Available online at http://scik.org
J. Math. Comput. Sci. 5 (2015), No. 5, 728-736
ISSN: 1927-5307

CONVERGENCE AND STABILITY THEOREMS FOR A FASTER ITERATIVE SCHEME FOR A GENERAL CLASS OF CONTRACTIVE-LIKE OPERATORS

RAMESH PRASAD PATHAK[1], SAMIR DASHPUTRE[2*], SHAILESH DHAR DIWAN[3], RAJLAXMI GUPTA[1]

[1]Department of Applied Mathematics, Naitional Institute of Technology, Raipur, 492001, India

[2] Department of Applied Mathematics, Shri Shankaracharya Technical Campus,

Shri Shankaracharya Group of Institutions, (F.E.T), Junwani, Bhilai, 490020, India

[3]Sant Guru Ghasidas Govt. P. G. College, Kurud, Dhamtari, 493663, India

Abstract. The purpose of this paper is to prove strong convergence and stability results for $S-$iteration procedure which is faster than Picard iteration procedure for the general class of contractive-like operators introduced by Bosede and Rhoades [4] in a real normed linear space. Our results generalize and improve a multitude of results in the literature, including the recent results of Akewe and Okeke [2] and many others for the general class of contractive-like operators using faster iterative procedure.

Keywords: Strong convergence; S iteration procedure; Contractive-like operators.

2010 AMS Subject Classification: 47H09, 47H10, 54H25.

1. Introduction

In this paper, we assume that X is a real normed linear space and C is a nonempty subset of X. In this paper, we use $F(T)$ to denote the set of fixed points of T which is nonempty, i.e., $F(T) = \{x \in C : x = Tx\} \neq \phi$. Let $T : C \to C$ be a mapping. Let x_0 be a initial approximation

*Corresponding author

Received June 13, 2015

in C and

$$x_{n+1} = f(T, x_n) \tag{1.1}$$

define an iteration procedure which produces a sequence $\{x_n\}$ in C. Suppose $F(T) \neq \phi$ and the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$. Let $\{y_n\}$ be any sequence in X and $\{\varepsilon_n\}$ be a sequence in $[0, \infty)$ defined by

$$\varepsilon_n = \|y_{n+1} - f(T, y_n)\|.$$

If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = x^*$, then the iteration procedure $\{x_n\}$ defined by (1.1) is called $T-$stable or stable with respect to T.

Stability results established for several iteration procedures in metric spaces, normed linear spaces and Banach spaces are available in literature for single-valued mappings, (see e.g. [2]-[15] and references therein).

A pioneer work of Harder [7], Harder and Hicks [8, 9] where a concept of stable fixed point iteration procedure was introduced and studied several stability results for certain classes of nonlinear mappings. Harder and Hicks [9] revealed the importance of investigating the stability of various iteration schemes for various classes of nonlinear mappings. In [15], Rhoades continued the study of stability results by using more general contractive definition appears in the results of Harder and Hicks [8], Rhoades [14].

In [12], Osilike proved several stability results by using following contractive definition (1.2), which are generalizations and extensions of most of the results of Harder [7] and Rhoades [14, 15]: for each $x, y \in X$ there exists $a \in [0, 1)$ and $L \geq 0$ such that

$$\|Tx - Ty\| \leq \|x - y\| + L\|x - Tx\|. \tag{1.2}$$

In [10], Imoru and Olatinwo proved some stability results using the following general contractive definition: for each $x, y \in X$ there exists $\delta \in [0, 1)$ and a monotone increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\| + \phi(\|x - Tx\|). \tag{1.3}$$

In [4], Bosede and Rhoades made an assumption implied by (1.2) and the one which renders all generalizations of the form (1.3) pointless. That is if $x = p$ is a fixed point of T then (1.3) becomes

$$\|p - Ty\| \le \delta \|p - y\|, \tag{1.4}$$

for all $\delta \in [0,1)$ and for all $x, y \in X$.

