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CORRESPONDING DEVELOPABLE RULED SURFACES IN EUCLIDEAN 3-SPACE E^3

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Abstract. In this paper, motion of Darboux vector on two different space curves in Euclidean 3-space is investigated. The structure of the motion is based on ruled surfaces generated by Darboux vector $\vec{\omega}$. According to this, developability of the considered ruled surfaces are studied. An extensive comparison between these developable ruled surfaces is performed. Some relations between the corresponding curves on the developable ruled surfaces are performed. Finally, special correspondence is given and plotted.

Keywords: Ruled surface; Darboux vector; Frenet frame.

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1. Introduction

Ruled surfaces are one of the important and interesting subjects for line geometry in Euclidean and semi-Euclidean 3-space [1, 2, 3, 4, 5, 6]. Use of developable ruled surfaces has a long history [7, 8]. Real developable ruled surfaces have natural applications in many areas of engineering and manufacturing. For instance, an aircraft designer uses them to design the

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airplane wings, and a tinsmith uses them to connect two tubes of different shapes with planar segments of metal sheets. A developable surface is a surface that can be (locally) unrolled onto a flat plane without tearing or stretching it. If a developable surface lies in three-dimensional Euclidean space, and is complete, then it is necessarily ruled, but the converse is not always true. For instance, the cylinder and cone are developable, but the general hyperboloid of one sheet is not. More generally, any developable surface in three-dimensions is part of a complete ruled surface, and so itself must be locally ruled [9].

In the study of the fundamental theory and the characterizations of space curves, the corresponding relations between the curves are very interesting and important problem. The well-known Bertrand curves are characterized as a kind of such corresponding relation between the two curves. For the Bertrand curve $\vec{\alpha}$, it shares the normal lines with another curve $\vec{\beta}$, called Bertrand mate or Bertrand partner curve of $\vec{\beta}$ [10]. In [11] Y. Yayli studied the motion of Frenet Frame on two different space curves in Euclidean 3-space.

In this study, we classify the ruled surfaces generated by Darboux vector on a space curve. Furthermore, we study the relation between the corresponding developable ruled surfaces when their base curves are Bertrand and obtain some results related to the mean curvatures and curves on these surfaces.

2. Basic concepts and properties

Now we review some basic concepts on classical differential geometry of space curves in Euclidean space. For any two vectors $x=(x_1,x_2,x_3)$ and $y=(y_1,y_2,y_3)$, we denote $\langle x, y\rangle$ as the standard inner product. Let $\vec{\alpha} : \mathrm{I} \to R^3$ be a curve with $\vec{\alpha}'(s) \neq 0$, where $\vec{\alpha}'(s) = d\vec{\alpha}(s)/ds$. We also denote the norm of x by $\|x\|$. The *arc-length parameter s* of a curve $\vec{\alpha}$ is determined such that $\|\vec{\alpha}'(s)\| = 1$, where $\vec{\alpha}'(s) = d\vec{\alpha}(s)/ds$. Let us denote $\vec{e_1}(s) = \vec{\alpha}'(s)$ and we call $\vec{e_1}(s)$ *a unit tangent vector* of $\vec{\alpha}$ at s. We define the *curvature* of $\vec{\alpha}$ by $\kappa(s) = \|\vec{\alpha}''(s)\|$. If $\kappa(s) \neq 0$, then the *unit principal normal vector* $\vec{e_2}(s)$ of the curve $\vec{\alpha}$ at s is given by $\vec{\alpha}''(s) = \kappa(s)\ \vec{e_2}(s)$. The unit vector $\vec{e_3}(s) = \vec{e_1}(s) \wedge \vec{e_2}(s)$ is called the *unit binormal vector* of $\vec{\alpha}$ at s, where $\wedge$ stands the cross product in E^3. Then we have the following *Frenet-Serret* formulae

$$\begin{aligned} \vec{e}_1{}'(s) &= \kappa(s)\vec{e}_2(s), \\ \vec{e}_2{}'(s) &= -\kappa(s)\vec{e}_1(s) + \tau(s)\vec{e}_3(s), \\ \vec{e}_3{}'(s) &= -\tau(s)\vec{e}_2(s), \end{aligned} \tag{2.1}$$

where $\tau(s)$ is the *torsion* of the curve $\vec{\alpha}(s)$ at s [8].

