left(\frac{(t-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{|\lambda_k(t)|}{k!|\alpha_k|} \left(\frac{|\theta_k|(T-t_0)^{q-k}}{\Gamma(q-k+1)} + |\beta_k|\,(T-t_0) \right) \right) \le 1$, *for any positive constant* $\varepsilon > 0$, *we have the same result of Theorem 3.5.*

Assuming hypothesis (A), and letting $x \in C(J,\mathbb{R})$ such that $x = \lambda \Psi x$ for $0 < \lambda < 1$. Then, we have $\|x\| \le \|\Psi x\| \le L$. Hence, by virtue of Theorem 3.2, the following result follows.

Theorem 3.7. *Let Ψ be defined as in (5). If assumption (A) holds, then the problem (1) has a solution.*

Our next existence result is based on Krasnoselskii's fixed point Theorem 3.3, which needs the following assumption.

(C): Let $C_k \in \mathbb{R}^+$, $k = 0,1,2,3,4,5$, such that

$$\begin{cases} |g_k(t,x(t)) - g_k(t,y(t))| \le C_k\,|x(t)-y(t)|, k = 0,1,2,3,4, \\ |f(t,x(t)) - f(t,y(t))| \le C_5\,|x(t)-y(t)|, \end{cases}$$

for $t \in J, x, y \in C(J,\mathbb{R})$.

(D): Let $\mu_k \in C(J,\mathbb{R})$, $k = 0,1,2,3,4,5$, such that

$$\begin{cases} g_k(t,x(t)) \le \mu_k(t), k = 0,1,2,3,4, \\ f(t,x(t)) \le \mu_5(t), \end{cases}$$

for $t \in J, x, y \in C(J,\mathbb{R})$.

Theorem 3.8. *Let $f : J \times C(J,\mathbb{R}) \to C(J,\mathbb{R})$ be jointly continuous function. If assumptions (C) and (D) hold, then the problem (1) has a solution if*

$$\sum_{k=0}^{4} \frac{\|\lambda_k\|}{k!\,|\alpha_k|}\left(\frac{C_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + C_k\,|\beta_k|\,(T-t_0)\right) < 1. \tag{7}$$

Proof. Let $B_r = \{x \in C(J,\mathbb{R}) : \|x\| \le r\}$, for some fixed positive constant r that satisfying

$$r \ge \frac{\|\mu_5\|\,(T-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{\|\lambda_k\|}{k!\,|\alpha_k|}\left(\frac{\|\mu_5\|\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + \|\mu_k\|\,|\beta_k|\,(T-t_0)\right).$$

Define the operators Φ and Θ on B_r as

$$(\Phi x)(t) = \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))\,ds,$$

$$(\Theta x)(t) = \sum_{k=0}^{4} \frac{\lambda_k(t)}{k!\alpha_k}\left(\theta_k \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)} f(s,x(s))ds + \beta_k \int_{t_0}^{T} g_k(s,x(s))ds\right).$$

For $x, y \in B_r$, we find that

$$\|\Phi x + \Theta y\| \le r.$$

Thus, $\Phi x + \Theta y \in B_r$. Moreover, if $x, y \in B_r$, then

$$\begin{aligned} &|(\Theta y)(t) - (\Theta x)(t)| \\ \le\ & \sum_{k=0}^{4} \frac{|\lambda_k(t)|}{k!\,|\alpha_k|}\left(|\theta_k| \int_{t_0}^{T} \frac{(T-s)^{q-k-1}}{\Gamma(q-k)}\,|f(s,x(s))|\,ds + |\beta_k| \int_{t_0}^{T} |g_k(s,x(s))|\,ds\right). \end{aligned}$$

In accordance with (7), Θ is a contraction mapping on B_r.

Continuity of f implies the continuity of Φ. Also, Φ is uniformly bounded on B_r as $\|\Phi x\| \le \frac{\|\mu_5\|(T-t_0)^q}{\Gamma(q+1)}$. Next, we prove the compactness of Φ. Let $\sup_{(t,x)\in J\times Br} \|f(t,x)\| = f^* < \infty$, then, for $t_1, t_2 \in J$, we have

$$\begin{aligned} \|(\Phi x)(t_2) - (\Phi x)(t_1)\| &= \frac{1}{\Gamma(q)}\left\| \int_{t_0}^{t_1} \left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right] f(s,x(s))\,ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_2-s)^{q-1} f(s,x(s))\,ds \right\| \end{aligned}$$

$$\leq \frac{f^*}{\Gamma(q+1)}\left(2\left|t_2-t_1\right|^q+\left|(t_2-t_0)^q-(t_1-t_0)^q\right|\right),$$

which is independent of x and tends to zero as $t_2 \to t_1$. So Φ is relatively compact on B_r. Hence, by the Arzela-Ascoli theorem, Φ is compact on B_r. Thus all assumptions of Theorem 3.3 are satisfied. Therefore, the problem (1) has a solution. This completes the proof.

