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TRIVIAL GENERALIZED GROUP OF UNITS OF THE RING $\mathbf{Z}[i]/<\beta>$

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Abstract. In this article we study the structure of the generalized k th group of units, $U^k(R)$ of the quotient ring $R = Z[i]/<\beta>$. In particular, we consider the case where the generalized k th group of units, $U^k(R)$ is the trivial group for a fixed k.

Keywords: Gaussian primes; group of units; ring of Gaussian integers.

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1. Introduction

The fundamental theorem of finite abelian groups states that any finite abelian group G is isomorphic to a direct product of cyclic groups. That is, $G \cong \mathbf{Z}_{n_1} \times \mathbf{Z}_{n_2} \times \ldots \times \mathbf{Z}_{n_r}$. Hence, the group of units of a finite commutative ring with identity is isomorphic to a direct product of cyclic groups. The decomposition of U_n, the group of units of Z_n, into a product of cyclic groups of prime power order is given in the following theorem.

Theorem 1.1. Let $n = p_1^{a_1}.p_2^{a_2}...p_r^{a_r}$ be the decomposition of n into product of distinct prime powers. Then, $U_n \cong U_{p_1^{a_1}} \times U_{p_2^{a_2}} \times \ldots \times U_{p_r^{a_r}}$. Moreover,

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(1) $U_2 \cong \mathbf{Z}_1$

(2) $U_{2^2} \cong \mathbf{Z}_2$

(3) $U_{2^a} \cong \mathbf{Z}_2 \times \mathbf{Z}_{2^{a-2}}$, where $a > 2$.

(4) $U_{p^a} \cong \mathbf{Z}_{p-1} \times \mathbf{Z}_{p^{a-1}}$, where p is an odd prime.

If R is a finite commutative ring with identity, then $U(R)$ denotes its group of units. It is well known that if R decomposes as a direct sum of rings, $R = R_1 \oplus R_2 ... \oplus R_r$, then $U(R) \cong U(R_1) \times U(R_2) \times ... \times U(R_r)$. In [2], El-Kassar and Chehade generalized the concept of the group of units as follows: the multiplicative group $U(R)$ support a ring structure by defining the operations $\oplus$ and $\otimes$ on $U(R)$ that makes $(U(R), \oplus, \otimes)$ a ring isomorphic to $U(R_1) \oplus U(R_2) \oplus ... \oplus U(R_r)$. The ring $U(R_1) \oplus U(R_2) \oplus ... \oplus U(R_r)$ will be denoted by $R^2 \cong U(R)$ and R^1 denote the ring R. They defined $U^2(R)$ to be the group of units of the ring $R^2 \cong U(R)$ so that $U^2(R) = U(R^2) \cong U(U(R))$ and in general $U^m(R) = U(R^m) \cong U(U^{m-1}(R))$. The group $U^m(R)$ is the mth group of units of the ring R. That is, if $R = R_1 \oplus R_2 \oplus ... \oplus R_r$, then $U^m(R) \cong U^m(R_1) \times U^m(R_2) \times ... \times U^m(R_r)$.

Theorem 1.2. Let γ_1, γ_2, ..., γ_r be distinct Gaussian prime integers and let $\beta = \prod_{j=1}^{r} \gamma_j^{n_j}$, then $\mathbf{Z}[i]/ < \beta > \cong \mathbf{Z}[i]/ < \gamma_1^{n_1} > \oplus \mathbf{Z}[i]/ < \gamma_2^{n_2} > \oplus ... \oplus \mathbf{Z}[i]/ < \gamma_r^{n_r} >$ and $U^k(\mathbf{Z}[i]/ < \beta >) \cong U^k\left(\mathbf{Z}[i]/ < \gamma_1^{n_1} >\right) \times U^k\left(\mathbf{Z}[i]/ < \gamma_2^{n_2} >\right) \times ... \times U^k\left(\mathbf{Z}[i]/ < \gamma_r^{n_r} >\right)$. We may also write $U^k(\beta) \cong U^k\left(\gamma_1^{n_1}\right) \times U^k\left(\gamma_2^{n_2}\right) \times ... \times U^k\left(\gamma_r^{n_r}\right)$.