In [5], Chidume and Olaleru gave several examples to show that the class of mappings satisfying (1.4) is more general than that of (1.2) and (1.3), provided the fixed point exists.

The following example shows that every contraction map with a fixed point satisfies inequality (1.4), but converse is not true.

Example 1.1. [5] Let $X = \ell_\infty$ with the norm defined by $\|x\|_\infty = \max\limits_{1 \le i \le n} |x_i|$, and $C = \{x \in \ell_\infty : \|x\| \le 1\}$. Let $T : C \to C$ be mapping defined by

$$Tx = \frac{9}{10}(0, x_1^2, x_2^2, \cdots),$$

for $x = (x_1, x_2, \cdots) \in C$. Then

(1) T is continuous.

(2) $Tp = p$ implies $p = 0$.

(3) T satisfy (1.4). Indeed,

$$\begin{aligned} \|Tx - p\|_\infty &= \frac{9}{10}\|0, x_1^2, x_2^2, \cdots\|_\infty \\ &\le \frac{9}{10}\|0, x_1, x_2, \cdots\|_\infty \\ &= \frac{9}{10}\|x - p\|_\infty, \end{aligned}$$

for all $x \in C$ (since $p = 0$).

(4) T is not a contraction map. To see that, let $x = (\frac{3}{4}, \frac{3}{4}, \cdots)$ and $y = (\frac{1}{2}, \frac{1}{2}, \cdots)$. Clearly, $x, y \in C$, also

$$\begin{aligned} \|x - y\|_\infty &= \|(\frac{3}{4}, \frac{3}{4}, \cdots) - (\frac{1}{2}, \frac{1}{2}, \cdots)\|_\infty \\ &\le \|\frac{1}{4}, \frac{1}{4}, \cdots\|_\infty = \frac{1}{4}. \end{aligned}$$

On the other hand

$$\begin{aligned} T(x) &= T(\frac{3}{4}, \frac{3}{4}, \cdots) \\ &= \frac{9}{10}(0, \frac{9}{16}, \frac{9}{16}, \cdots), \end{aligned}$$

and

$$\begin{aligned} T(y) &= T(\frac{1}{2}, \frac{1}{2}, \cdots) \\ &= \frac{9}{10}(0, \frac{1}{4}, \frac{1}{4}, \cdots), \end{aligned}$$

so that, we have

$$\begin{aligned} \|Tx - Ty\|_\infty &= \frac{9}{10}\|(0, \frac{9}{16}, \frac{9}{16}, \cdots) - (0, \frac{1}{4}, \frac{1}{4}, \cdots)\|_\infty \\ &= \frac{45}{160}. \end{aligned}$$

Suppose, there exists $a \in [0,1)$, then by definition of contraction mapping, we have

$$\|Tx - Ty\|_\infty \leq a\ \|x - y\|_\infty$$
$$\frac{45}{160} \leq \frac{a}{4}$$

for the above choice of x and y. But then this implies that $a \geq \frac{180}{160} > 1$, so, T is not a contraction.

Most recently, Akewe and Okeke [2] used definition (1.4) and proved strong convergence and stability results for Picard-Mann hybrid iterative schemes in a real normed linear space as follows:

Theorem 1.2. [2] *Let X be a real normed linear space and $T : X \to X$ be a map with a fixed point p satisfying the condition (1.4). For arbitrary $x_0 \in X$, let $\{x_n\}$ be sequence defined by*

$$\begin{aligned} x_{n+1} &= Ty_n \\ y_n &= (1-\alpha_n)x_n + \alpha_n Tx_n \quad \textit{for all } n \in \mathbb{N}, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then

(1) $\{x_n\}$ converges strongly to p;

(2) iteration procedure defined by (1.5) is T-stable.

