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TRANSLATION SURFACES ACCORDING TO BISHOP FRAME IN EUCLIDEAN 3-SPACE

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Abstract: In this paper, by using non-planer space curves the translation surfaces are investigated according to Bishop frames in Euclidean 3-space. We have calculated the first and second fundamental quantities. We give the gaussian and mean curvatures of these surfaces according to the first and second Bishop curvatures in Euclidean 3-space. Also, using the Bishop curvatures of generating curves of translation surface, we give the necessary and sufficient conditions for generating curves to be geodesic and asymptotic curves on the surface.

Keywords: Translation Surfaces, Bishop Frame, Curvature, Darboux Frame, Bishop Curvatures.

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1. Introduction

Bishop frame, which is also called alternative or parallel frame of the curves, was introduced by L. R. Bishop in 1975 by means of parallel vector fields, [1].

A practical application of Bishop frames is that they are used in the area of Biology and Computer Graphics. For example, it may be possible to compute information about the shope of sequences of DNA using a curve defined by the Bishop frame. The Bishop frame may also provide a new way to control virtual cameras in computer animatons,[2].

Translation surfaces have been investigated by some differential geometers. In recent

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years several authors have used these translation surfaces in their investigations in, [3,4]. In one of the latest papers, in [5,6] the authors studied the translation surfaces according to Frenet frames in Euclidean 3-space and 3-dimensional Minkowski space. Also, their introduced the Darboux frame of the genarator curves of the translation surfaces in Minkowski 3-space,[6]. H. Liu has given the classification of the translation surfaces with constant mean curvature or constant Gaussian curvature in 3-dimensional Euclidean space and 3-dimensional Minkowski space,[7]. In [8] the authors have studied the second fundamental form of translation surfaces in

E^3. In this paper, by using non-planer space curves the translation surfaces are investigated according to Bishop frames in Euclidean 3-space. We have calculated the first and second fundamental quantities. We give the gaussian and mean curvatures of these surfaces according to Bishop curvatures in Euclidean 3-space. Also, using the Bishop curvatures of generating curves of translation surface, we give the necessary and sufficient conditions for generating curves to be geodesic and asymptotic curves on the surface.

2. Preliminaries

A surface can be generated from two space curves by translating either one of them parallel to itself in such a way that each of its points describes a curve that is a translation of the other curve. Let $M(u,v)$ be a translation surface in Euclidean 3-space. Then $M(u,v)$ is parametrized by

$$M(u,v)=\alpha(u)+\beta(v) \tag{2.1}$$

where (α) and (β) being unit-speed space curves of the arc-length parameters u and v, respectively.

The first fundamental form I of the surface is defined by

$$I=Edu^2+2Fdudv+Gdv^2 \tag{2.2}$$

where $E=\langle M_u,M_u\rangle$, $F=\langle M_u,M_v\rangle$ and $G=\langle M_v,M_v\rangle$, $M_u=\dfrac{\partial M(u,v)}{\partial u}$ are the coefficients of I. Also, The first fundamental form II of the surface M is defined by

$$II = ldu^2 + 2mdudv + ndv^2 \tag{2.3}$$

where $l = \det(M_u, M_v, M_{uu})$, $m = \det(M_u, M_v, M_{uv})$ and $n = \det(M_u, M_v, M_{vv})$ are the coefficiants of II.

Then, the Gaussian curvature K and mean curvature H of the surface M is given by [9],

$$K = \frac{ln - m^2}{EG - F^2}, \qquad H = \frac{En - 2Fm - Gl}{2\left(EG - F^2\right)}. \tag{2.4}$$

Let now $\alpha(s)$ be a regular curve in Euclidean 3-space. $\{T, N, B\}$, $\{T, N_1, N_2\}$ and $\{T, g, U\}$ be the Frenet frame field, the Bishop frame and the Darboux frame of (α) with the first and second k_1 , k_2 bishop curvatures, respectively. In Darboux frame U is the unit normal of the surface M along (α) and g is a unit vector given by $g = U \times T$.

Bishop frame formulas of (α) curve is expressed as

$$\begin{aligned} \frac{dT}{ds} &= k_1 N_1 + k_2 N_2 \\ \frac{dN_1}{ds} &= -k_1 T \\ \frac{dN_2}{ds} &= -k_2 T. \end{aligned} \tag{2.5}$$

Here, we shall call the set $\{T, N_1, N_2\}$ as Bishop frame and k_1 , k_2 are called the first and second bishop curvatures (α) curve, respectively,[1].