Throughout this paper,

- m, n and r always denote positive integers,
- p and p_j always denote prime integers that are congruent to 3 modulo 4,
- γ and γ_j always denote Gaussian prime integers,
- π and π_j always denote Gaussian prime integers of the form $a + bi$ where a and b are non zero integers,
- $q = \pi\overline{\pi}$ and $q_j = \pi_j\overline{\pi}_j$ always.
- q and q_j always denote prime integers that are congruent to 1 modulo 4,
- S_1, S_2 and S_3 always denote the sets $\{1,2,3,4,5,6,7,8,9\}$, $\{1,2\}$ and $\{5,13\}$ respectively.

The problem of classifying the group of units of an arbitrary finite commutative ring with identity is an open problem. However, the problem is solved for certain classes. In the case when

$R=\mathbf{Z}_n$, it is well-known that U_n is cyclic if and only if $n=2,\,4,\,p^\alpha,q^\alpha,\,2p^\alpha$ or $2q^\alpha$. Also, Cross [1] showed that the group of units of the quotient ring of Gaussian integers, $U(\mathbf{Z}[i]/<\beta>)$, is cyclic if and only if $\beta=1+i,\,(1+i)^2,\,(1+i)^3,\,p,\,(1+i)p,\,\pi^n,\,(1+i)\pi^n$. Cross did not mention when the group of units $U(\mathbf{Z}[i]/<\beta>)=U(\beta)$ is trivial. The goal of this paper is to study trivial case of the group of units of the ring $\mathbf{Z}[i]/<\beta>$ and its generalization $U^m(\beta)$ in the special cases when $m=2$ or 3.

It is well known that the Boolean rings are the only rings with trivial group of units. El-Kassar and Chehade considered the problem of determining all rings R for which $U^m(R)$ is the trivial group for a fixed m. They completely solved the problem of determining all rings $\mathbf{Z}_n$ with a trivial second group of units and they showed that if $U^2(\mathbf{Z}_n)$ is trivial, then n must be a product of at most two prime power factors when $n>1$. El-Kassar and Chehade proved that $U^2(\mathbf{Z}_n)\cong\{0\}$ if and only if $n=2,3,4,6,8,12$ or 24. They also considered the problem of determining the values of m for which the mth group of units of $\mathbf{Z}_n$ is trivial for some special values of m and n.

2. Trivial group of Units

The structure of the group of units of $Z[i]/<\beta>$ is given by Cross as stated in the below theorem.

Theorem 2.1.

(1) $U(\pi^n)\cong\mathbf{Z}_{q^n-q^{n-1}}$.
(2) $U(p^n)\cong\mathbf{Z}_{p^{n-1}}\times\mathbf{Z}_{p^{n-1}}\times\mathbf{Z}_{p^2-1}$.
(3) $U((1+i)^n)\cong\mathbf{Z}_{2^{m-1}}\times\mathbf{Z}_{2^{m-2}}\times\mathbf{Z}_4$ if $n=2m$.
(4) $U((1+i)^n)\cong\mathbf{Z}_{2^{m-1}}\times\mathbf{Z}_{2^{m-1}}\times\mathbf{Z}_4$ if $n=2m+1$.

Note that $U(\mathbf{Z}[i]/<1+i>)\cong U(\mathbf{Z}_2)\cong\{0\}$.

Lemma 2.2. $U((1+i)^n)$ is not trivial for every $n\geq 2$.

Proof. For $n=2$, we have $U\left(\mathbf{Z}[i]/\left\langle(1+i)^2\right\rangle\right)\cong U(\mathbf{Z}_2[i])\cong\mathbf{Z}_2\ncong\{0\}$. If $n>2$, then

$$U\left((1+i)^n\right) \cong \begin{cases} \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_4 \text{ if } n = 2m+1 \\ \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_{2^{m-2}} \times \mathbf{Z}_4 \text{ if } n = 2m \end{cases}$$

and since $\mathbf{Z}_4$ is non trivial, then $U\left((1+i)^n\right)$ is non trivial for every $n \geq 2$.

Lemma 2.3. $U(p^n)$ is non trivial for every n.

