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FINITENESS PROPERTY OF DEFORMED REVOLUTION SURFACES IN E^3 (PART II)

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Abstract. The motivation of the present work is to develop the finiteness property of deformed revolution surfaces in E^3 in our work [1]. The finiteness property of the mean and Gaussian curvatures flow for the revolution surfaces in E^3 is studied. Finally, general example for such property is presented.

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1. Introduction

Algebraic geometry studies varieties which are defined locally as the common zero sets of polynomials. Also, one can define the degree of an algebraic variety by its algebraic structure, where the concept of degree plays a fundamental role. On the other hand, according to Nash's embedding theorem, every Riemannian manifold can be realized as a Riemannian submanifold in some Euclidean space with sufficiently higher codimension. However, one lacks the notion of the degree for Riemannian submanifolds in Euclidean spaces [2].

Inspired by the above simple observation, Bang-Yen Chen introduced in the late 1970's the notions of "order" and "type" for submanifolds of Euclidean spaces and used them to introduce the notion of finite type submanifolds. Just like minimal submanifolds, submanifolds of

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finite type can be characterized by a spectral variational principle; namely, as critical points of directional deformations [3].

On one hand, the notion of finite type submanifolds provides a very natural way to apply spectral geometry to study submanifolds. On the other hand, one can also apply the theory of finite type submanifolds to investigate the spectral geometry of submanifolds. The first results on submanifolds of finite type were collected in [4, 5]. A list of twelve open problems and three conjectures on submanifolds of finite type was published in [6]. Furthermore, a detailed report of the progress on this theory was presented in [7]. Recently, in [8] was studied frenet surfaces with pointwise $1-$type Gauss map. Also, the study of finite type submanifolds, in particular, of biharmonic submanifolds, have received a growing attention with many progresses since the beginning of this century. In [2], was provided a detailed account of recent development on the problems and conjectures listed in [6].

One of the most interesting and profound aspects of classical differential geometry is its interplay with the calculus of variations. The calculus of variations have their roots in the very origins of subject, such as, for instance, in the theory of minimal surfaces. More recently, the variational principles which give rise to the field equations of the general theory of relativity have suggested the systematic investigation of a seemingly new type of variational problem. In the case of the earlier applications one is, at least implicitly, concerned with a multiple integral in the calculus of variations. In additional, the normal variational problem on general surfaces and hyperruled surfaces were studied by some geometers, specifically one may cite [9]-[20].

The mean curvature flow has many physical problems in the nature, starting from the well-known Poisson-Laplace theorem which relates, the pressure and the mean curvature flow of a surface immersed in a liquid until the capillary theory [21]. The theory of the Gaussian curvature flow has been generalized to a class of nonconvex surfaces. For example, in [22] the existence and the uniqueness of a viscosity solution to the PDE. were studied that describes the time evolution of a nonconvex graph by a convexified Gaussian curvature. Also, in [23] the existence and the uniqueness of the motion (or time evolution) of a nonconvex compact set were discussed which evolves by a convexified Gaussian curvature in E^n $(n \geq 2)$. And, in [24], surface mesh fairing was studied by the Gaussian curvature flow.

The main aim in this paper is to study the effectiveness of the normal variation in deferent directions of revolution surfaces in Euclidean 3−space E^3 for finiteness property. This aim determine whether the property of finiteness for surfaces in E^3 remains the same or not. And we find that the deformation depends on the ϕ function where we deal with some revolution surfaces. Finally, we prove that the variation of surfaces preserves the property of finiteness for some surfaces and does not preserve that property for other surfaces.

2. Basic concepts

In this section, we review some basic definitions and relations. Let a surface $M : \mathbf{X} = \mathbf{X}(u,v)$ in an Euclidean 3−space E^3. The map $\mathbf{G} : M \to S^2(1) \subset E^3$ which sends each point of M to the unit normal vector to M at the point is called the Gauss map of a surface M; where $S^2(1)$ denotes the unit sphere of E^3. The standard unit normal vector field $\mathbf{G}$ on the surface M can be defined by:

$$\mathbf{G} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{| \mathbf{X}_u \times \mathbf{X}_v |}, \tag{1}$$

where $\mathbf{X}_u$ and $\mathbf{X}_v$ are the first partial derivatives with respect to the parameters of $\mathbf{X}$.

Definition 2.1. [25, 26] *Let M be an $n-$dimensional surface. Then the Laplacian Δ operator (or Laplacian-Beitrami operator) associated with the induced metric on M is a mapping which sends any differentiable function f to the function Δf of the form*

$$\Delta = -\frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i} (\sqrt{g}\, g^{ij} \frac{\partial}{\partial x_j}), \tag{2}$$

where x_i is the local coordinate on M, (g_{ij}) is the matrix of the Riemannian metric g on M where $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$.

The mean curvature H of the surface is defined by

$$H = \frac{1}{2} \sum_{i,j=1}^{2} g^{ij} L_{ij}, \tag{3}$$

where L_{ij} are the coefficients of the second fundamental form.

An isometric immersion $\mathbf{X}: M \to E^3$ of a submanifold M in E^3 is said to be of finite type if $\mathbf{X}$ identified with the position vector field of M in E^3 can be expressed as a finite sum of eigen vectors of the Laplacian Δ of M, that is,

$$\mathbf{X} = X_0 + \sum_{i=1}^{j} X_i, \tag{4}$$

where X_0 is a constant map and $X_1, X_2, \cdots, X_j$ non-constant maps such that

$$\Delta X_i = \lambda_i X_i, \quad \lambda_i \in R, \quad 1 \le i \le j. \tag{5}$$

If $\lambda_1, \lambda_2, ..., \lambda_j$ are different eigen values, then M is said to be of $j-$type. If in particular, one of λ_i is zero then M is said to be of null $j-$type. If all coordinate function of E^3, restricted to M, are of finite type, then M is said to be of finite type. Otherwise, M is said to be of infinite type. Similarly, a smooth map ϕ of an $2-$dimensional Riemannian manifold M of E^3 is said to be of finite type if ϕ is a finite sum of E^3-valued eigen functions of Δ [4, 5].