The relation between Bishop frame and Darboux frame is given as follows,

$$\begin{bmatrix} T \\ g \\ U \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Sigma & \sin\Sigma \\ 0 & -\sin\Sigma & \cos\Sigma \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix} \tag{2.6}$$

here $\Sigma = \psi + \theta$, ψ is the angle between vectors of N and g, θ is the angle between vectors of N and N_1, [10].

Also, the relation between Bishop curvatures and the normal curvature of (α) curve is given as follows [10],

$$k_n = -k_1 \sin\Sigma + k_2 \cos\Sigma . \tag{2.7}$$

The relation between Bishop curvatures and the geodesic curvature of (α) curve is given as follows [10],

$$k_g = -k_1 \sin\Sigma + k_2 \cos\Sigma . \tag{2.8}$$

Now, we can write the following important definitions:

Definition 2.1. For a curve (α) lying on a surface, the following are well-known:

(1) (α) curve is an asymptotic line of surface if and only if normal curvature k_n vanishes.

(2) (α) curve is an geodesic line of surface if and only if geodesic curvature k_g vanishes, [9].

3. Translation Surfaces with Space Curves According to Bishop Frame in Euclidean 3-Space

Let $M(u,v)$ be a translation surface in Euclidean 3-space. Then $M(u,v)$ is parametrized by

$$M(u,v) = \alpha(u) + \beta(v) \tag{3.1}$$

where (α) and (β) being unit-speed space curves of the arc-length parameters u and v, respectively. Let $\{T_\alpha, N_\alpha, B_\alpha\}$, $\{T_\alpha, N_1^\alpha, N_2^\alpha\}$ and $\{T_\alpha, g_\alpha, U\}$ be the Frenet frame field, the Bishop frame and the Darboux frame of (α) with the first and second k_1^α , k_2^α bishop curvatures, respectively. In Darboux frame U is the unit normal of the surface M along (α) and g_α is a unit vector given by $g_\alpha = U \times T_\alpha$. Also, let $\{T_\beta, N_\beta, B_\beta\}$, $\{T_\beta, N_1^\beta, N_2^\beta\}$ and

$\left\{T_\beta, g_\beta, U\right\}$ be the Frenet frame field, the Bishop frame and the Darboux frame of (β) with k_1^β , k_2^β the first and second bishop curvatures, respectively.

The relation between Bishop frame and Darboux frame is given as follows,

$$\begin{bmatrix} T_\alpha \\ g_\alpha \\ U \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Sigma_\alpha & \sin\Sigma_\alpha \\ 0 & -\sin\Sigma_\alpha & \cos\Sigma_\alpha \end{bmatrix} \begin{bmatrix} T_\alpha \\ N_1^\alpha \\ N_2^\alpha \end{bmatrix} \tag{3.2}$$

here $\Sigma_\alpha = \psi_\alpha + \theta_\alpha$, ψ_α is the angle between vectors of N_α and g_α, θ_α is the angle between vectors of N_α and N_1^α.

Similary, The relation between Bishop frame and Darboux frame of curve(β) is given as follows,

$$\begin{bmatrix} T_\beta \\ g_\beta \\ U \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Sigma_\beta & \sin\Sigma_\beta \\ 0 & -\sin\Sigma_\beta & \cos\Sigma_\beta \end{bmatrix} \begin{bmatrix} T_\beta \\ N_1^\beta \\ N_2^\beta \end{bmatrix} \tag{3.3}$$

here $\Sigma_\beta = \psi_\beta + \theta_\beta$, ψ_β is the angle between vectors of N_β and g_β, θ_β is the angle between vectors of N_β and N_1^β.

The unit normal of the surface M can be defined by

$$U(u,v) = \frac{M_u \times M_v}{\left\| M_u \times M_v \right\|} = \frac{1}{\sin\varphi} T_\alpha \times T_\beta \tag{3.4}$$

where $\varphi(u)$ is the angle between tangent vectors of (α) and (β).

The first fundamental form I of the surface is

$$I = du^2 + 2\cos\varphi\, du dv + dv^2 \tag{3.5}$$

where $E = 1,\ F = \cos\varphi$ and $G = 1$ are the coefficients of I .