Proof. $U(p^n) \cong \mathbf{Z}_{p^{n-1}} \times \mathbf{Z}_{p^{n-1}} \times \mathbf{Z}_{p^2-1}$, so $U(p^n)$ is trivial if and only if $\mathbf{Z}_{p^{n-1}}$ and $\mathbf{Z}_{p^2-1}$ are trivial. But $Z_{p^{n-1}}$ is trivial for every prime p if $n = 1$ while that $\mathbf{Z}_{p^2-1}$ is trivial only if $p^2 = 2$. This contradicts that p is a prime integer.

Lemma 2.4. $U(\pi^n)$ is non trivial for every n.

Proof. $U(\pi^n) \cong \mathbf{Z}_{q^n - q^{n-1}}$ and $\mathbf{Z}_{q^n - q^{n-1}}$ is trivial if and only if $q^n - q^{n-1} = 4k(4k+1)^{n-1} = 1$. The last equation has no integer solution for any integer k.

Using the previous lemmas, we obtain the following theorem.

Theorem 2.5. If β is a Gaussian prime power integer other than $1+i$, then $U(\beta)$ is non trivial.

3. Trivial Second group of Units

In this section, we find all values of β such that $U^2(\beta)$ is trivial. Note that
$U^2\left(\mathbf{Z}[i]/\left\langle(1+i)^2\right\rangle\right) \cong U^2\left(\mathbf{Z}_2[i]\right) \cong U\left(\mathbf{Z}_2\right) \cong \{0\}$.

Lemma 3.1. $U^2((1+i)^n)$ is non trivial for every $n \geq 3$.

Proof. $U\left((1+i)^n\right) \cong \begin{cases} \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_4 \text{ if } n = 2m+1 \\ \mathbf{Z}_{2^{m-1}} \times \mathbf{Z}_{2^{m-2}} \times \mathbf{Z}_4 \text{ if } n = 2m \end{cases}$.

Then, $U^2\left((1+i)^n\right) \cong \begin{cases} U(\mathbf{Z}_{2^{m-1}}) \times U(\mathbf{Z}_{2^{m-1}}) \times U(\mathbf{Z}_4) \text{ if } n = 2m+1 \\ U(\mathbf{Z}_{2^{m-1}}) \times U(\mathbf{Z}_{2^{m-2}}) \times U(\mathbf{Z}_4) \text{ if } n = 2m \end{cases}$

but $U(\mathbf{Z}_4) \cong \mathbf{Z}_2 \ncong \{0\}$.

Lemma 3.2. $U^2(p^n)$ is non trivial for every n.

Proof. $U^2(p^n) \cong U(\mathbf{Z}_{p^{n-1}}) \times U(\mathbf{Z}_{p^{n-1}}) \times U(Z_{p^2-1})$. Since $p^2 - 1 > 1$, then $U(\mathbf{Z}_{p^2-1}) \ncong \{0\}$ and $U(\mathbf{Z}_{p^2-1}) \cong \{0\}$ if and only if $p^2 - 1 = 2$. Hence $p = \sqrt{3}$ and the proof is complete.

Lemma 3.3. $U^2(\pi^n)$ is non trivial for every n.

Proof. Let $t = q^n - q^{n-1}$, then $U(\pi^n) \cong \mathbf{Z}_t$ and $U^2(\pi^n) \cong U(\mathbf{Z}_t)$. In [2], the authors showed that $U(\mathbf{Z}_t) \cong \{0\}$ if and only if $t = 2$. For $t = 2$, we have $2k(4k+1)^{n-1} = 1$ and this equation has no integer solution.

Using the preceding lemmas and note, we have the following theorem that is a direct consequence of theorem 2.5

Theorem 3.4. If $\beta = \prod_{j=1}^{r} \gamma_j^{n_j}$ with $n_r \geq 1$ and $\gamma_1, \gamma_2, ..., \gamma_r$ are distinct, then $U^2(\beta)$ is trivial if and only if $\beta = (1+i)$ or $(1+i)^2$.