On the other hand, Agarwal et al. [1] introduced $S-$ iteration procedure whose rate of convergence is similar to the Picard iteration procedure and faster than other fixed point iteration procedures (see Theorem 3.2 [1]).

Let C be a nonempty convex subset of a normed space X and let $T : C \to C$ be a nonlinear mapping. Then for $x_0 \in C$, the $S-$iteration procedure is defined by:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n T y_n \\ y_n &= (1-\alpha_n)x_n + \alpha_n T x_n \quad \text{for all } n \in \mathbb{N}, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying certain conditions.

The purpose of this paper is to prove strong convergence and stability results for $S-$iteration procedure which is faster than Picard iteration procedure for the general class of contractive-like operators introduced by Bosede and Rhoades [4] in a real normed linear space. Our results generalize and improve a multitude of results in the literature, including the recent results of Akewe and Okeke [2] and many others for the general class of contractive-like operators using faster iterative procedure.

2. Main results

Now, we start with the following useful Lemma for our main result.

Lemma 2.1. [3] *Let δ be a real number satisfying $0 \le \delta < 1$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that* $\lim_{n\to\infty} \varepsilon_n = 0$. *Then, for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \le \delta u_n + \varepsilon_n, \quad \textit{for } \; n = 0,1,2,\cdots,$$

then $\lim_{n\to\infty} u_n = 0$.

Theorem 2.2. *Let X be a real normed linear space and C be a nonempty subset of X. Let $T : C \to C$ be a general class of contractive-like operators with a fixed point p satisfying the condition (1.4). Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences such that* $0 \le \alpha_n, \beta_n \le 1$ *and* $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$. *For given $x_0 \in C$, sequence $\{x_n\}$ is defined by (1.6). Let $\{y_n\}$ be a sequence in X and define a sequence $\{\varepsilon_n\}$ in* $[0,\infty)$ *by*

$$\begin{aligned} s_n &= (1-\beta_n)y_n + \beta_n T y_n \\ \varepsilon_n &= \|y_{n+1} - (1-\alpha_n)Ty_n - \alpha_n T s_n\|, \quad n \in \mathbb{N}, \end{aligned} \tag{2.1}$$

Then we have the following:

(a) The sequence $\{x_n\}$ converges strongly to fixed point p of T.

(b) For $p \in F(T)$, let $\{y_n\}$ be sequence such that

$$\|y_{n+1} - p\| \leq \delta\|y_n - p\| + \varepsilon_n.$$

(c) $\lim_{n\to\infty} y_n = p$ *if and only if* $\lim_{n\to\infty} \varepsilon_n = 0$, *that is $S-$iteration procedure defined by (1.6) is $T-$stable.*

Proof. (a) Let $p \in F(T)$, in the view of (1.4) and (1.6), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)Tx_n + \alpha_n Tz_n - p\| \\ &\leq (1-\alpha_n)\|Tx_n - p\| + \alpha_n\|Tz_n - p\| \\ &\leq \delta\Big[(1-\alpha_n)\|x_n - p\| + \alpha_n\|z_n - p\|\Big], \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \|z_n - p\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \delta\beta_n\|x_n - p\| \\ &\leq (1-\beta_n(1-\delta))\|x_n - p\|. \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \delta\Big[(1-\alpha_n)\|x_n - p\| + \alpha_n(1-\beta_n(1-\delta))\|x_n - p\|\Big] \\ &\leq \delta(1-\alpha_n\beta_n(1-\delta))|\, x_n - p\| \\ &\leq \delta\|x_n - p\|. \end{aligned}$$

By using the fact that $0 \leq \delta < 1, 0 \leq \alpha_n, \beta_n \leq 1$, $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$ and with the help of Lemma (2.1), we have $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$, that is sequence $\{x_n\}$ converges strongly to p.