2.1. Ruled surfaces

A ruled surface in differential geometry is a surface formed by a motion of a straight line. The lines that belongs to this surface are called generators(rullings), and every curve that intersects all the generators is called a base curve(directrix). The ruled surface in Euclidean 3-space is given by the parametrization

$$\phi(s,v) = \vec{\alpha}(s) + v\vec{L}(s), \quad \|\vec{L}(s)\| = 1, \tag{2.2}$$

where $\vec{\alpha}(s)$ is the base curve and it is a differential curve parameterized by its arc-length in Euclidean 3-space that is

$$\langle \vec{\alpha}'(s), \vec{\alpha}'(s) \rangle = 1,$$

and $\vec{L}$(s) is the generator vector of the ruled surface such that $\vec{L}$(s) is orthogonal to the tangent vector field of the base curve. If the tangent plane is constant along the generator, then the ruled surface is called developable ruled surface [11]. Let $\vec{n}$ denote the unit normal vector field on the surface (2.2) which is given by

$$\vec{n} = \frac{\phi_s \wedge \phi_v}{|\phi_s \wedge \phi_v|}, \qquad \phi_s = \frac{\partial \phi}{\partial s}, \qquad \phi_v = \frac{\partial \phi}{\partial v} \tag{2.3}$$

Then the metric I of the surface (2.2) is defined by

$$I = g_{11}ds^2 + 2g_{12}dsdv + g_{22}dv^2, \tag{2.4}$$

with differentiable coefficients

$$g_{11} = \langle \phi_s, \phi_s \rangle, \quad g_{12} = \langle \phi_s, \phi_v \rangle, \quad g_{22} = \langle \phi_v, \phi_v \rangle, \tag{2.5}$$

where $\langle , \rangle$ is the Euclidean inner product in E^3 and the discriminate g of the first fundamental form is

$$g = Det(g_{ij}) = g_{11}g_{22} - (g_{12})^2. \tag{2.6}$$

The second fundamental form II of $\phi(s,v)$ is given as

$$II = h_{11}ds^2 + 2h_{12}dsdv + h_{22}dv^2, \tag{2.7}$$

with differentiable coefficients

$$h_{11} = \langle \phi_{ss}, \vec{n} \rangle, \quad h_{12} = \langle \phi_{sv}, \vec{n} \rangle, \quad h_{22} = \langle \phi_{vv}, \vec{n} \rangle. \tag{2.8}$$

The discriminate h of the second fundamental form is

$$h = Det(h_{ij}) = h_{11}h_{22} - (h_{12})^2. \tag{2.9}$$

For the parametrization of the ruled surface (2.2), we have the mean curvature H and the Gaussian curvature K as in the following

$$H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2g}, \quad K = \frac{h}{g}. \tag{2.10}$$

Definition 2.1. The parameter of distribution λ of the ruled surface (2.2) is defined as the limit of the ratio of the shortest distance between the two rulings and their angulated angle which is given by [12]

$$\lambda = \frac{\det(\vec{\alpha}', \vec{L}, \vec{L}')}{\langle \vec{L}', \vec{L}' \rangle}. \tag{2.11}$$

The ruled surface (2.2) is developable if its distribution parameter λ is zero, i.e., $\lambda = 0$.

Definition 2.2. For any unit speed curve

$$\vec{\alpha}(s) : I \subset R \to R^3,$$

Darboux vector field of $\vec{\alpha}$ is defined as a unit vector in the form [13]

$$\vec{\omega}(s) = \frac{\tau(s)\vec{e}_1(s) + \kappa(s)\vec{e}_3(s)}{\sqrt{\kappa^2 + \tau^2}}. \tag{2.12}$$

Furthermore, one can see that

$$\vec{\omega}'(s) = \sigma(s)(\kappa\vec{e}_1 - \tau\vec{e}_3)(s), \tag{2.13}$$

where

$$\sigma(s) = \Big(\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}(\frac{\tau}{\kappa})'\Big)(s) \tag{2.14}$$

3. Darboux vector and ruled surfaces

The ruled surface generated by the Darboux vector of a space curve $\vec{\alpha}(s)$ has the parametrization

$$\phi(s,v) = \vec{\alpha}(s) + v\vec{\omega}(s), \tag{3.1}$$

where $\phi(s,v)$ is a developable ruled surface, see [14, 15]. Let H be a moving space and $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the Frenet-Frame field along the curve $\vec{\alpha}(s)$: I$\rightarrow$ E^3, then we represent the moving line space H as follows

$$H = Sp\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}_{\alpha(s)},$$

where $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ are the tangent, principal normal and binormal vectors, respectively. Consider a curve $\vec{\beta}(\bar{s})$ in the moving space H, such that the tangent vector of $\vec{\beta}(\bar{s})$ is $\vec{\beta}'(\bar{s})$, i.e.