Let M be a connected surface in E^3. Then the position vector $\mathbf{X}$ and the mean curvature vector $\mathbf{H}$ of M in E^3 satisfy [5]

$$\Delta \mathbf{X} = -2\mathbf{H}, \tag{6}$$

where $\mathbf{H} = H\,\mathbf{G}$. This formula yields the following well-known result: A surface M in E^3 is minimal if and only if all coordinate functions of E^3, restricted to M, are harmonic functions, that is,

$$\Delta \mathbf{X} = 0. \tag{7}$$

We recall theorem of T.Takahashi [27] and [7] which states that a submanifold M of a Euclidean space is of $1-$type, i.e., the position vector field of the submanifold in the Euclidean space satisfies the differential equation

$$\Delta \mathbf{X} = \lambda \mathbf{X}, \tag{8}$$

for some real number λ, if and only if either the submanifold is a minimal submanifold of the Euclidean space $(\lambda = 0)$ or it is a minimal submanifold of a hypersphere of the Euclidean space centered at the origin $(\lambda \neq 0)$.

We will mention the following known result for later use.

Proposition 2.1. [2, 5, 28, 29] *Let M be a $j-$type $(j = 1, 2, ...)$ surfaces whose spectral decomposition is given by Eq.* (4). *If we put* [28]

$$P(T) = \prod_{i=1}^{j} (T - \lambda_i), \tag{9}$$

then

$$P(\Delta)\,(\mathbf{X} - \mathbf{X}_0) = 0. \tag{10}$$

We can rewrite the previous equation as follows

$$\Delta^{j+1}\,\mathbf{X} + d_1\,\Delta^j\,\mathbf{X} + \cdots + d_j\,\Delta\mathbf{X} = 0, \tag{11}$$

where $d_1, d_2, ..., d_j$ are constants for some $j \geq 1$.
And the monic polynomial P is called the minimal polynomial which plays a very important role to find out whether or nor a surface is of finite type.

Definition 2.2. [9, 10, 15] *Let $\mathbf{X} : U \to R^{n+1}$ be a parameterized $n-$surface in R^{n+1}. A variation of $\mathbf{X}$ is a smooth map $\overline{\mathbf{X}} : U \times [0, 1] \to R^{n+1}$ with the property that $\overline{\mathbf{X}}(u^i, 0) = \mathbf{X}(u^i)$ for all $u^i \in U$. Thus a variation surrounds the $n-$surface $\mathbf{X}$ with a family of singular $n-$surface $\overline{\mathbf{X}}_t : U \to R^{n+1}$ defined by*

$$\overline{M} : \overline{\mathbf{X}}_t(u^i) = \overline{\mathbf{X}}\,(u^i, t) = \mathbf{X}(u^i) + t\,\phi(u^i)\,\mathbf{G}(u^i), \quad i = 1, 2,\ u^i = (u, v). \tag{12}$$

where ϕ is a smooth function along $\mathbf{X}$ and $\mathbf{G}$ is the Gauss map of $\mathbf{X}$, is called a normal variation of $\mathbf{X}$, where t is a parameter where $t \in [0, 1]$. The family of revolution surfaces represented by $\overline{\mathbf{X}}(u^i, t)$ is called a deformable revolution surfaces resulting from $\mathbf{X}(u^i)$ by the normal variation.

3. **Deformation for revolution surfaces in E^3**

In this section, we shall describe and derive the fundamental quantities $\overline{g}_{ij}$, $\overline{g}^{ij}$, and $\overline{g}$ after normal variation. Thus general formula of Laplacian Δ of the normal variation for revolution surfaces is derived.

Let M be a connected revolution surface which is generated by a plane curve $\alpha(u)$ when it is rotated around a straight line in the same plane. Let the plane be xz and the line be $z-$axis. Then, the parametrization of the plane curve takes the following form [30]

$$\alpha(u) = \{f(u)\,,\, h(u)\}. \tag{13}$$

Hence the parametrization of M is usually given by [31]

$$\mathbf{X}(u,v) = \{f(u)\,\cos v\,,\, f(u)\,\sin v\,,\, h(u)\}. \tag{14}$$

The unite normal vector field on M is

$$\mathbf{G} = -\frac{1}{\sqrt{\gamma}}\{h'(u)\,\cos v\,,\, h'(u)\,\sin v\,,\, -f'(u)\}, \quad ' = \frac{\partial}{\partial u}, \tag{15}$$

where $\gamma = f'^2(u) + h'^2(u) \neq 0$.

The metric (g_{ij}) and the contravariant metric (g^{ij}) can be written as

$$(g_{ij}) = diag\,(\gamma, f^2), \quad (g^{ij}) = diag\left(\frac{1}{\gamma}, \frac{1}{f^2}\right), \quad g = \gamma f^2. \tag{16}$$

After little calculations, we can get the Laplacian Δ as the following

$$\Delta = \frac{1}{2f^2\gamma^2}\left(f\,(f\,\gamma' - 2\,\gamma f')\frac{\partial}{\partial u} - 2\,f^2\,\gamma\frac{\partial^2}{\partial u^2} - 2\,\gamma^2\,\frac{\partial^2}{\partial v^2}\right). \tag{17}$$

Using Eqs. (12)-(15) the normal variation of M in E^3 associated with ϕ is given by

$$\overline{\mathbf{X}}(u,v,t) = \frac{1}{\sqrt{\gamma}}\left\{(f\sqrt{\gamma} - t\,\phi\,h')\cos v\,,\, (f\sqrt{\gamma} - t\,\phi\,h')\,\sin v\,,\, h\sqrt{\gamma} + t\,\phi\,f'\right\}. \tag{18}$$

We can write the above equation as the following

$$\overline{\mathbf{X}}(u,v,t) = \left\{\,\overline{f}(u)\cos v\,,\, \overline{f}(u)\,\sin v\,,\, \overline{h}(u)\,\right\}, \tag{19}$$

where $\overline{f}(u) = \frac{1}{\sqrt{\gamma}}\big(f\sqrt{\gamma} - t\,\phi\,h'\big)$ and $\overline{h}(u) = \frac{1}{\sqrt{\gamma}}\big(h\sqrt{\gamma} + t\,\phi\,f'\big)$.

Thus, we have the following corollary.