4. Trivial Third group of Units

Since the second group of units, $U^2(\beta)$, is not trivial except for $\beta = (1+i)$ or $(1+i)^2$, then it is natural to study the trivial case for the higher order generalized group of units. In this section, we study in particular the trivial case for the third group of units, $U^3(\beta)$, of the quotient ring $\mathbf{Z}[i]/<\beta>$. We prove that $U^3(\beta)$ is trivial if and only if β is a product of at most six distinct Gaussian prime integers. In our work we study the cases(each alone) where β is divisible by

(1) One Gaussian prime.
(2) Two distinct Gaussian primes.
(3) Three distinct Gaussian primes.
(4) Four distinct Gaussian primes.
(5) Five distinct Gaussian primes.
(6) Six distinct Gaussian primes.

A complete characterization for the trivial case of $U^3(\beta)$ is given.

4.1 Prime Power Factor

In this subsection, the trivial case for the third group of units will be studied when β is divisible by one prime factor.

Lemma 4.1.1. If $\beta = (1+i)^n$, then $U^3(\beta)$ is trivial if and only if $n \in S_1$.

Proof. If $n \leq 2$, then $U^2((1+i)^n) \cong \{0\}$ and hence $U^3((1+i)^n) \cong \{0\}$. For $n = 3$, we have $U((1+i)^3) \cong \mathbf{Z}_4$ and hence $U^3((1+i)^3)$ is trivial. Also, $U((1+i)^4) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_4$ and $U^3((1+i)^4) \cong \{0\}$. Now, if $n > 4$, then

$$\begin{aligned} U^3((1+i)^n) &\cong \begin{cases} U^2(\mathbf{Z}_{2^{m-1}}) \times U^2(\mathbf{Z}_{2^{m-1}}) \times U^2(\mathbf{Z}_4) & \text{if } n = 2m+1 \\ U^2(\mathbf{Z}_{2^{m-1}}) \times U^2(\mathbf{Z}_{2^{m-2}}) \times U^2(\mathbf{Z}_4) & \text{if } n = 2m \end{cases} \\ &\cong \begin{cases} U^2(\mathbf{Z}_{2^{m-1}}) \times U^2(\mathbf{Z}_{2^{m-1}}) & \text{if } n = 2m+1 \\ U^2(\mathbf{Z}_{2^{m-1}}) \times U^2(\mathbf{Z}_{2^{m-2}}) & \text{if } n = 2m \end{cases} \end{aligned}$$

Case n is odd ($n = 2m+1$):

$U^2(\mathbf{Z}_{2^{m-1}}) \cong \{0\}$ if and only if $m-1 = 1$, 2 or 3. Hence, $n = 5, 7$ or 9.

Case n is even ($n = 2m$):

$U^3((1+i)^n) \cong \{0\}$ if and only if $U^2(\mathbf{Z}_{2^{m-1}}) \cong \{0\} \cong U^2(\mathbf{Z}_{2^{m-2}})$ if and only if $m = 3$ or 4 and hence $n = 6$ or 8.

Lemma 4.1.2. If $\beta = p^n$, then $U^3(\beta)$ is trivial if and only if $p = 3$ and $n \in S_2$.

Proof. $U^3(p^n) \cong U^2(\mathbf{Z}_{p^{n-1}}) \times U^2(\mathbf{Z}_{p^{n-1}}) \times U^2(\mathbf{Z}_{p^2-1})$. Note that $U^3(p) \cong U^2(\mathbf{Z}_{p^2-1})$. But $p^2 - 1 = 8(2k^2 + 3k + 1)$.

If $k = 0$, then $p = 3$ and $U^2(\mathbf{Z}_{p^2-1}) = U^2(\mathbf{Z}_8) \cong \{0\}$.

If $k \geq 1$, then $p^2 - 1 \geq 48$ and hence $U^2(\mathbf{Z}_{p^2-1}) \ncong \{0\}$.

For $n \geq 2$, then $U^2(\mathbf{Z}_{p^{n-1}}) \cong \{0\}$ if and only if $p = 3$ and $n = 2$.

Lemma 4.1.3. If $\beta = \pi^n$, then $U^3(\beta)$ is trivial if and only if $n = 1$ and $q \in S_3$.