$$\vec{\beta} : J \to E^3,$$

$$\bar{s} \mapsto \vec{\beta}(\bar{s}), \ \ \bar{s} = \bar{s}(s).$$

At this time, Frenet-Frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ of $\vec{\alpha}(s)$ can be thought on the curve $\vec{\beta}(\bar{s})$. Then, we can get ruled surface that produced by the Darboux vector of the Frenet-Frame of the curve $\vec{\alpha}(s)$ as in the following

$$\bar{\phi}(\bar{s},\bar{v}) = \vec{\beta}(\bar{s}) + \bar{v}\vec{\omega}(\bar{s}), \ \ \bar{s} = \bar{s}(s), \ \ \bar{v} = \bar{v}(v). \tag{3.2}$$

The two coordinate systems $\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\{O'; \vec{e'_1}, \vec{e'_2}, \vec{e'_3}\}$ are orthogonal coordinate systems in E^3 which represent the moving space H and the fixed space H' respectively, see Figure (1). Let express the displacements (H/H') of H with respect to H'. During the one parameter spatial motion H/H', the Darboux vector of the moving space H generates, in generally, a ruled surface in the fixed spaceH'.

Now, for the base curve $\vec{\beta}(\bar{s})$ of the ruled surface (3.2), we can write

$$\vec{\beta}'(\bar{s}) = \frac{d\vec{\beta}}{d\bar{s}}\frac{d\bar{s}}{ds} = M_1\vec{e}_1(\bar{s}) + M_2\vec{e}_2(\bar{s}) + M_3\vec{e}_3(\bar{s}), \ \ ' = \frac{d}{ds}. \tag{3.3}$$

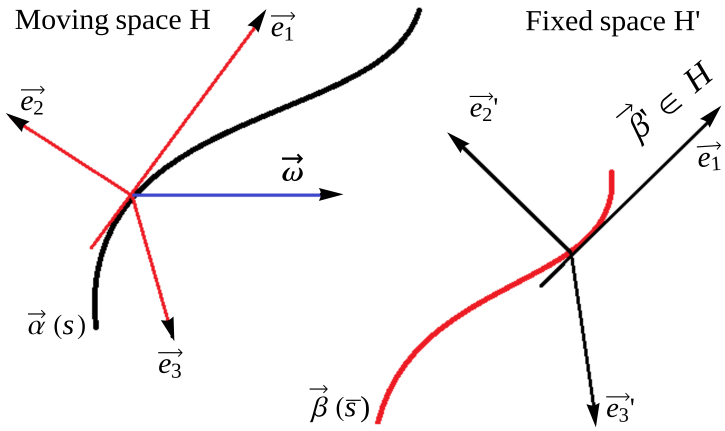

FIGURE 1. The moving and the fixed spaces H and H'.

Without lose of generality, we take $\bar{s} = s$ and $\bar{v} = v$. Thus the parameter of distribution of the ruled surface (3.2) is given by Eq. (2.11) as follows

$$\lambda = \frac{\det(\vec{\beta}', \vec{\omega}, \vec{\omega}')}{\langle \vec{\omega}', \vec{\omega}' \rangle}. \tag{3.4}$$

From Eqs. (2.13), (2.14) and (3.4), one can see the following

$$\det(\vec{\beta}', \vec{\omega}, \vec{\omega}') = \sqrt{\kappa^2(s) + \tau^2(s)}\, M_2\, \sigma(s). \tag{3.5}$$

According to Eq. (3.5), let investigate special cases for developability.

3.1 Special cases for developability

The ruled surface (3.2) is developable $(\lambda = 0)$ if

(1) σ(s) = 0 $\Leftrightarrow \frac{\tau}{\kappa}$=const, implies that $\vec{\alpha}$(s) is a helix curve.

(2) $M_2 = 0 \Rightarrow$ the vector $\vec{\beta}'$ lies on the rectifying plane of the curve $\vec{\alpha}$, i.e.,

$$\vec{\beta}' = M_1 \vec{e}_1 + M_3 \vec{e}_3.$$

The last case can be degenerate into three cases as follows

(i) $M_1 = 1$ and $M_3 = 0 \Rightarrow \vec{\beta}$(s) = $\vec{\alpha}$(s) (trivial case).