Corollary 3.1. *The deformed surface $\overline{M}$ is a revolution surface if and only if $t = constant$ and $\phi = \phi(u)$.*

Here, and in the sequel, we will omit $O(t^2)$, $O(t^3)$, ... because it does not affect the results. Thus, the metric $\left(\overline{g}_{ij}\right)$ and the contravariant metric $\left(\overline{g}^{ij}\right)$ of $\overline{M}$ are given by

$$\begin{aligned}
\left(\overline{g}_{ij}\right) &= \frac{1}{\sqrt{\gamma}} diag\left(\gamma^{\frac{3}{2}} + 2t\,\phi\,\varepsilon\,,\, f\,(f\sqrt{\gamma} - 2t\,\phi\,h')\right),\\
\left(\overline{g}^{ij}\right) &= \frac{1}{\overline{g}\sqrt{\gamma}} diag\left(f\,(f\sqrt{\gamma} - 2t\,\phi\,h')\,,\, \gamma^{\frac{3}{2}} + 2t\,\phi\,\varepsilon\right),\\
\overline{g} &= \frac{1}{\sqrt{\gamma}}\left(f^2\gamma^{\frac{3}{2}} - 2t\,f\,\phi\,(\gamma h' - f\,\varepsilon)\right),
\end{aligned} \tag{20}$$

where $\varepsilon = h' f'' - f' h''$.

After a long straight-forward computations, we reach to general formula of the Laplacian $\overline{\Delta}$ for the deformed surface $\overline{M}$ as the following

$$\begin{aligned}
\overline{\Delta} = \frac{1}{2\overline{g}^2\gamma^{\frac{3}{2}}} &\Big(\big(f\gamma^{\frac{3}{2}}\,(f\,\overline{g}_u - 4\overline{g}\,f') + t\,\big(2\overline{g}\,(\phi\,(h'\,(2\gamma f' - f\gamma') + 2f\gamma h'')\\
&+ 2f\gamma h'\,\phi_u)\big) - 2f\gamma\phi\,h'\,\overline{g}_u\big)\frac{\partial}{\partial u} + \big(\gamma^{\frac{5}{2}}\,\overline{g}_v + 2t\,\varepsilon\,\gamma(\phi\,\overline{g}_v - 2\overline{g}\,\phi_v)\big)\frac{\partial}{\partial v}\\
&+ (4t\,f\,\overline{g}\,\gamma\phi\,h' - 2f^2\overline{g}\,\gamma^{\frac{3}{2}})\frac{\partial^2}{\partial u^2} - 2\,(\overline{g}\,\gamma^{\frac{5}{2}} + 2t\,\overline{g}\,\varepsilon\,\gamma\phi)\frac{\partial^2}{\partial v^2}\Big),
\end{aligned} \tag{21}$$

where $\phi_i = \frac{\partial\phi}{\partial u^i}$ and $\overline{g}_i = \frac{\partial\overline{g}}{\partial u^i}$.

Remark 3.1. *After little computations, one can see $\overline{\Delta}\,|_{t=0} = \Delta$ which gives Eq* (17)*.*

4. Finiteness property of the mean curvature flow

In the following, we deal with two cases of revolution surfaces which have worked under the effect of normal variation where mean curvature flow is a term that is used to describe the variation of this surfaces whose function ϕ is given by the mean curvature [21]. Thus, in view of the parametrization (12) one can see that $\frac{\partial\overline{\mathbf{X}}}{\partial t} = H\,\mathbf{G}$. Then the finiteness property is studied before and after the deformation and it is noticed that the finiteness property is not affected by the deformation.

Case 4.1. If we put $f(u) = u$ and $h(u) = u^3$. Then the parametrization of revolution surface in Eq. (14) takes the following form

$$\mathbf{X}(u,v) = \{u \cos v\,,\, u \sin v\,,\, u^3\}. \tag{22}$$

Then the unite normal vector field of M is given by

$$\mathbf{G} = \frac{1}{\sqrt{Q}}\{-3u^2 \cos v\,,\, -3u^2 \sin v\,,\, 1\}, \tag{23}$$

where $Q = 1+9u^4 \neq 0$. Therefore, we get

$$(g_{ij}) = diag\left(Q\,,\, u^2\right), \quad (g^{ij}) = diag\left(\frac{1}{Q}, \frac{1}{u^2}\right), \quad g = u^2 Q. \tag{24}$$

The Laplacian Δ of M can be expressed as follows

$$\Delta = \frac{1}{u^2 Q^2}(u \frac{\partial}{\partial u} - u^2 Q \frac{\partial^2}{\partial u^2} - Q^2 \frac{\partial^2}{\partial v^2}). \tag{25}$$

Hence, the mean curvature function is given by

$$H = \frac{9\, u\, \xi}{2\, Q^{\frac{3}{2}}}, \quad \xi = 1 + 3\, u^4. \tag{26}$$

Let X_1, X_2, and X_3 be the three components functions of $\mathbf{X}$. Then, we will take

$$X_1 = u \cos v. \tag{27}$$

Consequently,

$$\Delta X_1 = \frac{R(u)}{Q(u)^2} \cos v, \tag{28}$$

where $R(u) = 27\, u^3\, \xi$.

Inspired by the reference [32] we offer the following lemma which we can prove it by mathematical induction. Here and in the sequel, for convenient, replace deg instead of degree.

Lemma 4.1. *If R and Q are polynomials in u, and* $\deg R = r$ *Then,*

$$\Delta\left(\frac{R}{Q} \cos v\right) = \frac{\widehat{R}}{u^2 Q^5} \cos v, \tag{29}$$

where $\widehat{R}$ is a polynomial in u and $\deg \widehat{R} \leq r+16$, Applying the above Lemma and after straightforward calculations, we get

$$\Delta^j X_1 = \frac{R_j}{u^2 Q^{3j-1}} \cos v. \tag{30}$$

Therefore, if j goes up by one, the degree of the numerator of $\Delta^j X_1$ goes up by at most 10, while the degree of the denominator goes up by 12. Hence the decomposition (11) can never be zero. Therefore, M is infinite type and this result agrees with the results in paper [28].

