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EXISTENCE OF PERIODIC SOLUTIONS OF A GENERALIZED LIENARD EQUATION

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Abstract. Conditions under which the existence of periodic solution of a generalized Lienard equation are introduced. The elements of direct Lyapunov method permits us to obtain the existence criteria of cycles.

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1. Introduction

The study of the generalized Lienard equations of the form

$$\ddot{x}+\phi(x,\dot{x})\dot{x}+g(x)=0 \tag{1.1}$$

where $\dot{(\cdot)}=\frac{d}{dt}$, holds an important place in the theory of dynamical systems. A special case of this kind of differential equation is of the form

$$\ddot{x}+f(x)\dot{x}^2+g(x)\dot{x}+h(x)=0 \tag{1.2}$$

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which is sometime called in the literature as Langmiur equation [4],[5]. The Langmiur equation governs the space-change current in an electron tube is a special case, as in the equation for the brachistochrone.

For the Lienard equation, the classical theorems on the existence of periodic orbits is well known [2]. Here we are interested in the conditions under which equation (1.1) has a periodic solution when the system has only one unstable equilibrium point. The main tool of this work is to construct a piecewise-smooth transversal closed curve surrounding an unstable singular point.

In this paper we generalize the approach used in [3] which was only for a special case $\phi(x)$. The technique developed is based on use of an artificial closed piecewise-smooth curve of the families of transversal curves and special Lyapunov type functions and then applying Poincare-Bendixon theorem [1] led the existence criteria of cycles. This work admits different extension and allow us to deal with a more general the term $\phi(x,y)$.

In section 2 we present the main result Theorem1 which gathers the Lemmas introduced before it.

2. MAIN RESULTS

Equation (1.1) is usually studied by means of an equivalent plane differential system

$$
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= -\phi(x,y)y - g(x)
\end{aligned}
\tag{2.1}
$$

We assume that, the functions $\phi(x,y)$ and $g(x)$ are continuous on the region $(a,+\infty)\times(\alpha,\beta)$ and $(a,+\infty)$ respectively for some suitable chosen real numbers a,α and β, and for certain numbers r_1,r_2 and x_0 such that $a<r_1\leq x_0\leq r_2$ and the following hypotheses are satisfied

$$H1: \qquad \lim_{x\to a} g(x) = -\lim_{x\to\infty} g(x) = -\infty$$

$$\lim_{x\to a}\int_{x_0}^{x} g(u)du = \lim_{x\to\infty}\int_{x_0}^{x} g(u)du = \infty$$

$$H2: \qquad \phi(x,y)>0 \quad \text{on} \quad (a,r_1)\times(\alpha,\beta)\cup(r_2,\infty)\times(\alpha,\beta)$$

$$\int_{r_1}^{r_2} \phi(x,y)dx \geq 0 \quad \text{for all} \quad y \in (\alpha, \beta)$$

Consider a pair of numbers $c_1 \in (a, r_1)$ and $c_2 \in (r_2, \infty)$ such that c_1 is sufficiently close to a, c_2 is sufficiently large provided

$$\int_{c_1}^{c_2} g(x)dx = 0 \tag{2.2}$$

Without loss of generality, we may put

$$\begin{aligned} g(x) &< 0, \quad \text{for all } x \in [c_1, r_1] \\ g(x) &> 0, \quad \text{for all } x \in [r_2, c_2] \end{aligned} \tag{2.3}$$

Consider of the following seven Lyapunov functions

$$V_1(x,y) = y^2 + 2\int_{x_0}^{x} g(u)du,$$

$$V_2(x,y) = \left(y + \int_{r_1}^{x} \phi(u, y_0)du \right)^2 + 2\int_{x_0}^{x} g(u)du$$

$$V_3(x,y) = \left(y + \int_{r_2}^{x} \phi(u, y_0)du \right)^2 + 2\int_{x_0}^{x} g(u)du$$

$$V_4(x,y) = V_2(x,y) - \varepsilon(x - r_1), \qquad V_5(x,y) = V_3(x,y) - \varepsilon(x - r_2)$$

$$V_6(x,y) = V_3(x,y) + \varepsilon(x - r_2), \qquad V_7(x,y) = V_2(x,y) + \varepsilon(x - r_1)$$

Here y_0 is any number $0 < y_0 < \beta$ and ε is a certain sufficiently small number.