(ii) $M_1 = 0$ and $M_3 = 1 \Rightarrow \vec{\dot{\beta}}$(s) = $\vec{e}_3$ (the binormal of the curve $\vec{\alpha}$).

(iii) $M_1 \neq 0$ and $M_3 \neq 0$, hence if $\vec{\alpha}$(s) and $\vec{\beta}$(s) are Bertrand curves, then $\vec{\beta}$(s) can be represented as

$$\vec{\beta}(s) = \vec{\alpha}(s) + a\vec{e}_2, \;\; a = const. \tag{3.6}$$

From (2.1), we have

$$\vec{\beta}'(s) = (1 - a\kappa)\vec{e}_1 + a\tau\vec{e}_3, \tag{3.7}$$

where M_1=1-aκ and M_3=aτ.

From the above results, we have the following

Lemma 3.1. *The ruled surface $\bar{\phi}(\bar{s},\bar{v}) = \vec{\beta}(\bar{s}) + \bar{v}\omega(s)$, which is obtained by the Darboux vector$\vec{\omega}(s)$ of the Frenet-frame of a curve $\vec{\alpha}(s)$, is developable if one of the following cases is satisfied*

(1) $\vec{\alpha}(s)$ is a helix curve.

(2) $\vec{\beta}(s)$ lies in the rectifying plane of the curve $\vec{\alpha}(s)$.

(3) $\vec{\alpha}(s)$ and $\vec{\beta}(s)$ are Bertrand curves.

4. Corresponding developable ruled surfaces

In this section, we study the developable ruled surfaces $\phi(s,v)$ and $\bar{\phi}(\bar{s},\bar{v})$ such that the base curves are of type Bertrand. For the developable ruled surfaces

$$\phi(s,v) = \vec{\alpha}(s) + v\vec{\omega}(s), \tag{4.1}$$

$$\bar{\phi}(\bar{s},\bar{v}) = \vec{\beta}(\bar{s}) + \bar{v}\vec{\omega}(\bar{s}), \tag{4.2}$$

where

$$\vec{\beta}(s) = M_1\vec{e}_1 + M_3\vec{e}_3, \quad (\bar{v} = v, \bar{s} = s). \tag{4.3}$$

Since the curves $\vec{\alpha}$(s)and $\vec{\beta}$(s) are Bertrand , then the curve $\vec{\beta}$(s) can be represented as the following

$$\vec{\beta}(s) = \vec{\alpha}(s) + a\vec{e}_2. \tag{4.4}$$

Thus, the corresponding developable ruled surface is given as

$$\bar{\phi}(s,v) = \phi(s,v) + a\vec{e}_2. \tag{4.5}$$

The tangents to the parametric curves on the surface $\bar{\phi}$ are given as

$$\bar{\phi}_s = \phi_s + a(\tau\vec{e}_3 - \kappa\vec{e}_1). \tag{4.6}$$

$$\bar{\phi}_v = \phi_v = \vec{\omega} = \frac{1}{\sqrt{\kappa^2+\tau^2}}\Big(\tau\vec{e}_1 + \kappa\vec{e}_3\Big). \tag{4.7}$$

Thus, using Eqs. (2.5) and (4.5) we can get the first fundamental quantities as follows

$$g_{\bar{1}1} = g_{11} + (\tau^2+\kappa^2)(a^2 - 2av\sigma) - 2a\kappa. \tag{4.8}$$

Also,

$$g_{\bar{1}2} = g_{12} = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}, \qquad g_{\bar{2}2} = g_{22} = 1. \tag{4.9}$$

Therefore, the discriminate $\bar{g}$ of the first fundamental form on the surface $\bar{\phi}$ as follows

$$\bar{g} = g + (\tau^2+\kappa^2)(a^2 - 2av\sigma) - 2a\kappa, \quad a > 2v\sigma. \tag{4.10}$$

Hence, we have the following corollary

Corollary 4.1. *The first fundamental forms for the two corresponding developable ruled surface are related by the quadratic form*

$$\bar{I} = I + \big((\tau^2+\kappa^2)(a^2 - 2av\sigma) - 2a\kappa\big)ds^2, \ \ a > 2v\sigma. \tag{4.11}$$

In addition, let $\vec{n}$ and $\vec{\bar{n}}$ and be the unit normal vectors for the corresponding developable ruled surfaces (4.1) and (4.2), respectively, then by using (2.3), one can get the following