Proof. $U^3(\pi^n) \cong U^2(\mathbf{Z}_t)$, where $t = 4k(4k+1)^{n-1}$. But $U^2(\mathbf{Z}_t) \cong \{0\}$ if and only if $t = 2, 3, 4, 6, 8, 12$ or 24. But t is even, so $t = 3$ is dismissed. No integer solution for the cases where $t = 2$ or 6. The cases where $t = 8$ and 24 give $q = 9$ and 25 respectively and hence rejected. If $t = 4$, then $k(4k+1)^{n-1} = 1$. Hence $k = n = 1$ and $q = 5$. For $t = 12$, we get $k = 3$ and $n = 1$ and hence $q = 13$.

The below theorem illustrates the trivial case for the third group of units of the ring $\mathbf{Z}[i]/<\beta>$ when β is divisible by one Gaussian prime integer.

Theorem 4.1.4. If $\beta = \gamma^n$, then $U^3(\beta)$ is trivial if and only if

(1) $\beta=(1+i)^n$ with $n\in S_1$;

(2) $\beta=3$ or 3^2;

(3) $\beta=(1\pm 2i)$ or $(2\pm 3i)$.

4.2 Two Prime Power Factors

The trivial case for the third group of units when β is divisible by two prime factors is studied.

Lemma 4.2.1. If $\beta=(1+i)^n p^m$, then $U^3(\beta)$ is trivial if and only if $p=3$, $n\in S_1$ and $m\in S_2$.

Proof. Since $U^3(\beta)\cong U^3((1+i)^n)\times U^3(p^m)$, then the proof follows by using theorem 4.1.4.

The proof of the next three lemmas follows directly from theorem 4.1.4.

Lemma 4.2.2. If $\beta=(1+i)^n\pi^r$, then $U^3(\beta)$ is trivial if and only if $r=1$, $n\in S_1$ and $q\in S_3$.

Lemma 4.2.3. If $\beta=p^m\pi^r$, then $U^3(\beta)$ is trivial if and only if $\beta=3\pi$ or $3^2\pi$ with $q\in S_3$.

Lemma 4.2.4. If $\beta=\pi_1^{r_1}\pi_2^{r_2}$, then $U^3(\beta)$ is trivial if and only if $\beta=(1+2i)(1-2i),(2+3i)(2-3i),(1\pm 2i)(2\pm 3i)$.

Lemma 4.2.5. If $\beta=p_1^{m_1}p_2^{m_2}$, then $U^3(\beta)$ is non trivial.

Proof. $U^3(\beta)$ is trivial if both $U^3(p_1^{m_1})$ and $U^3(p_2^{m_2})$ are trivial and by theorem 4.1.4, we have $p_1=p_2=3$ which contradicts that p_1 and p_2 are distinct.

The above results are summarized in the following theorem.

Theorem 4.2.6. Let $n\in S_1,\ m\in S_2$ and $q,\ q_1,\ q_2\in S_3$. If $\beta=\gamma_1^{n_1}\gamma_2^{n_2}$, then $U^3(\beta)$ is trivial if and only if one of the following is true:

(1) $\beta=3^m(1+i)^n$.

(2) $\beta=(1+i)^n\pi$.

(3) $\beta=3^m\pi$.

(4) $\beta=\pi_1\pi_2$.

4.3 Three Prime Power Factors

We study the trivial case for the third group of units when β is divisible by three distinct prime factors.

Lemma 4.3.1. If $\beta = (1+i)^n \prod_{j=1}^{2} p_j^{m_j}$, then $U^3(\beta)$ is non trivial.

Proof. $U^3(\beta) \cong U^3((1+i)^n) \times U^3(p_1^{m_1} p_2^{m_2})$ and by lemma 4.2.5, $U^3(p_1^{m_1} p_2^{m_2})$ is non trivial.

The proof of the next lemma arises directly from theorem 4.1.4.

Lemma 4.3.2. If $\beta = (1+i)^n \pi_1^{r_1} \pi_2^{r_2}$, then $U^3(\beta)$ is trivial if and only if $\beta = (1+i)^n(1+2i)(1-2i)$, $(1+i)^n(2+3i)(2-3i)$, $(1+i)^n(1 \pm 2i)(2 \pm 3i)$ with $n \in S_1$.