Consider the following eight regions

$$R_1 = \{x \in [c_1, r_1], y \geq 0, V_1(x,y) \leq V_1(c_1, 0)\}$$

$$R_2 = \{x \in [r_1, x_0], y \geq 0, V_4(x,y) \leq V_2(r_1, y_1)\}$$

$$R_3 = \{x \in [x_0, r_2], y \geq 0, V_5(x,y) \leq V_3(r_2, y_2)\}$$

$$R_4 = \{x \in [r_2, c_2], y \geq 0, V_3(x,y) \leq V_3(c_2, 0)\}$$

$$R_5 = \{x \in [r_2, c_2], y \leq 0, V_1(x,y) \leq V_1(c_2, 0)\}$$

$$R_6 = \{x \in [x_0, r_2], y \leq 0, V_6(x,y) \leq V_3(r_2, y_3)\}$$

$$R_7 = \{x \in [r_1, x_0], y \le 0, V_7(x,y) \le V_2(r_1, y_4)\}$$

$$R_8 = \{x \in [c_1, r_1], y \le 0, V_2(x,y) \le V_2(c_1, 0)\}$$

where $y_1 > 0$, $y_2 > 0$, $y_3 < 0$, $y_4 < 0$ are solutions of the following square equations

$$y_1: \quad V_1(r_1, y_1) = V_1(c_1, 0), \qquad y_2: \quad V_3(r_2, y_2) = V_3(c_2, 0)$$

$$y_3: \quad V_1(r_2, y_3) = V_1(c_2, 0), \qquad y_4: \quad V_2(r_1, y_4) = V_2(c_1, 0)$$

Lemma 1. *The derivatives $\dot{V}_j(x,y)$ along the solutions of system (2.1) for $y \ne 0$, $x \ne r_j$, satisfy the following inequalities*

$$\dot{V}_1 < 0 \quad on \quad R_1 \cup R_5, \qquad \dot{V}_2 < 0 \quad on \quad R_8, \qquad \dot{V}_3 < 0 \quad on \quad R_4,$$

$$\dot{V}_4 < 0 \quad on \quad R_2, \qquad \dot{V}_5 < 0 \quad on \quad R_3, \qquad \dot{V}_6 < 0 \quad on \quad R_6,$$

$$\dot{V}_7 < 0 \quad on \quad R_7$$

Proof. For the derivatives of the functions $V_j(x,y)$, $j = 1,2,3$ along the solutions of system (2.1) we have the following relations

$$\dot{V}_1 = -2\phi(x,y)y^2, \qquad \dot{V}_2 = -2g(x)\int_{r_1}^{x} \phi(u,y)du, \qquad \dot{V}_3 = -2g(x)\int_{r_2}^{x} \phi(u,y)du,$$

It is clear that these three functions all satisfy the required inequality on the regions $R_1 \cup R_2$, R_8, R_4 respectively.

To clarify that $\dot{V}_4 < 0$, $\dot{V}_5 < 0$, $\dot{V}_6 < 0$, $\dot{V}_7 < 0$ on the sets R_2, R_3, R_6, R_7, respectively, see the following

We have

$$\dot{V}_4 = -2g(x)\int_{r_1}^{x} \phi(u,y)du - \varepsilon y, \qquad \dot{V}_5 = -2g(x)\int_{r_2}^{x} \phi(u,y)du - \varepsilon y$$

$$\dot{V}_6 = -2g(x)\int_{r_2}^{x} \phi(u,y)du + \varepsilon y, \qquad \dot{V}_7 = -2g(x)\int_{r_1}^{x} \phi(u,y)du + \varepsilon y$$