Corollary 4.2. *The unit normal vectors for the corresponding developable ruled surfaces (4.5) are in one to one corresponding, i.e.,*

$$\vec{\bar{n}} = \vec{n} = \vec{e}_2. \tag{4.12}$$

The second derivative for the function $\bar{\phi}$ are in the form

$$\bar{\phi}_{ss} = \phi_{ss} + a\big(-\kappa'\vec{e}_1 - (\tau^2+\kappa^2)\vec{e}_2 + \tau'\vec{e}_3\big). \tag{4.13}$$

Also,

$$\bar{\phi}_{sv} = \phi_{sv} = \vec{\omega}' = \sigma(\kappa\vec{e}_1 - \tau\vec{e}_3), \quad \bar{\phi}_{vv} = \phi_{vv} = 0. \tag{4.14}$$

Using (2.8), we can get the second fundamental quantities as follows

$$\bar{h}_{11} = h_{11} - a(\tau^2 + \kappa^2), \quad \bar{h}_{12} = h_{12} = 0, \quad \bar{h}_{22} = h_{22} = 0. \tag{4.15}$$

Making use (2.7), we can get the following

Corollary 4.3. *The second fundamental forms for the corresponding developable ruled surfaces are related through the quadratic form*

$$\bar{I}I = II - a(\tau^2 + \kappa^2)ds^2 \tag{4.16}$$

4.1. Mean curvature

Let H and $\bar{H}$ be the mean curvatures for the corresponding developable ruled surfaces $\phi(s,v)$ and $\bar{\phi}(s,v)$ respectively. Subsequently, we can give the relation between H and $\bar{H}$ by using Eq. (2.10) as follows

$$\bar{H} = \frac{H - \frac{a\mu}{2}}{1 + a^2\mu - 4aH}, \tag{4.17}$$

where

$$H = \frac{\kappa + v\sigma(\tau^2 + \kappa^2)}{2g}, \tag{4.18}$$

and $\mu = \frac{\tau^2+\kappa^2}{g}, \quad 4aH \neq 1 + a^2\mu.$

According to Eq.(4.17), let discuss the following two cases

(i) If $\bar{H} = 0$ ($\bar{\phi}$ is minimal), then

$$H = \frac{a\mu}{2}. \tag{4.19}$$

Hence, case (i) gives the following

Corollary 4.4. *The distance between the corresponding developable surface $\phi(s,v)$ and $\bar{\phi}(s,v)$ such that $\bar{\phi}(s,v)$ is a part of a plane is given by*

$$a = \frac{2H}{\mu}. \tag{4.20}$$

(ii) If $H \leq 0$, then

$$H - \frac{a\mu}{2} < 0, \quad \textit{and} \quad 1 + a^2\mu - 4aH > 0, \tag{4.21}$$

From Eq. (4.17), we can get

$$\bar{H} < 0. \tag{4.22}$$

Hence, case(ii) gives the following

Corollary 4.5. *If the mean curvature of $\phi(s,v)$ is non-positive ($H \leq 0$), then the mean curvature for the corresponding developable surface $\bar{\phi}(s,v)$ is negative ($\bar{H} < 0$).*

4.2. Orthogonal trajectory of the rulings

If a point P displaced orthogonally along the ruling $\vec{\omega}$ to a neighbouring point P_0, then we have an orthogonal trajectory

$$\vec{\gamma} : I \to \phi(s,v)$$

Let $\vec{\gamma}$ and $\vec{\bar{\gamma}}$ be the orthogonal trajectory of the ruling on the corresponding developable ruled surfaces $\phi(s,v)$ and $\bar{\phi}(s,v)$ respectively, so it can be expressed as

$$\vec{\gamma}(s) = \vec{\alpha}(s) + v(s)\vec{\omega}(s), \tag{4.23}$$

$$\vec{\bar{\gamma}}(s) = \vec{\beta}(s) + \bar{v}(s)\vec{\omega}(s). \tag{4.24}$$