In lemma 4.2.5, we see that $U^3(p_r^{m_r} p_s^{m_s})$ is non trivial when $r \neq s$. Then $U^3(\beta)$ is non trivial when $\beta = \prod_{j=1}^{3} p_j^{m_j}$ or $\beta = \left(\prod_{j=1}^{2} p_j^{m_j} \right) \pi^r$.

Using lemmas 2.4 and 4.2.4, the following lemma arise.

Lemma 4.3.3. If $\beta = p^m \prod_{j=1}^{2} \pi_j^{r_j}$, then $U^3(\beta)$ is trivial if and only if $\beta = 3^m(1+2i)(1-2i)$, $3^m(2+3i)(2-3i)$, $3^m(1 \pm 2i)(2 \pm 3i)$ with $m \in S_2$.

Lemma 4.3.4. If $\beta = \prod_{j=1}^{3} \pi_j^{r_j}$, then $U^3(\beta)$ is trivial if and only if one of the following is true:

(1) $\beta = (1-2i)(1+2i)(2 \pm 3i)$.

(2) $\beta = (1 \pm 2i)(2-3i)(2+3i)$.

Proof. $U^3(\beta) \cong U^3(\pi_1^{r_1}) \times U^3(\pi_2^{r_2}) \times U^3(\pi_3^{r_3})$ and by lemma 4.1.3, we have $U^3(\pi_s^{r_s})$ is trivial if $r_s = 1$ with $\pi_s = 1 \pm 2i$ or $\pi_s = 2 \pm 3i$. Since π_1, π_2 and π_3 are distinct, then $\beta = (1-2i)(1+2i)(2 \pm 3i)$ or $\beta = (1 \pm 2i)(2-3i)(2+3i)$.

Lemma 4.3.5. If $\beta = (1+i)^n p^m \pi^r$, then $U^3(\beta)$ is trivial if and only if $\beta = 3^m(1+i)^n \pi$ with $n \in S_1$, $m \in S_2$ and $q \in S_3$.

Proof. Follows directly from lemmas 4.1.1, 4.1.2 and 4.1.3.

Summarizing lemmas 4.3.1 to 4.3.5, we get the following general result.

Theorem 4.3.6. Let $n \in S_1,\ m \in S_2,\ j \in \{1,2,3\}$ and $q,\ q_j \in S_3$. If $\beta = \prod_{j=1}^{3} \gamma_j^{n_j}$, then $U^3(\beta)$ is trivial if and only if β equals one of the following:

(1) $3^m(1+i)^n \pi$.

(2) $(1+i)^n \pi_1 \pi_2$.

(3) $3^m\pi_1\pi_2$.

(4) $\beta = \pi_1\pi_2\pi_3$.

4.4 Four Prime Power Factors

The case where β is product of four distinct Gaussian prime powers is given next. The proof can be obtained in a similar manner used in the proof of the previous lemmas.

Lemma 4.4.1. If $\beta = (1+i)^n\prod_{j=1}^{3}p_j^{m_j}$, then $U^3(\beta)$ is non trivial.

Lemma 4.4.2. If $\beta = (1+i)\left(\prod_{j=1}^{2}p_j^{m_j}\right)\pi^r$, then $U^3(\beta)$ is non trivial.

Lemma 4.4.3. If $n \in S_1$ and $\beta = (1+i)^n p^m \prod_{j=1}^{2}\pi_j^{r_j}$, then $U^3(\beta)$ is trivial if and only if

(1) $\beta = 3(1+i)^n(1\pm 2i)(2\pm 3i)$.

(2) $\beta = 3^2(1+i)^n(1\pm 2i)(2\pm 3i)$.

Lemma 4.4.4. Let $n \in S_1$and let $\beta = (1+i)^n\prod_{j=1}^{3}\pi_j^{r_j}$, then $U^3(\beta)$ is trivial if and only if

(1) $\beta = (1+i)^n(1-2i)(1+2i)(2\pm 3i)$.

(2) $\beta = (1+i)^n(1\pm 2i)(2-3i)(2+3i)$.

Lemma 4.4.5. If $\beta = \prod_{j=1}^{4}p_j^{m_j}$, then $U^3(\beta)$ is non trivial.