Hold fixed the arbitrary $\varepsilon > 0$, we choose c_1 so much closer to a and c_2 sufficiently large, that the minimal values of $|y|$ on the intersection of the constructed closed curve and the band $\{x \in [r_1, r_2]\}$ are more than

$$\frac{1}{\varepsilon} \max_{x \in [r_1, r_2]} 2 \left| g(x) \int_{r_1}^{x} \phi(u,y) du \right|, \qquad \text{and} \qquad \frac{1}{\varepsilon} \max_{x \in [r_1, r_2]} 2 \left| g(x) \int_{r_2}^{x} \phi(u,y) du \right|$$

This implies the required inequalities $\dot{V}_j < 0, \ \ j = 4,5,6,7.$ ∎

Define the numbers y_5, y_6, y_7, and y_8 as follows

y_5 is a positive solution of the equation $V_4(x_0, y_5) = V_2(r_1, y_1)$

y_6 is a positive solution of the equation $V_5(x_0, y_6) = V_3(r_2, y_2)$

y_7 is a negative solution of the equation $V_6(x_0, y_7) = V_3(r_2, y_3)$

y_8 is a negative solution of the equation $V_7(x_0, y_8) = V_2(r_1, y_4)$

Lemma 2. $y_5 < y_6$ *and* $y_7 > y_8$

Proof. First we prove that $y_5 < y_6.$

We have

$$V_4(x_0, y_5) = V_2(r_1, y_1)$$

Therefore

$$\left(y_5 + \int_{r_1}^{x_0} \phi(x,y) dx \right)^2 - \varepsilon(x_0, r_1) = y_1^2 + 2 \int_{x_0}^{r_1} g(x) dx$$

From this we can write the positive value of y_5 as follows

$$y_5 = \left(\varepsilon(x_0 - r_1) + y_1^2 + 2 \int_{x_0}^{r_1} g(x) dx \right)^{\frac{1}{2}} + \int_{x_0}^{r_1} \phi(x,y) dx \tag{2.4}$$

But from the condition

$$\int_{r_1}^{r_2} \phi(x,y) dx \geq 0 , \qquad \text{for all } \ y \in (\alpha, \beta)$$

we get

$$\int_{x_0}^{r_1} \phi(x,y)dx \leq \int_{x_0}^{r_2} \phi(x,y)dx$$

The equation 2.4 implies the following inequality

$$y_5 \leq \left(\varepsilon(x_0 - r_1) + y_1^2 + 2\int_{x_0}^{r_1} g(x)dx\right)^{\frac{1}{2}} + \int_{x_0}^{r_2} \phi(x,y)dx$$

On the other hand from

$$V_5(x_0, y_6) = V_3(r_2, y_2)$$

we get

$$y_6 = \left(\varepsilon(x_0 - r_2) + y_2^2 + 2\int_{x_0}^{r2} g(x)dx\right)^{\frac{1}{2}} + \int_{x_0}^{r_2} \phi(x,y)dx \tag{2.5}$$

Now if we choose ε such that

$$0 < \varepsilon < \frac{1}{r_2 - r_1}\left(\int_{r_2}^{c_2} \phi(x,y)dx\right)^2$$

Hence

$$0 < \varepsilon(r_1 - r_2) + \left(\int_{r_2}^{c_2} \phi(x,y)dx\right)^2$$

Therefore

$$2\int_{x_0}^{c_2} g(x)dx < \varepsilon(r_1 - r_2) + \left(\int_{r_2}^{c_2} \phi(x,y)dx\right)^2 + 2\int_{x_0}^{c_2} g(x)dx \tag{2.6}$$

From the condition

$$\int_{c_1}^{c_2} g(x)dx = 0$$

we get

$$\int_{x_0}^{c_1} g(x)dx = \int_{x_0}^{c_2} g(x)dx$$

Hence the inequality 2.6 will be

$$2\int_{x_0}^{c_1} g(x)dx < \varepsilon(r_1 - r_2) + \left(\int_{r_2}^{c_2} \phi(x,y)dx\right)^2 + 2\int_{x_0}^{c_2} g(x)dx \tag{2.7}$$