The existence and uniqueness result can be obtained by the well-known Banach fixed point theorem.

Theorem 3.9. *Let $f, g_k : J\times C(J,\mathbb{R}) \to C(J,\mathbb{R}), k = 0,1,2,3,4,$ be jointly continuous functions satisfying the hypothesis (C). Then the problem (1) has a unique solution if*

$$\frac{C_5(T-t_0)^q}{\Gamma(q+1)}+\sum_{k=0}^{4}\frac{\|\lambda_k\|}{k!\,|\alpha_k|}\left(\frac{C_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)}+C_k\,|\beta_k|\,(T-t_0)\right)<1$$

Proof. Setting $\sup_{t\in J}|g_k(t,0)| = M_k$, and $\sup_{t\in J}|f(t,0)| = M_5$, and selecting

$$r \geq \frac{\frac{M_5(T-t_0)^q}{\Gamma(q+1)}+\sum_{k=0}^{4}\frac{\|\lambda_k\|}{k!|\alpha_k|}\left(\frac{M_5|\theta_k|(T-t_0)^{q-k}}{\Gamma(q-k+1)}+M_k\,|\beta_k|\,(T-t_0)\right)}{1-\gamma}$$

where

$$\gamma=\frac{C_5(T-t_0)^q}{\Gamma(q+1)}+\sum_{k=0}^{4}\frac{\|\lambda_k\|}{k!\,|\alpha_k|}\left(\frac{C_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)}+C_k\,|\beta_k|\,(T-t_0)\right).$$

If $x\in B_r=\{x\in C(J,\mathbb{R}) : \|x\|\leq r\}$, then

$$\begin{aligned}
|(\Psi x)(t)| \leq\; & \max_{t\in J}\left\{\int_{t_0}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds\right.\\
&+\sum_{k=0}^{4}\frac{|\theta_k|}{k!\,|\alpha_k|}\,|\lambda_k(t)|\int_{t_0}^{T}\frac{(T-s)^{q-k-1}}{\Gamma(q-k)}(|f(s,x(s))-f(s,0)|+|f(s,0)|)ds\\
&\left.+\sum_{k=0}^{4}\frac{|\beta_k|}{k!\,|\alpha_k|}\,|\lambda_k(t)|\int_{t_0}^{T}(|g_k(s,x(s))-g_k(s,0)|+|g_k(s,0)|)ds\right\}\\
\leq\; & \frac{M_5(T-t_0)^q}{\Gamma(q+1)}+\sum_{k=0}^{4}\frac{\|\lambda_k\|}{k!\,|\alpha_k|}\left(\frac{M_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)}+M_k\,|\beta_k|\,(T-t_0)\right)\\
&+\left(\frac{C_5(T-t_0)^q}{\Gamma(q+1)}+\sum_{k=0}^{4}\frac{\|\lambda_k\|}{k!\,|\alpha_k|}\left(\frac{C_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)}+C_k\,|\beta_k|\,(T-t_0)\right)\right)r
\end{aligned}$$

$$\leq \quad (1-\gamma)r+\gamma r = r.$$

Hence Ψ maps the subset B_r into itself. Now, for $x,y \in B_r$, and $t \in J$, we obtain

$$\begin{aligned}
&|(\Psi x)(t)-(\Psi y)(t)| \\
\leq\ & \max_{t\in J}\Bigg\{\int_{t_0}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|\,ds \\
&+\sum_{k=0}^{4}\frac{|\theta_k|}{k!\,|\alpha_k|}|\lambda_k(t)|\int_{t_0}^{T}\frac{(T-s)^{q-k-1}}{\Gamma(q-k)}|f(s,x(s))-f(s,y(s))|\,ds \\
&+\sum_{k=0}^{4}\frac{|\beta_k|}{k!\,|\alpha_k|}|\lambda_k(t)|\int_{t_0}^{T}|g_k(s,x(s))-g_k(s,y(s))|\,ds\Bigg\} \\
\leq\ & \max_{t\in J}\left(\frac{C_5(t-t_0)^q}{\Gamma(q+1)}ds+\sum_{k=0}^{4}\frac{|\lambda_k(t)|}{k!\,|\alpha_k|}\left(\frac{C_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)}+C_k\,|\beta_k|\,(T-t_0)\right)\right)\|x-y\| \\
\leq\ & \gamma\|x-y\|,
\end{aligned}$$

Since γ depends only upon the parameters involved in the problem. Therefore, Ψ is a contraction operator. Hence, the conclusion of the theorem follows by the Banach fixed point theorem. This finishes the proof.

The last result of existence problems is due to the Leray-Schauder degree theorem. The following hypothesis is sufficient for the next theorem.