Let $\overline{M}$ be the surface after variation by mean curvature flow, i.e., $\phi = H$ in the parametrization (12). Then, it can be parameterized by

$$\overline{\mathbf{X}}(u,v,t) = \frac{1}{2Q^2}\{2uQ^2 \cos v - tR\cos v, 2uQ^2 \sin v - tR\sin v, 2Q^2u^3 + 9tu\xi\}. \tag{31}$$

Then the unite normal vector field of $\overline{M}$ is given by

$$\begin{aligned}\overline{\mathbf{G}} =& \frac{1}{2Q(Q^3u-3Rt\xi)}\Big\{-3\cos v\left(2Q^3u^2-3t\left(3u^4(81u^8+63u^4+19)-1\right)\right),\\ & -3\sin v\Big(2Q^3u^2-3t\big(3u^4(81u^8+63u^4+19)-1\big)\Big), 2\big(Q^3+54tu^2(3u^4-1)\big)\Big\}.\end{aligned} \tag{32}$$

Hence, we have

$$\begin{aligned}(\overline{g}_{ij}) &= \frac{1}{Q^2}\operatorname{diag}\left(Q^3-54tu^2\xi, u^2Q^2-tuR\right),\\ (\overline{g}^{ij}) &= \frac{1}{Q^3u^2-3tuR\xi}\operatorname{diag}\Big(u^2Q^2-tuR, Q^3-54tu^2\xi\Big),\\ \overline{g} &= \frac{1}{Q^2}\big(u^2Q^3-3tuR(1+3u^4)\big).\end{aligned} \tag{33}$$

Using Eq. (2), we obtain that the Laplacian $\overline{\Delta}$ of $\overline{M}$ can be written as

$$\begin{aligned}\overline{\Delta} =& \frac{Q\Big(Q^3(9u^4-1)-81tu^2\big(u^4(3Ru-19)-1\big)\Big)}{u(Q^3-81tu^2\xi^2)^2}\frac{\partial}{\partial u}+\Big(\frac{27tu^2\xi-Q^2}{Q^3-81tu^2\xi^2}\Big)\frac{\partial^2}{\partial u^2}\\ &-\frac{Q^3-54tu^2\xi}{uQ\sqrt{\Big(t\big(\frac{4}{Q^2}-9u^4-4\big)+u^2Q\Big)\Big(Q^3-81tu^2\xi^2\Big)}}\frac{\partial^2}{\partial v^2}.\end{aligned} \tag{34}$$

If $\overline{X}_3$ denotes the third coordinate function of $\overline{\mathbf{X}}$, then we get

$$\overline{\Delta}\,\overline{X}_3 = \frac{\overline{R}(u,t)}{\overline{Q}(u,t)}, \tag{35}$$

where,

$$\overline{R}(u\,,t) = t\,\left(354294\,u^{20} + 334611\,u^{16} + 52488\,u^{12} + 47142\,u^{8} + 4914\,u^{4} - 9\right) - 354294\,u^{22} - 275562\,u^{18} - 78732\,u^{14} - 10692\,u^{10} - 702\,u^{6} - 18\,u^{2},$$

and

$$\overline{Q}(u\,,t) = t\,\left(-2125764\,u^{23} - 2125764\,u^{19} - 787320\,u^{15} - 134136\,u^{11} - 10692\,u^{7} - 324\,u^{3}\right) + 1062882\,u^{25} + 708588\,u^{21} + 196830\,u^{17} + 29160\,u^{13} + 2430\,u^{9} + 108\,u^{5} + 2\,u.$$

Hence, we have the proof of the following lemma:

Lemma 4.2. *If $\overline{R}$ and $\overline{Q}$ are polynomials in $u\,,t$ where $\deg\overline{R} = \overline{r}$, and $\deg\overline{Q} = \overline{q}$. Then,*

$$\overline{\Delta}\left(\frac{\overline{R}(u,t)}{\overline{Q}(u,t)}\right) = \frac{\widehat{\overline{R}}(u,t)}{\widehat{\overline{Q}}(u,t)}, \tag{36}$$

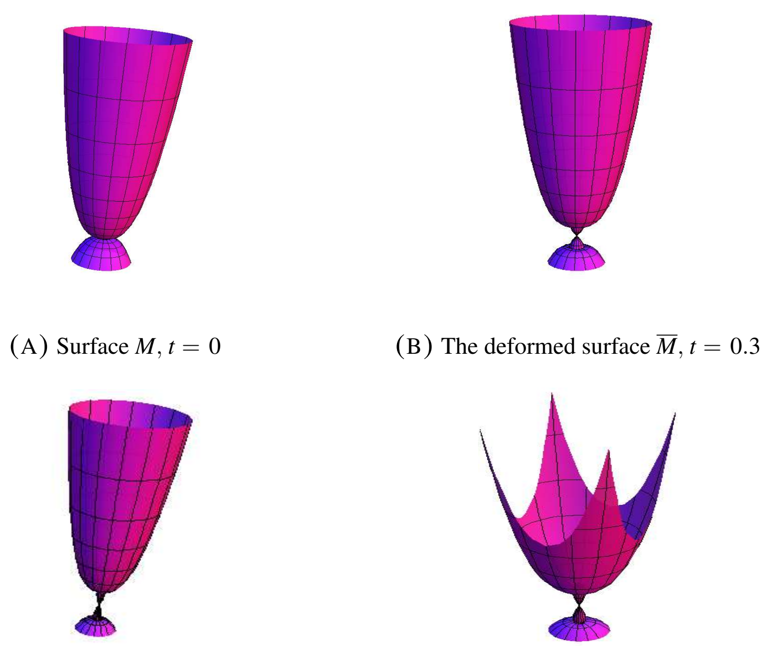
where $\widehat{\overline{R}}(u,t)$ and $\widehat{\overline{Q}}(u,t)$ are polynomials in u, t where $\deg\widehat{\overline{R}} \leq \overline{r} + 2\overline{q} + 19$ and $\deg\widehat{\overline{Q}} \leq 3\,\overline{q} + 25$ where $\overline{r} < \overline{q}$.

Then, we observe that the degree of denominator is larger than the degree of numerator. Hence the decomposition (11) can never be zero. Then $\overline{M}$ is infinite type.

From the above results we easily deduce the following consequence:

Corollary 4.1. *The mean curvature flow of the deformed revolution surface preserves the property of infiniteness.*

See Figure 1.