(b) Form (2.1), we have

$$\begin{aligned} \|y_{n+1} - p\| &\leq \|y_{n+1} - (1-\alpha_n)Ty_n - \alpha_n Ts_n\| + \|(1-\alpha_n)Ty_n + \alpha_n Ts_n - p\| \\ &\leq \varepsilon_n + \|(1-\alpha_n)Ty_n + \alpha_n Ts_n - p\| \\ &\leq \varepsilon_n + \delta\Big[(1-\alpha_n)\|y_n - p\| + \alpha_n\|s_n - p\|\Big], \end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\|s_n-p\| &= \|(1-\beta_n)y_n+\beta_n Ty_n-p\| \\ &\leq (1-\beta_n)\|y_n-p\|+\beta_n\|Ty_n-p\| \\ &\leq (1-\beta_n)\|y_n-p\|+\delta\beta_n\|y_n-p\| \\ &\leq (1-\beta_n(1-\delta)]\|y_n-p\|.\end{aligned} \tag{2.5}$$

Substituting (2.5) into (2.4), we have

$$\begin{aligned}\|y_{n+1}-p\| &\leq \varepsilon_n+\delta\Big[(1-\alpha_n)\|y_n-p\|+\alpha_n(1-\beta_n(1-\delta))\|y_n-p\|\Big] \\ &\leq \varepsilon_n+\delta(1-\alpha_n\beta_n(1-\delta))\|y_n-p\| \\ &\leq \varepsilon_n+\delta\|y_n-p\|,\end{aligned} \tag{2.6}$$

for all $n \geq 0$, i.e, (b) holds.

(c) Suppose that $\lim_{n\to\infty} y_n = p$. Using (2.1), we have

$$\begin{aligned}\varepsilon_n &= \|y_{n+1}-(1-\alpha_n)Ty_n-\alpha_n Ts_n\| \\ &\leq \|y_{n+1}-p\|+\|(1-\alpha_n)Ty_n+\alpha_n Ts_n-p\| \\ &\leq \|y_{n+1}-p\|+(1-\alpha_n)\|Ty_n-p\|+\alpha_n\|Ts_n-p\| \\ &\leq \|y_{n+1}-p\|+(1-\alpha_n)\delta\|y_n-p\|+\delta\alpha_n\|s_n-p\|, \\ &\leq \|y_{n+1}-p\|+\delta\Big[(1-\alpha_n)\|y_n-p\|+\alpha_n\|s_n-p\|\Big]\end{aligned} \tag{2.7}$$

form (2.5), we have

$$\begin{aligned}\varepsilon_n &\leq \|y_{n+1}-p\|+\delta\Big[(1-\alpha_n)\|y_n-p\|+\alpha_n(1-\beta_n(1-\delta)\Big]\|y_n-p\| \\ &\leq \|y_{n+1}-p\|+\delta\Big[1-\alpha_n\beta_n(1-\delta)\Big]\|y_n-p\| \\ &\to 0 \ \text{ as } \ n\to\infty.\end{aligned}$$

Conversely, suppose that $\lim_{n\to\infty}\varepsilon_n = 0$. By using the fact that $0 \leq \delta < 1, 0 \leq \alpha_n, \beta_n \leq 1$, $\sum_{n=1}^{\infty}\alpha_n\beta_n = \infty$ and with the help of Lemma (2.1), we have $\lim_{n\to\infty} y_n = p$. This completes the proof.

The following example shows that the iteration procedure (1.6) is $T-$stable.