Lemma 4.4.6. If $\beta = \left(\prod_{j=1}^{3}p_j^{m_j}\right)\pi^r$, then $U^3(\beta)$ is non trivial.

Lemma 4.4.7. If $\beta = \left(\prod_{j=1}^{2}p_j^{m_j}\right)\left(\prod_{j=1}^{2}\pi_j^{r_j}\right)$, then $U^3(\beta)$ is non trivial.

Lemma 4.4.8. If $\beta = p^m\prod_{j=1}^{3}\pi_j^{r_j}$, then $U^3(\beta)$ is trivial if and only if $\beta = 3^m(1-2i)(1+2i)(2\pm 3i)$ or $3^m(1\pm 2i)(2-3i)(2+3i)$ with $m \in S_2$.

Lemma 4.4.9. If $\beta = \prod_{j=1}^{4}\pi_j^{r_j}$, then $U^3(\beta)$ is trivial if and only if $\beta = (1-2i)(1+2i)(2-3i)(2+3i)$.

The following general result illustrates lemmas 4.4.1 to 4.4.9.

Theorem 4.4.10. Let $n \in S_1,\ m \in S_2,\ j \in \{1,2,3,4\}$ and $q,\ q_j \in S_3$. If $\beta = \prod_{j=1}^{4} \gamma_j^{n_j}$, then $U^3(\beta)$ is trivial if and only if one of the following is true:

(1) $\beta = 3^m (1+i)^n \prod_{j=1}^{2} \pi_j$.

(2) $\beta = (1+i)^n \prod_{j=1}^{3} \pi_j$.

(3) $\beta = 3^m \prod_{j=1}^{3} \pi_j$.

(4) $\beta = \prod_{j=1}^{4} \pi_j$.

4.5 Five Prime Power Factors

In this subsection, β is divisible by five distinct prime factors. The proof of the below lemmas will be omitted.

Lemma 4.5.1. If $\beta = (1+i)^n \prod_{j=1}^{4} p_j^{m_j}$, then $U^3(\beta)$ is non trivial.

Lemma 4.5.2. If $\beta = (1+i)^n \pi^r \prod_{j=1}^{3} p_j^{m_j}$, then $U^3(\beta)$ is non trivial.

Lemma 4.5.3. If $\beta = (1+i)^n \left(\prod_{j=1}^{2} \pi_j^{r_j} \right) \left(\prod_{j=1}^{2} p_j^{m_j} \right)$, then $U^3(\beta)$ is non trivial.

Lemma 4.5.4. Let $n \in S_1,\ m \in S_2$ and let $\beta = (1+i)^n p^m \prod_{j=1}^{3} \pi_j^{r_j}$. Then $U^3(\beta)$ is trivial if and only if $\beta = 3^m (1+i)^n (1-2i)(1+2i)(2 \pm 3i)$ or $\beta = 3^m (1+i)^n (1 \pm 2i)(2-3i)(2+3i)$.

Lemma 4.5.5. Let $n \in S_1$ and let $\beta = (1+i)^n \prod_{j=1}^{4} \pi_j^{r_j}$. Then $U^3(\beta)$ is trivial if and only if $\beta = (1+i)^n (1-2i)(1+2i)(2-3i)(2+3i)$.

Lemma 4.5.6. If $\beta = \prod_{j=1}^{5} \pi_j^{r_j}$, then $U^3(\beta)$ is non trivial.

Lemma 4.5.7. Let $m \in S_2$ and let $\beta = p^m \prod_{j=1}^{4} \pi_j^{r_j}$. Then $U^3(\beta)$ is trivial if and only if
$\beta = 3^m (1-2i)(1+2i)(2-3i)(2+3i)$.

Lemma 4.5.8. If $\beta = \left(\prod_{j=1}^{2} p_j^{m_j} \right) \left(\prod_{j=1}^{3} \pi_j^{r_j} \right)$, then $U^3(\beta)$ is non trivial.

Lemma 4.5.9. If $\beta = \left(\prod_{j=1}^{3} p_j^{m_j}\right)\left(\prod_{j=1}^{2} \pi_j^{r_j}\right)$, then $U^3(\beta)$ is non trivial.