(A) Surface M, $t = 0$

(B) The deformed surface $\overline{M}$, $t = 0.3$

(C) The deformed surface $\overline{M}$, $t = 0.6$

(D) The deformed surface $\overline{M}$, $t = 0.7$

FIGURE 1. The deformed surface : $u \in [-1, 2]$, $v \in [-\pi, \pi]$

$f(u) = u$, $h(u) = u^3$, $\phi = H$

From the previous figure we get the following result:

Corollary 4.2. *The effect of the mean curvature flow of the deformed revolution surface is very strong at* $t > 0.6$.

Case 4.2. If we put $f(u) = a$ and $h(u) = c\,u$ where a, c are constants. Then the parametrization of revolution surface in Eq. (14) gives revolution cylinder as the following

$$\mathbf{X}(u,v) = \{a \cos v\,,\, a \sin v\,,\, c\,u\}, \quad a,\, c \neq 0. \tag{37}$$

The unite normal vector field of M is

$$\mathbf{G} = -\{\cos v\,,\, \sin v\,, 0\}. \tag{38}$$

Thus, we get

$$(g_{ij}) = diag\left(c^2\,,\, a^2\right), \quad (g^{ij}) = diag\left(\frac{1}{c^2}\,,\, \frac{1}{a^2}\right), \quad g = a^2\,c^2. \tag{39}$$

Then, mean curvature function is given by $H = \frac{1}{2a}$. Therefore,

$$\Delta \mathbf{X} = -(\frac{1}{c^2}\frac{\partial^2}{\partial u^2} + \frac{1}{a^2}\frac{\partial^2}{\partial v^2}). \tag{40}$$

Solving the following equation for λ.

$$\Delta \mathbf{X} - \lambda\, \mathbf{X} = 0. \tag{41}$$

Hence, the eigenvalues of Δ take the following values

$$\lambda_1 = \lambda_2 = \frac{1}{a^2} \quad \text{and} \quad \lambda_3 = 0. \tag{42}$$

That is, the revolution cylinder is null $2-$type as well know, see [8, 29].

Here, we show the effect of the finiteness property for the deformed revolution cylinder by mean curvature flow.

Let $\overline{M}$ be a surface after variation by function of the mean curvature, that is $\phi = H$. Then $\overline{M}$ has a parametrization as the following

$$\overline{\mathbf{X}}(u,v,t) = \frac{1}{2a}\{(2a^2-t)\cos v, (2a^2-t)\sin v, 2acu\}.$$

One can see $\overline{M}$ is a family of revolution cylinders. Then, the unit normal vector field is given by

$$\overline{\mathbf{G}} = \{\cos v, \sin v, 0\} = -\mathbf{G}. \tag{43}$$

Therefore, we have

$$(\overline{g}_{ij}) = diag\left(c^2, \xi_1\right), \quad (\overline{g}^{ij}) = diag\left(\frac{1}{c^2}, \frac{1}{\xi_1}\right), \quad \overline{g} = c^2\xi_1, \quad \xi_1 = a^2 - t. \tag{44}$$

Direct computations, we can find the Laplacian $\overline{\Delta}$ of $\overline{M}$ as in the following

$$\overline{\Delta} = -\left(\frac{1}{c^2}\frac{\partial^2}{\partial u^2} + \frac{1}{\xi_1}\frac{\partial^2}{\partial v^2}\right). \tag{45}$$

The mean curvature function of $\overline{\mathbf{X}}$ is given by

$$\overline{H} = \frac{1}{4a\xi_1}(t-2a^2). \tag{46}$$

Consequently,

$$\Delta\overline{\mathbf{X}} = \frac{1}{2a\xi_1}\left\{(2a^2-t)\cos v, (2a^2-t)\sin v, 0\right\}. \tag{47}$$

Solving the following equation for $\overline{\lambda}$

$$\Delta \overline{\mathbf{X}} - \overline{\lambda}\, \overline{\mathbf{X}} = 0. \tag{48}$$

We have

$$-\frac{1}{2\,a\,\xi_1}\Big\{(2\,a^2 - t)\,(\overline{\lambda}\,\xi_1 - 1)\,\cos v\,,\,(2\,a^2 - t)\,(\overline{\lambda}\xi_1 - 1)\,\sin v\,,\,c\,a\,\xi_1\,\overline{\lambda}\,u\Big\} = 0.$$

Thus, one can get

$$\overline{\lambda}_1 = \overline{\lambda}_2 = \frac{1}{\xi_1}, \quad \overline{\lambda}_3 = 0. \tag{49}$$

Corollary 4.3. *If we put $t = 0$ in Eq.* (49)*, we get Eq.* (42) *which gives the same result of papers* [8, 29]*, for original cylinder.*

We conclude that, there is a family of the deformed revolution cylinder which is null $2-$type.

Corollary 4.4. *The mean curvature flow of deformed revolution cylinder preserves the property of finiteness.*

See Figure 2.

(A) Revolution cylinder M, $t = 0$ (B) The deformed cylinder $\overline{M}$, $t = 0.6$

FIGURE 2. The deformed revolution cylinder $u \in [0, 2\,\pi]$, $v \in [-\pi, \pi]$
$f(u) = a$, $h(u) = c\,u$, $\phi = H$, $a = 2$, $c = 3$

Corollary 4.5. *The effect of the mean curvature flow of the deformed revolution cylinder is very weak $\forall t > 0$, where the geometric properties are hereditary.*

Remark 4.1. *Observe if $a = c$, then the revolution cylinder will be isothermal surface.*

5. Finiteness property of isothermal revolution surfaces

In this section, we focus on the isothermal revolution surfaces for finiteness property for the Gaussian curvature flow of revolution surfaces where Gaussian curvature flow is a term that is used to describe the variation of surface whose function ϕ is given by the Gaussian curvature [24]. Thus, in view of the parametrization (12) one can see $\frac{\partial \overline{\mathbf{X}}}{\partial t} = K\,\mathbf{G}$.