From $V_3(r_2,y_2) = V_3(c_2,0)$, we get

$$y_2^2 + 2\int_{x_0}^{r_2} g(x)dx = \left(\int_{r_2}^{c_2} \phi(x,y)dx\right)^2 + 2\int_{x_0}^{c_2} g(x)dx$$

Then the inequality 2.7 will be

$$2\int_{x_0}^{c_1} g(x)dx < \varepsilon(r_1 - r_2) + y_2^2 + 2\int_{x_0}^{r_2} g(x)dx \tag{2.8}$$

From $V_1(r_1,y_1) = V_1(c_1,0)$, we get

$$y_1^2 + 2\int_{x_0}^{r_1} g(x)dx = 2\int_{x_0}^{c_1} g(x)dx$$

Inequality will be

$$y_1^2 + 2\int_{x_0}^{r_1} g(x)dx < \varepsilon(r_1 - r_2) + y_2^2 + 2\int_{x_0}^{r_2} g(x)dx$$

Then

$$\varepsilon(x_0 - r_1) + y_1^2 + 2\int_{x_0}^{r_1} g(x)dx < \varepsilon(x_0 - r_2) + y_2^2 + 2\int_{x_0}^{r_2} g(x)dx$$

Both sides are positive, therefore

$$\left(\varepsilon(x_0 - r_1) + y_1^2 + 2\int_{x_0}^{r_1} g(x)dx\right)^{\frac{1}{2}} - \int_{r_1}^{x_0} \phi(x,y)dx <$$

$$\left(\varepsilon(x_0 - r_2) + y_2^2 + 2\int_{x_0}^{r_2} g(x)dx\right)^{\frac{1}{2}} + \int_{x_0}^{r_2} \phi(x,y)dx$$

This means $y_5 < y_6$.

To prove $y_7 > y_8$, we follow similar steps but here we choose ε to be

$$\varepsilon > \frac{1}{r_2 - r_1}\left(\int_{r_1}^{c_1} \phi(x,y)dx\right)^2$$

which proves the second assertion of Lemma (the details can be sent on request). ∎

Note, so far we have obtained a disconnected transversal piecewise-smooth curve. This curve consists of two connected parts one to the left of the point x_0 of a shape $\subset$, let us call it C_1, which passes through the points (x_0,y$_8$), (r_1,y$_4$), (c_1,0), (r_1,y$_1$) , and (x_0,y$_5$). The other part is of the shape $\supset$, let us call it C_2, and it is on the right of the point x_0 and passes through the points (x_0,y$_7$), (r_2,y$_3$), (c_2,0), (r_2,y$_2$) and (x_0,y$_6$). Recall that, we have $0 < y_5 < y_6$, and $y_8 < y_7 < 0$. So there are two line segments one L_1 connecting the endpoint (x_0, y_5) of the part C_1 with endpoint (x_0, y_6) of the part C_2. The other line segment L_2 connecting the endpoint (x_0, y_7) of the part C_2 with endpoint (x_0, y_8) of the part C_1. The vector field of the system (2.1) on the line segment L_1 is directing towards right and on the line segment L_2 is directing towards left. Therefore the curve $C_1 \cup L_1 \cup C_2 \cup L_2$ is connected closed transversal piecewise-smooth curve.

Then, consequently, if we apply Poincare-Bendixon theorem [1], we can state and prove the following theorem.

Theorem 1. *For system (2.1), if conditions $H1$ and $H2$ are valid then in the phase space in the region $R = \{(x,y) \in (a,\infty) \times (\alpha,\beta)\}$ there is a piecewise-smooth transversal closed curve which intersects the straight line $y = 0$ at the certain points $a < c_1 < r_1$ and $r_2 < c_2$. If in addition, in region R, system* (2.1) *has only one unstable focal equilibrium, then the system has a periodic solution.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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