Lemma 4.5.10. If $\beta = \left(\prod_{j=1}^{4} p_j^{m_j}\right)\pi^r$, then $U^3(\beta)$ is non trivial.

Lemma 4.5.11. If $\beta = \prod_{j=1}^{5} p_j^{m_j}$, then $U^3(\beta)$ is non trivial.

Combining lemmas 4.5.1 to 4.5.11, the following general result is obtained.

Theorem 4.5.12. Let $n \in S_1,\ m \in S_2,\ j \in \{1,2,3,4\}$ and $q,\ q_j \in S_3$ and let $\beta = \prod_{j=1}^{5} \gamma_j^{n_j}$. Then $U^3(\beta)$ is trivial if and only if one of the following is true:

(1) $\beta = 3^m(1+i)^n \prod_{j=1}^{3} \pi_j$.

(2) $\beta = (1+i)^n \prod_{j=1}^{4} \pi_j$.

(3) $\beta = 3^m \prod_{j=1}^{4} \pi_j$.

4.6 Six Prime Power Factors

Following the same argument in the above subsections, we consider the case where β is divisible by six distinct prime factors. The following theorem shows up.

Theorem 4.6.1. Let $n \in S_1,\ m \in S_2,\ j \in \{1,2,3,4\}$ and $q,\ q_j \in S_3$ and let $\beta = \prod_{j=1}^{6} \gamma_j^{n_j}$. Then $U^3(\beta)$ is trivial if and only if $\beta = 3^m(1+i)^n \prod_{j=1}^{4} \pi_j$.

5. Main Result

Now, it is clear that $U^3(\beta)$ is non trivial if β is a product of more than six prime power factors. Combining all the theorems stated in this article, the following main result is obtained.

Theorem 5.1. Let $n \in S_1,\ m \in S_2$ and let $\beta = \prod_{j=1}^{r} \gamma_j^{n_j}$. Then $U^3(\beta)$ is trivial if and only if $r \leq 6$ and one of the following is true:

(1) $\beta = (1+i)^n$;

(2) $\beta = 3^m$;

(3) $\beta = (1 \pm 2i)$ or $(2 \pm 3i)$;

(4) $\beta = 3^m(1+i)^n$;

(5) $\beta = (1+i)^n(1 \pm 2i)$ or $(1+i)^n(2 \pm 3i)$;

(6) $\beta = 3^m(1 \pm 2i)$ or $3^m(2 \pm 3i)$;

(7) $\beta = (1 \pm 2i)(2 \pm 3i)$;

(8) $\beta = 3^m(1+i)^n(1 \pm 2i)$ or $3^m(1+i)^n(2 \pm 3i)$;

(9) $\beta = (1+i)^n(1 \pm 2i)(2 \pm 3i)$;

(10) $\beta = 3^m(1 \pm 2i)(2 \pm 3i)$;

(11) $\beta = (1-2i)(1+2i)(2 \pm 3i)$;

(12) $\beta = (1 \pm 2i)(2-3i)(2+3i)$;

(13) $\beta = 3^m(1+i)^n(1 \pm 2i)(2 \pm 3i)$;

(14) $\beta = (1+i)^n(1-2i)(1+2i)(2 \pm 3i)$;

(15) $\beta = (1+i)^n(1 \pm 2i)(2-3i)(2+3i)$;

(16) $\beta = 3^m(1-2i)(1+2i)(2 \pm 3i)$;

(17) $\beta = 3^m(1 \pm 2i)(2-3i)(2+3i)$;

(18) $\beta = (1-2i)(1+2i)(2-3i)(2+3i)$;

(19) $\beta = 3^m(1+i)^n(1-2i)(1+2i)(2 \pm 3i)$;

(20) $\beta = 3^m(1+i)^n(1 \pm 2i)(2-3i)(2+3i)$;

(21) $\beta = (1+i)^n(1-2i)(1+2i)(2-3i)(2+3i)$;

(22) $\beta = 3^m(1-2i)(1+2i)(2-3i)(2+3i)$;

(23) $\beta = 3^m(1+i)^n(1-2i)(1+2i)(2-3i)(2+3i)$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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