Case 5.1. If we put $f = f(u)$ and $h(u) = u$ in the parametrization (14) for M to be an isothermal surface, we get

(50) $$1 + f'^2(u) = f^2(u).$$

Solving the above differential equation gives

(51) $$f(u) = \frac{1}{2}(e^{u\pm c} + e^{-(u\pm c)}) = \cosh(u \pm c), \quad \omega_1 = c - u, \quad c = \text{ constant.}$$

Here, we can rewrite the parametrization of revolution surface in (14) as in the following form

(52) $$\mathbf{X}(u,v) = \{\cosh \omega_1 \cos v\,, \cosh \omega_1 \sin v\,, u\}.$$

The unite normal vector field of M is

(53) $$\mathbf{G} = -\{\operatorname{sech} \omega_1 \cos v\,, \operatorname{sech} \omega_1 \sin v\,, \tanh \omega_1\}.$$

Consequently,

(54) $$(g_{ij}) = \cosh^2 \omega_1\, daig\,(1\,,\,1)\,, \quad (g^{ij}) = \operatorname{sech}^2 \omega_1\, daig\,(1\,,\,1)\,, \quad g = \cosh^4 \omega_1.$$

Then, the Gaussian and mean curvature functions are given by $K = -\operatorname{sech}^4 \omega_1$ and $H = 0$, respectively. Therefore, $\Delta\mathbf{X} = 0$. Thus this surface is $1-$type [30].

Let $\overline{M}$ be a surface after variation by function of the Gaussian curvature, that is $\phi = K$. Then the parametrization of $\overline{M}$ is defined as

(55)
$$\overline{\mathbf{X}}(u,v,t) = \{\cos v\,(\cosh^6 \omega_1 + t)\operatorname{sech}^5 \omega_1,\, \sin v\,(\cosh^6 \omega_1 + t)\operatorname{sech}^5 \omega_1,\, u + t\operatorname{sech}^4 \omega_1 \tanh \omega_1\}.$$

Hence, the unite normal vector field of $\overline{M}$ is given by

$$\overline{\mathbf{G}} = \left\{ -\cos v\, \mathrm{sech}\omega_1 \left(4t \tanh^2 \omega_1 \,\mathrm{sech}^4\omega_1 + 1\right), -\sin v\, \mathrm{sech}\omega_1 \left(4t \tanh^2 \omega_1 \mathrm{sech}^4\omega_1 + 1\right) \right.$$
$$\left. , \tanh\omega_1 \left(4t\, \mathrm{sech}^6\omega_1 - 1\right) \right\}. \tag{56}$$

Therefore,

$$(\overline{g}_{ij}) = daig\left(\cosh^2 \omega_1 - 2t\, \mathrm{sech}^4\, \omega_1 , \cosh^2\, \omega_1 + 2t\, \mathrm{sech}^4\, \omega_1\right), \quad \overline{g} = \cosh^4\, \omega_1,$$
$$(\overline{g}^{ij}) = \mathrm{sech}^2\, \omega_1\, daig\left(1 + 2t\, \mathrm{sech}^6\, \omega_1 , 1 - 2t\, \mathrm{sech}^6\, \omega_1\right). \tag{57}$$

Then, the formula of the Laplacian $\overline{\Delta}$ of $\overline{M}$ is given by

(58)
$$\overline{\Delta} = \mathrm{sech}^2\, \omega_1 \left(12t\, \mathrm{sech}^6\, \omega_1 \tanh\, \omega_1 \frac{\partial}{\partial u} - \left(1 + 2t\, \mathrm{sech}^6\, \omega_1\right) \frac{\partial^2}{\partial u^2} - \left(1 - 2t\, \mathrm{sech}^6\, \omega_1\right) \frac{\partial^2}{\partial v^2}\right).$$

The mean curvature function of $\overline{M}$ is given by

$$\overline{H} = t\, \mathrm{sech}^6\, \omega_1\, (9\, \mathrm{sech}^2\, \omega_1 - 8). \tag{59}$$

Let $\overline{X}_1$, $\overline{X}_2$, and $\overline{X}_3$ be the three components functions of $\overline{\mathbf{X}}$. Then, we take

$$\overline{X}_3 = u + t\, \mathrm{sech}^4\, \omega_1 \tanh\, \omega_1. \tag{60}$$

Therefore,

$$\overline{\Delta}\, \overline{X}_3 = 2t\, \mathrm{sech}^9\, \omega_1 \left(7 \sinh\, \omega_1 - 2 \sinh 3\omega_1\right), \tag{61}$$

and

$$\overline{\Delta}^2\, \overline{X}_3 = 4t\, \mathrm{sech}^{13}\, \omega_1 \left(295 \sinh\, \omega_1 - 101 \sinh 3\omega_1 + 9 \sinh 5\omega_1\right). \tag{62}$$

Using mathematical induction, we find that $\forall\, j$, $\overline{\Delta}^j\, \overline{X}_3$ has the following structure

$$\overline{\Delta}^j\, \overline{X}_3 = 2^j\, t\, \mathrm{sech}^{4j+5}\, \omega_1 \left(c_{j1} \sinh\, \omega_1 + c_{j2} \sinh 3\omega_1 + \cdots + c_{j(j+1)} \sinh(2j+1)\omega_1\right). \tag{63}$$

Assume the variation of surface is of finite type, then by the decomposition (11), we get

$$2^{j+1}\, t\, \mathrm{sech}^{4j+9}\, \omega_1 \left(c_{(j+1)1} \sinh\, \omega_1 + c_{(j+1)2} \sinh 3\omega_1 + \cdots\right.$$
$$\left. + c_{(j+1)(j+1)} \sinh(2j+3)\omega_1\right) + d_1\, 2^j\, t\, \mathrm{sech}^{4j+5}\, \omega_1 \left(c_{j1} \sinh\, \omega_1 + c_{j2} \sinh 3\omega_1\right. \tag{64}$$
$$\left. + \cdots + c_{j(j+1)} \sinh(2j+1)\omega_1\right) + \cdots + d_j\, 2t\, \mathrm{sech}^9\, \omega_1 \left(7 \sinh\, \omega_1 - 2 \sinh 3\omega_1\right) = 0.$$