Example 2.3 *Let $\mathbb{R}$ denotes set of real numbers and $C = [0,1] \subset \mathbb{R}$ with usual norm. Let $T : C \to C$ be a mapping defined by $Tx = \frac{x}{2}$.*

Indeed, T satisfy (1.4) with $F(T) = 0$. Take $p = 0$ and $y_n = \frac{1}{n}$, for each $n \geq 0$. Then $\lim_{n\to\infty} y_n = 0$. We see that

$$\begin{aligned}
\varepsilon_n &= \left| y_{n+1} - (1-\alpha_n)Ty_n - \alpha_n T[(1-\beta_n)y_n + \beta_n Ty_n] \right| \\
&= \left| y_{n+1} - (1-\alpha_n)Ty_n - \alpha_n T\left[\frac{(2-\beta_n)y_n}{2}\right] \right| \\
&= \left| y_{n+1} - (1-\alpha_n)Ty_n - \alpha_n \frac{(2-\beta_n)y_n}{4} \right| \\
&= \left| y_{n+1} - (1-\alpha_n)\frac{y_n}{2} - \alpha_n \frac{(2-\beta_n)y_n}{4} \right| \\
&\leq \left| \frac{n-1}{2n(n+1)} \right| + \left| \frac{\alpha_n\beta_n}{4n} \right| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

We have $\lim_{n\to\infty} \varepsilon_n = 0$. Hence, iteration procedure (1.6) is T stable.

Conclusion The iteration procedure defined by (1.6) relatively faster and independent iteration procedure as compared many iteration procedures appears in the literatures, so that our Theorem 2.2 generalize and improve a multitude of results in the literature, including the recent results of Akewe and Okeke [2] and many others for the general class of contractive-like operators.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] R. P. Agarwal, D. O'Regan, D. R. Sahu, Iterative construction of fixed points of nearly asymptotically non-expansive mappings, J. Convex Anal. 8 (2007), 61-79.

[2] H. Akewe, G. A. Okeke, Convergence and stability theorems for the Picard-Mann hybrid iterative scheme for a general class of contractive-like operators, Fixed Point Theory Appl. 2015 (2015), 66.

[3] V. Berinde, On the stability of some fixed point procedures, Bul. Stiint. - Univ. BaiaMare, Ser. B Fasc. Mat.-Inform. 18 (2002), 7-14.

[4] A. O. Bosede, B. E Rhoades, Stability of Picard and Mann iteration for a general class of functions, J. Adv. Math. Stud. 3 (2010), 1-3.

[5] C. E. Chidume, J. O. Olaleru, Picard iteration process for a general class of contractive mappings, J. Niger. Math. Soc. 33 (2014), 19-23.

[6] R. H. Haghi, M. Postolache, Sh. Rezapour, On T-stability of the Picard iteration for generalized phi-contraction map-pings, Abstr. Appl. Anal. 2012 (2012), Article ID 658971.

[7] A. M. Harder, Fixed point theory and stability results for fixed point iteration procedure, Ph.D. Thesis, University of Missouri Rolla 1987.

[8] A. M. Harder, T. L. Hicks, A stable iteration procedure for nonexpansive mappings, Math. Japonica 33 (1998), 687-692.

[9] A. M. Harder, T. L. Hicks, Stability results for fixed point iteration procedures, Math. Japonica 33 (1998), 693-706.

[10] C. O. Imoru, M. O. Olatinwo, On the stability of Picard and Mann iteration, Carpath. J. Math. 19 (2003), 155-160.

[11] M. O. Olatinwo, M. Postolache, Stability results for Jungck-type iterative processes in convex metric spaces, Appl. Math. Comput. 218 (2012), 6727-6732.

[12] M. O. Osilike, Stability results for Ishikawa fixed point iteration procedure, Indian J. Pure Appl. Math. 26 (1995), 937-941.

[13] M. O. Osilike, Stability of the Mann and Ishikawa iteration procedure for $\phi-$ strongly pseuduocontractive and nonlinear equations of the $\phi-$ strongly accretive type, J. Math. Anal. Appl. 227 (1998), 319-334.

[14] B. E. Rhoades, Fixed point theorems and stability results for fixed point iteration procedures, Indian J. Pure Appl. Math. 21 (1990), 1-9.

[15] B. E. Rhoades, Fixed point theorems and stability results for fixed point iteration procedures -II, Indian J. Pure Appl. Math. 24 (1193), 691-703.