(E): Let $A_k, B_k, k=0,1,2,3,4,5,$ be positive constants satisfying

$$\begin{cases} |g_k(t,x(t))| \leq A_k\,|x(t)|+B_k, k=0,1,2,3,4 \\ |f(t,x(t))| \leq A_5\,|x(t)|+B_5 \end{cases}$$

for $t \in J, x \in C(J,\mathbb{R})$. Moreover, assume

$$\begin{cases} B = \frac{B_5(T-t_0)^q}{\Gamma(q+1)}+\sum\limits_{k=0}^{4}\frac{\|\lambda_k\|}{k!|\alpha_k|}\left(\frac{B_5|\theta_k|(T-t_0)^{q-k}}{\Gamma(q-k+1)}+B_k\,|\beta_k|\,(T-t_0)\right) \\ A = \frac{A_5(T-t_0)^q}{\Gamma(q+1)}+\sum\limits_{k=0}^{4}\frac{\|\lambda_k\|}{k!|\alpha_k|}\left(\frac{A_5|\theta_k|(T-t_0)^{q-k}}{\Gamma(q-k+1)}+A_k\,|\beta_k|\,(T-t_0)\right) < 1 \end{cases}$$

Theorem 3.10. *Let $f, g_k : J\times C(J,\mathbb{R}) \to C(J,\mathbb{R})$ be jointly continuous functions. If hypothesis (E) holds, then the problem (1) has at least one solution.*

Proof. Define a fixed point problem by

$$x = \Psi x, \tag{8}$$

where Ψ is defined by (5). Then we only need to prove the existence of at least one solution $x \in C(J,\mathbb{R})$ satisfying (8). Define a suitable ball $B_r \subset C(J,\mathbb{R})$ with radius $r > 0$ as $B_r = \{x \in C(J,\mathbb{R}) : \|x\| < r\}$, where r will be fixed later. Then, it is sufficient to show that $\Psi : \overline{B_r} \to C(J,\mathbb{R})$ satisfies

$$0 \notin (I - \lambda\Psi)(\partial B_r) \tag{9}$$

for any $\lambda \in [0,1]$. Here I denotes the identity operator. Let us define the homotopy

$$h_\lambda(x) = x - \lambda\Psi x, x \in X,\ \lambda \in [0,1].$$

Then, following the same steps of proof Lemma 3.4, $h_\lambda = I - \lambda\Psi$ is completely continuous. Using condition (9), and the homotopy invariance property of Leray-Schauder degrees, we have

$$\begin{aligned} deg(h_\lambda, B_r, 0) &= deg((I - \lambda\Psi), B_r, 0) = deg(h_1, B_r, 0) \\ &= deg(h_0, B_r, 0) = deg(I, B_r, 0) = 1 \neq 0,,\ 0 \in B_r. \end{aligned}$$

Hence, there is at least one $x \in B_r$, such that equation (8) is true. It remains to find the constant r satisfying (9). Therefore, for any $t \in J$, and $x \in B_r$ satisfying $x = \lambda\Psi x$ for some $\lambda \in [0,1]$, we have

$$\begin{aligned} |x(t)| &= |\lambda\Psi x(t)| \\ &\leq \frac{B_5(T-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{\|\lambda_k\|}{k!\,|\alpha_k|} \left(\frac{B_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + B_k\,|\beta_k|\,(T-t_0) \right) \\ &\quad \left(\frac{A_5(T-t_0)^q}{\Gamma(q+1)} + \sum_{k=0}^{4} \frac{\|\lambda_k\|}{k!\,|\alpha_k|} \left(\frac{A_5\,|\theta_k|\,(T-t_0)^{q-k}}{\Gamma(q-k+1)} + A_k\,|\beta_k|\,(T-t_0) \right) \right) \|x\| \\ &= B + A\,\|x\|. \end{aligned}$$

Hence

$$\|x\| \leq \frac{B}{1-A}.$$

Choosing $r > \frac{A}{1-B}$, then equation (9) holds. This completes the proof.

Example 3.11. *Consider the problem*

$$\text{(10)} \qquad \begin{cases} {}^cD_1^{4.2}x(t) = \frac{1}{(e^t+2)^3}\frac{|x(t)|}{|x(t)|+1}, t \in [1,2], \\ x^{(k)}(0) + x^{(k)}(1) = \frac{1}{2^{k+1}}\int\limits_1^2 \frac{\sin x(t)}{(t+2)^3} ds, k = 0,1,2,3,4. \end{cases}$$

where x is a real valued function defined on $J = [1,2]$. *In accordance with all above hypotheses, simple calculations led to*

$$\begin{cases} L_k = C_k = \sigma = \frac{1}{8}, \\ \mu_k = \frac{1}{(t+2)^3}, k = 0,1,2,3,4, \mu_5 = \frac{1}{(t+2)^3} \\ \tau = 0.665 < 1, A = \gamma = \frac{1}{8}\tau < 1. \end{cases}$$

Hence, using any of above theorems, the existence of solution for the problem (10) can be obtained.

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