For the developable ruled surface $\phi(s,v)$, the condition that the point P be displaced orthogonally to the ruling is

$$\langle \frac{d\vec{\gamma}}{ds}, \vec{\omega}(s) \rangle = 0. \tag{4.25}$$

Taking the derivative of Eq. (4.23) and using Eq. (2.12), one can get the following

$$v' = \frac{-\tau}{\sqrt{\kappa^2 + \tau^2}}. \tag{4.26}$$

Integrating, we have

$$v = \int \frac{-\tau}{\sqrt{\kappa^2 + \tau^2}} ds + c. \tag{4.27}$$

On the other hand, by a similar way, for the developable ruled surface $\bar{\phi}(s,v)$, by using Eqs. (2.12), (4.4) and (4.23), we get

$$\bar{v} = \int \frac{-\tau}{\sqrt{\kappa^2 + \tau^2}} ds + c. \tag{4.28}$$

Substituting υ into Eqs. (4.23) and (4.24) we have

$$\vec{\gamma}(s) = \vec{\alpha}(s) - \vec{\omega}(s) \int \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} ds + c, \tag{4.29}$$

$$\vec{\bar{\gamma}}(s) = \vec{\beta}(s) - \vec{\omega}(s) \int \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} ds + c. \tag{4.30}$$

Subtracting the Eqs. (4.29) and (4.30) we get

$$|\vec{\bar{\gamma}}(s) - \vec{\gamma}(s)| = \vec{\beta}(s) - \vec{\alpha}(s) = a. \tag{4.31}$$

Hence, we have the following

Corollary 4.6. *The distance between the orthogonal trajectory of the ruling on the corresponding developable ruled surface $\phi(s,v)$ and $\bar{\phi}(s,v)$ is a constant distance, equal to the distance between base curves.*

5. Corresponding curves on developable ruled surfaces

In this section, we study the relation between geodesic torsion, geodesic curvature and normal curvature of the corresponding curves on the developable ruled surfaces. Consider $\vec{\Gamma}(s)$ and $\vec{\bar{\Gamma}}(s)$ be two regular curves on the developable ruled surfaces ϕ and $\bar{\phi}$ respectively, so they are expressed as

$$\vec{\Gamma}(s) = \vec{\alpha}(s) + v(s)\vec{\omega}(s), \tag{5.1}$$

$$\vec{\bar{\Gamma}}(s) = \vec{\beta}(s) + v(s)\vec{\omega}(s). \tag{5.2}$$

Since $\vec{\alpha}$(s) and $\vec{\beta}$(s) are Bertrand curves, then by using equation (4.4), we have the corresponding curves on developable ruled surfaces as in the following

$$\vec{\bar{\Gamma}}(s) = \vec{\Gamma}(s) + a\vec{e}_2 \tag{5.3}$$

5.1. The geodesic torsion

Let τ_g and $\bar{\tau}_g$ be the geodesic torsion of the curves $\vec{\Gamma}$ and $\vec{\bar{\Gamma}}$, respectively, since the curve $\vec{\bar{\Gamma}}$ is a regular curve on a surface $\bar{\phi}$ in E^3, $\vec{n}$ and $\vec{\bar{x}}$ are the unit normal vectors field of the ϕ and $\bar{\phi}$

respectively, then the geodesic torsion for the curve $\vec{\bar{\Gamma}}$ is given by

$$\bar{\tau}_g = \frac{[\vec{\bar{\Gamma}}', \vec{\bar{n}}, \vec{\bar{n}}']}{|\vec{\bar{\Gamma}}'|}. \tag{5.4}$$

The above Eq. can be written in the following form

$$|\vec{\bar{\Gamma}}'|\ \bar{\tau}_g = \langle \vec{\bar{\Gamma}}', (\vec{\bar{n}} \wedge \vec{\bar{n}}') \rangle. \tag{5.5}$$

Taking the derivative of (5.3), we have

$$\vec{\bar{\Gamma}}' = \vec{\Gamma}' + a(\tau \vec{e}_3 - \kappa \vec{e}_1). \tag{5.6}$$

Using Eqs. (4.12) and (5.6), Eq. (5.5) becomes

$$|\vec{\bar{\Gamma}}'|\ \bar{\tau}_g = [\vec{\Gamma}', \vec{n}, \vec{n}'] + a\langle (\tau \vec{e}_3 - \kappa \vec{e}_1), (\tau \vec{e}_1 + \kappa \vec{e}_3) \rangle. \tag{5.7}$$

Hence, one can get

$$\bar{\tau}_g = \frac{|\vec{\Gamma}'|}{|\vec{\bar{\Gamma}}'|}\ \tau_g. \tag{5.8}$$

According to Eq. (5.8), we have the following

Corollary 5.1. *If $\vec{\Gamma}$ is a principal line on the surface $\phi(s,v)$, then the corresponding curve $\vec{\bar{\Gamma}}$ on the surface $\bar{\phi}(s,v)$ is a principal line.*