One can rewrite the last equation as the following

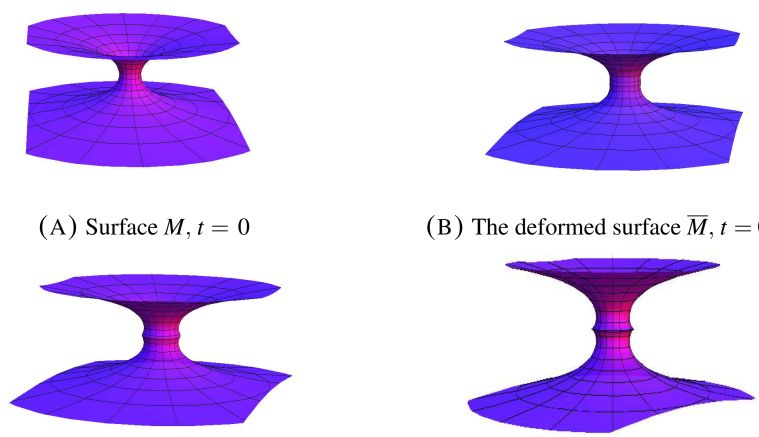
$$t\,\operatorname{sech}^{4j+9}\omega_1 \sinh(2j+3)\omega_1 + P_1(\operatorname{sech}\omega_1, t)\sinh(2j+1)\omega_1 + \cdots$$
$$+ P_{j-1}(\operatorname{sech}3\omega_1, t)\sinh\omega_1 + P_j(\operatorname{sech}\omega_1, t)\sinh\omega_1 = 0, \tag{65}$$

where P_j is a polynomial in $\operatorname{sech}\omega_1$ and t. Since $\sinh\omega_1$, $\sinh 3\omega_1$ and $\cdots \sinh(2j+3)\,\omega_1$ are linearly independent functions of ω_1, we obtain from the above equation the first term in left vanishes that means $t\,\operatorname{sech}^{4j+9}\omega_1 = 0$, and this is a contradiction. Then the deformed surface is infinite type.

Corollary 5.1. *If we put* $t = 0$, *in the formula* (59) *we have* $\overline{H} = 0$, *which gives minimal original surface.*

Corollary 5.2. *The Gaussian curvature flow of the deformed isothermal surface does not preserve the property of finiteness.*

The following Figure 3, shows the deformation that has occurred to the isothermal surface.



(A) Surface M, $t = 0$ (B) The deformed surface $\overline{M}$, $t = 0.3$

(C) The deformed surface $\overline{M}$, $t = 0.6$ (D) The deformed surface $\overline{M}$, $t = 0.999$

FIGURE 3. The deformed isothermal surface : $u \in [-0.3, 6]$, $v \in [0, 2\pi]$
$f(u) = \cosh(u - c)$, $h(u) = u$, $\phi = K$, $c = 3$

Corollary 5.3. *The effect of the Gaussian curvature flow of the deformed isothermal surface is very strong at* $t \geq 0.6$, *i.e., the geometric properties are not hereditary properties.*

In view of Eqs. (57) we give the following remark:

Remark 5.1. *The deformed surface can not be isothermal as its original surface.*

6. General example

Finally in this section, we study the normal variation under the effect of general function.

Case 6.1. If we put $f(u) = u^2$ and $h(u) = u$. Then the parametrization of revolution surface in Eq. (14) takes the form

$$\mathbf{X}(u,v) = \{u^2 \cos v\,,\, u^2 \sin v\,,\, u\}. \tag{66}$$

The unite normal vector field of M is

$$\mathbf{G} = \frac{1}{\phi}\{-\cos v\,,\, -\sin v\,,\, 2u\}, \tag{67}$$

where $\phi = \sqrt{1+4u^2}$. Hence, we have

$$(g_{ij}) = diag\left(\phi^2\,,\, u^4\right), \quad (g^{ij}) = diag\left(\frac{1}{\phi^2},\frac{1}{u^4}\right), \quad g = u^4\,\phi^2. \tag{68}$$

$$\Delta = \frac{-1}{u^4\,\phi^4}\Big(2u^3\left(1+2u^2\right)\frac{\partial}{\partial u} + u^4\,\phi^2\,\frac{\partial^2}{\partial u^2} + \phi^4\frac{\partial^2}{\partial v^2}\Big). \tag{69}$$

We study Δ for third component of $\mathbf{X}$. Thus

$$\Delta X_3 = -\frac{2}{u\,\phi^4}\,(2u^2+1). \tag{70}$$

The following lemma can be proved by mathematical induction.

Lemma 6.1. *If A is a polynomial in u and* $\deg A = r$, *then*

$$\Delta\left(\frac{A(u)}{u\,B^q(u)}\right) = \frac{\widehat{A}(u)}{u\,B^{q+3}(u)}. \tag{71}$$

where $\widehat{A}$ is a polynomial in u and $\deg \widehat{A} \leq r+2$.

Applying the above lemma, we can easily get

$$\Delta^j X_3 = \frac{A_j}{u\,B^{3j-1}}\cos v. \tag{72}$$

Then if j goes up by one, the degree of the numerator of $\Delta^j X_3$ goes up by at most 2, while the degree of the denominator goes up by 6. Hence the decomposition (11) can never be zero. Therefore, M is infinite type and this result is the same as in reference [28].

Under consideration that $\phi = \sqrt{1+4u^2}$ and using the parametrization (12) and (66), then $\overline{M}$ can be parameterized locally by

$$\overline{\mathbf{X}}(u,v,t) = \{(u^2-t)\cos v\,,\,(u^2-t)\sin v\,,\,(1+2t)\,u\}. \tag{73}$$

The unite normal vector field of $\overline{M}$ is given by

$$\overline{\mathbf{G}} = -\frac{1}{u\sqrt{u^2\phi^2-2t\,\xi_3}}\left\{\left(u^2+t\,(2u^2-1)\right)\cos v\,,\,\left(u^2+t\,(2u^2-1)\right)\sin v\,,\,2u\,(t-u^2)\right\}, \tag{74}$$

where, $\xi_3 = 1+2u^2$. Therefore, we obtain

$$\begin{aligned} (\overline{g}_{ij}) &= diag\,\left(\phi^2+4t\,,\,-(2t-u^2)\,u^2\right), \quad \overline{g} = \phi^2 u^4 - 2t\,u^2\,\xi_3, \\ (\overline{g}^{ij}) &= \frac{1}{u^2\left(2t\,\xi_3-u^2\,\phi^2\right)}\,diag\left(u^2(2t-u^2)\,,\,-4t-\phi^2\right). \end{aligned} \tag{75}$$