5.2. The geodesic curvature

Let k_g and $\bar{k_g}$ be the geodesic curvature for the curves $\vec{\Gamma}$ and $\vec{\bar{\Gamma}}$, respectively, then the geodesic curvature for the curve $\vec{\bar{\Gamma}}$ is given by

$$\bar{k}_g = \frac{[\vec{\bar{\Gamma}}'', \vec{\bar{\Gamma}}', \vec{\bar{n}}]}{|\vec{\bar{\Gamma}}'|^3}. \tag{5.9}$$

The above Eq. can be written in the following form

$$|\vec{\bar{\Gamma}}'|^3\ \bar{k}_g = \langle \vec{\bar{\Gamma}}'', (\vec{\bar{\Gamma}}' \wedge \vec{\bar{n}}') \rangle. \tag{5.10}$$

Taking the derivative of Eq. (5.6), we have

$$\vec{\bar{\Gamma}}'' = \vec{\Gamma}'' + a\big(-\kappa' \vec{e}_1 - (\tau^2 + \kappa^2)\vec{e}_2 + \tau' \vec{e}_3\big). \tag{5.11}$$

Substituting from Eqs. (4.12), (5.6) and (5.11) in Eq. (5.10), we get

$$\begin{aligned}\bar{k}_g = \frac{|\Gamma'|^3}{|\bar{\Gamma}'|^3}k_g + \frac{a}{|\bar{\Gamma}'|^3}\Big(-(\tau^2+\kappa^2)\Big(\frac{v'}{\sqrt{\tau^2+\kappa^2}}\Big)' - v'\tau(\sigma\kappa)' - v'\kappa(\sigma\tau)' \\ + v\sigma(\kappa\tau' - \kappa'\tau) - a(\kappa\tau)' + \tau'\Big).\end{aligned} \tag{5.12}$$

Corollary 5.2. *If $\vec{\Gamma}$ is a geodesic on the surface $\phi(s,v)$, then the geodesic curvature of the corresponding curve $\vec{bar}\Gamma$ on the surface $\bar{\phi}(s,v)$ is given by*

$$\bar{k}_g = \frac{a}{|\bar{\Gamma}'(s)|^3}\Big(-(\tau^2+\kappa^2)(\frac{v'}{\sqrt{\tau^2+\kappa^2}})' - v'\tau(\sigma\kappa)' - v'\kappa(\sigma\tau)' + v\sigma(\kappa\tau' - \kappa'\tau) - a(\kappa\tau)' + \tau'\Big).$$

5.3. The normal curvature

Let k_n and $\bar{k_n}$ be the normal curvature for the curves $\vec{\Gamma}$ and $\vec{\bar{\Gamma}}$ on the surfaces ϕ and $\bar{\phi}$ respectively, then the normal curvature for the curve $\vec{\bar{\Gamma}}$ is given by

$$\bar{k_n} = \frac{\langle \vec{\bar{\Gamma}}'', \vec{n}\rangle}{|\vec{\bar{\Gamma}}'|^2}. \tag{5.13}$$

The above Eq. can be written in the following form

$$|\vec{\bar{\Gamma}}'|^2\, \bar{k_n} = \langle \vec{\bar{\Gamma}}'', \vec{n}\rangle. \tag{5.14}$$

substituting from Eqs. (4.12) and (5.11) in Eq. (5.14), we get

$$|\vec{\bar{\Gamma}}'|^2\, \bar{k_n} = \langle \vec{\Gamma}'', \vec{n}\rangle - a(\tau^2+\kappa^2). \tag{5.15}$$

Thus

$$\bar{k_n} = \frac{|\vec{\Gamma}'|^2}{|\vec{\bar{\Gamma}}'|^2}k_n - \frac{a(\tau^2+\kappa^2)}{|\vec{\bar{\Gamma}}'|^2}. \tag{5.16}$$

According to Eq. (5.16) we have the following

Corollary 5.3. *If $\vec{\Gamma}$ is a asymptotic on the surface $\phi(s,v)$, then the normal curvature of the corresponding curve $\vec{\bar{\Gamma}}$ on the surface $\bar{\phi}(s,v)$ is given by* $\bar{k_n} = -\frac{a(\tau^2+\kappa^2)}{|\bar{\Gamma}'|^2}$.

Example *Consider the curve $\vec{\alpha}(s)$ given by*

$$\vec{\alpha}(s) = \{\cos(s), s, 1/s\},\ s \neq 0.$$

The curvature and the torsion of this curve are, respectively,

$$\kappa(s) = \frac{\sqrt{\psi_1}}{\psi_2}, \tag{5.17}$$

$$\tau(s) = \frac{2s\sin(s) - 6\cos(s)}{s^4\psi_1}, \tag{5.18}$$

where

$$\psi_1 = \frac{4 + s^6\cos(s)^2 + s\cos(s) + 2\sin(s)^2}{s^6}, \quad \psi_2 = (\frac{1}{s^4} + \cos(s)^2)^{\frac{3}{2}}.$$

The tangent, principal normal and binormal vectors of this curve are, respectively

$$\vec{e}_1(s) = \frac{1}{\psi_3}\Big\{-\sin(s), 1, -1\Big\}. \tag{5.19}$$

$$\begin{aligned}\vec{e}_2(s) = \frac{1}{\psi_3\psi_4}\Big\{ -\frac{1}{s^5}(s+s^5)\cos(s) - 2sin(s), \frac{2}{s^5} - \cos(s)\sin(s),\\ \frac{1}{s^3}(2 + s\cos(s)\sin(s) + 2\sin(s)^2)\Big\}.\end{aligned} \tag{5.20}$$

$$\vec{e}_3(s) = \psi_4\{\frac{2}{s^3}, s\cos(s) + 2\sin(s), \cos(s)\}, \tag{5.21}$$

where

$$\psi_3 = \sqrt{1 + \frac{1}{s^4} + \sin(s)^2}, \quad \psi_4 = \sqrt{\frac{36}{s^8} + \sin(s)^2}.$$

Therefore, the equation of the ruled surface generated by the Darboux vector $\vec{\omega}$ along the curve $\vec{\alpha}$ is

$$\phi(s,v) = \vec{\alpha}(s) + v\vec{\omega}(s), \tag{5.22}$$

where $\vec{\omega}(s)$ is the Darboux vector of the curve $\vec{\alpha}$. Hence the corresponding developable ruled surface is

$$\bar{\phi}(s,v) = \phi(s,v) + a\vec{e}_2. \tag{5.23}$$

Thus, Figure (2), shows the distance between the two corresponding developable ruled surfaces ϕ and $\bar{\phi}$ when the mean curvature for the developable ruled surface $\bar{\phi}$ is minimal($\bar{H} = 0$). Also, Figure (3), shows the distance between the two corresponding developable ruled surfaces ϕ and $\bar{\phi}$ when the mean curvature for the developable ruled surface $\bar{\phi}$ is non-positive($\bar{H} < 0$).

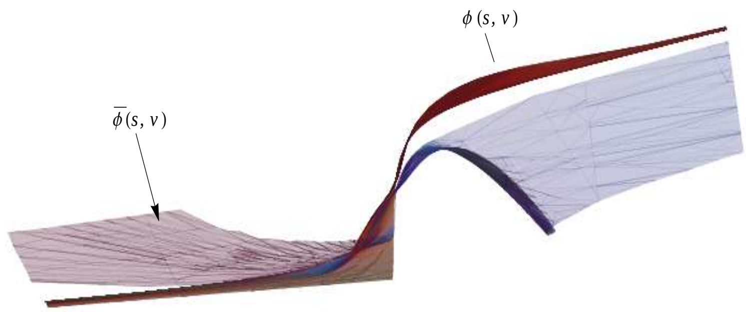

FIGURE 2. The correspondence between the developable surfaces ϕ and $\bar{\phi}(\bar{H}=0)$.

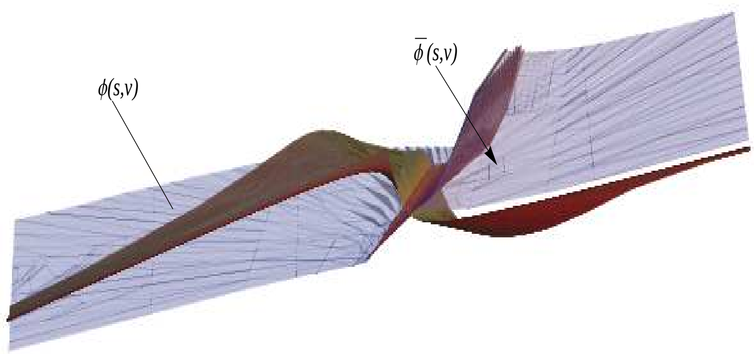

FIGURE 3. The correspondence between the developable surfaces ϕ and $\bar{\phi}(\bar{H}<0)$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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