Then, one can find the Laplacian $\overline{\Delta}$ of $\overline{M}$ as the following

$$\begin{aligned} \overline{\Delta} = &\frac{1}{u^2\left(u^2\phi^2-2t\,\xi_3\right)^2}\left(2u^3\left(3t-u^2\,\xi_3\right)\frac{\partial}{\partial u} + u^4\left(12tu^2+4t-u^2\,\phi^2\right)\frac{\partial^2}{\partial u^2}\right. \\ &\left. +\ \phi^2\left(2t-u^2\,\phi^2\right)\frac{\partial^2}{\partial v^2}\right). \end{aligned} \tag{76}$$

Let $\overline{X}_1$, $\overline{X}_2$, and $\overline{X}_3$ be the three components functions of $\overline{\mathbf{X}}$. Then,

$$\overline{X}_3 = (1+2t)\,u. \tag{77}$$

Therefore

$$\overline{\Delta}\overline{X}_3 = \frac{\overline{A}(u,t)}{\overline{B}(u,t)}, \tag{78}$$

where

$$\overline{A}(u,t) = -2\,u^3\,\xi_3 - 2t\,u\left(-3+2u^2\,\xi_3\right), \quad \overline{B}(u,t) = \left(u^2\,\phi^2 - 2t\,\xi_3\right)^2.$$

The following lemma can be proved by induction.

Lemma 6.2. *If* $\overline{A}$ *is a polynomial in* u *and* $\deg \overline{A} = r$, *then*

$$\Delta\left(\frac{\overline{A}(u,t)}{\overline{B}^{q}(u,t)}\right) = \frac{\widehat{\overline{A}}(u,t)}{\overline{B}^{q+5}(u,t)}, \tag{79}$$

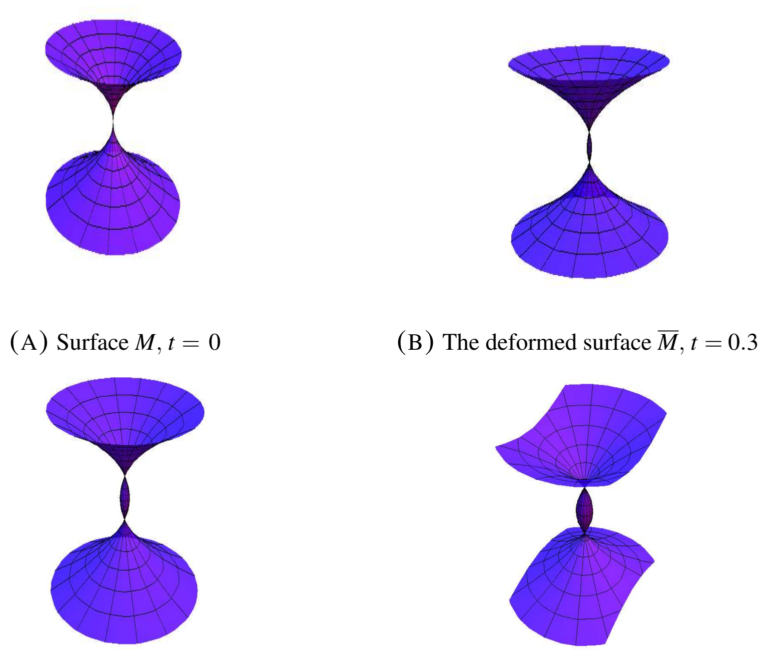
where $\widehat{\overline{A}}$ *is a polynomial in* u, t *and* $\deg\widehat{\overline{A}} \leq r+13$.

Applying the above Lemma, we obtain

$$\overline{\Delta}^{j}\,\overline{X}_3 = \frac{\overline{A}_j}{B^{5(j-1)}}, \quad j > 2. \tag{80}$$

One can observe that degree of denominator is larger than degree of numerator. Where if j goes up by one, the degree of the numerator of $\overline{\Delta}^{j}\,\overline{X}_3$ goes up by at most 16, while the degree of the denominator goes up by 20. Hence the decomposition (11) can never be zero. Therefore, $\overline{M}$ is infinite type.

See Figure 4, to note the deferent between surface before and after deformation.



(A) Surface M, $t = 0$ (B) The deformed surface $\overline{M}$, $t = 0.3$

(C) The deformed surface $\overline{M}$, $t = 0.6$ (D) The deformed surface $\overline{M}$, $t = 0.9$

FIGURE 4. The deformed surface: $u \in [-\pi, \pi]$, $v \in [0, 2\pi]$

$f(u) = u^2$, $h(u) = u$, $\phi = \sqrt{1+4u^2}$

Corollary 6.1. *The deformation of surface preserves the property of infiniteness at* $\phi = \sqrt{1+4u^2}$.

Corollary 6.2. *The intrinsic properties of surface do not change after deformation at* $\phi = \sqrt{1+4u^2}$, $\forall t \leq 0.6$.

Corollary 6.3. *The effect of normal variation for the surface is very strong at* $t > 0.6$, *i.e., the geometric properties are not hereditary properties.*

Combining the above corollaries we deduce the following theorem:

Theorem 6.1. *The normal variation of any surface does not necessarily preserve the property of finiteness of them.*

7. Conclusion

It is important to remark that the effect of the normal variation in deferent directions of the revolution surfaces of finiteness property is very weak in some cases. In other words, the deformed surfaces are still having some geometric properties which were before the deformation. In other cases, the effect of the normal variation is strong. In other words, the geometric properties of the deformed revolution surfaces are not hereditary properties. In the following, we give a summary of the studied cases previously:

(1) $f(u) = u$, $h(u) = u^3$ and $\phi = H$. Then, the surface M and the deformed surface $\overline{M}$ are infinite type.

(2) $f(u) = a$, $h(u) = c\,u$ and $\phi = H$. Therefore, M (revolution cylinder) is null $2-$type also the deformed surface $\overline{M}$ is a family of revolution cylinders dependent of value of t which is also of null $2-$type.

(3) $f(u) = \cosh(u+c)$, $h(u) = u$ and $\phi = K$, where c is constant. Hence, M (catenoid) is $1-$type and the deformed surface $\overline{M}$ is infinite type.

(4) $f(u) = u^2$, $h(u) = u$ and $\phi = \sqrt{1+4u^2}$. Then, M and its deformed surface $\overline{M}$ are infinite type.

The above four cases are translated to the Figures [1 - 4].

Conflict of Interests

The authors declare that there is no conflict of interests.

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