The first fundamental form II of the surface is

$$II = \left(-k_1^\alpha \sin\Sigma_\alpha + k_2^\alpha \cos\Sigma_\alpha\right) du^2 + \left(-k_1^\beta \sin\Sigma_\beta + k_2^\beta \cos\Sigma_\beta\right) dv^2 \tag{3.6}$$

where $l = -k_1^\alpha \sin\Sigma_\alpha + k_2^\alpha \cos\Sigma_\alpha$, $m = 0$ and $n = -k_1^\beta \sin\Sigma_\beta + k_2^\beta \cos\Sigma_\beta$ are the coefficiants of II.

On the other hand the Gaussian curvature K and mean curvature H of the surface are

$$K = \frac{\left(-k_1^{\alpha} \sin \Sigma_{\alpha} + k_2^{\alpha} \cos \Sigma_{\alpha}\right)\left(-k_1^{\beta} \sin \Sigma_{\beta} + k_2^{\beta} \cos \Sigma_{\beta}\right)}{\sin^2 \varphi} \tag{3.7}$$

where the relation between Bishop curvatures and the normal curvature of (α) curve and (β) curve is given as follows,

$$k_n^{\alpha} = -k_1^{\alpha} \sin \Sigma_{\alpha} + k_2^{\alpha} \cos \Sigma_{\alpha},\ k_n^{\beta} = -k_1^{\beta} \sin \Sigma_{\beta} + k_2^{\beta} \cos \Sigma_{\beta}. \tag{3.8}$$

Then, the Gaussian curvature K of the surface are

$$K = \frac{k_n^{\alpha} k_n^{\beta}}{\sin^2 \varphi} \tag{3.9}$$

and

$$H = \frac{k_n^{\alpha} + k_n^{\beta}}{2 \sin^2 \varphi}. \tag{3.10}$$

Theorem 3.1. *Gauss curvature of a translation surface generating by space curves according to Bishop frames is zero if and only if at least one of generator curves is an asymptotic line of surface.*

Proof. Let Gauss curvature be zero, then from **(**3.9**)**

$$k_n^{\alpha} k_n^{\beta} = 0 \tag{3.11}$$

so $k_n^{\alpha} = 0$ or $k_n^{\beta} = 0$. If k_n^{α} is zero, (α) is an asyptotic line. Similarly, if k_n^{β} is zero, (β) is an asyptotic line. Conversely, let (α) or (β) be an asyptotic line of surface. If (α) is an asymptotic line of surface, then $k_n^{\alpha} = 0$, $K = 0$ or if (β) is an asymptotic line of surface, then $k_n^{\beta} = 0$, $K = 0$.

Theorem 3.2. *Let (α) and (β) be space curves according to Bishop frame with nonzero curvatures and let (α) be an asypmtotic line. Translation surface is minimal if and only if (β) is an asymptotic line of surface too.*

Proof. If M is a translation surface with $K=0$, then M is a ruled surface or at least one of generator curves of surface is asyptotic line. Let M is a minimal translation surface with $K \neq 0$ and (α) ia an asymptotic line of translation surface.

Let mean curvature be zero, then (3.10)

$$k_n^\alpha + k_n^\beta = 0 . \tag{3.12}$$

Since (α) be an asymptotic line of surface, k_n^α is zero then (3.11), k_n^β is zero. So, (β) is an asymptotic line of translation surface. Hence, the following corollary can be given;

Corollary 3.3. *Let M be a translation surface generated by its asymptotic lines then M is a minimal surface if and only if*

$$k_n^\alpha + k_n^\beta = 0$$

is satisfies.

On the other hand, differentiating

$-k_1^\alpha \sin\Sigma_\alpha + k_2^\alpha \cos\Sigma_\alpha - k_1^\beta \sin\Sigma_\beta + k_2^\beta \cos\Sigma_\beta = 0$ with respect to u , then

$$-k_1^{\alpha'} \sin\Sigma_\alpha - k_1^\alpha \Sigma_\alpha^{'} \cos\Sigma_\alpha + k_2^{\alpha'} \cos\Sigma_\alpha - k_2^\alpha \Sigma_\alpha^{'} \sin\Sigma_\alpha = 0$$
$$-\Sigma_\alpha^{'} \left(k_1^\alpha \cos\Sigma_\alpha + k_2^\alpha \sin\Sigma_\alpha \right) - k_1^{\alpha'} \sin\Sigma_\alpha + k_2^{\alpha'} \cos\Sigma_\alpha = 0$$

$$-k_g^\alpha \Sigma_\alpha^{'} - k_1^{\alpha'} \sin\Sigma_\alpha + k_2^{\alpha'} \cos\Sigma_\alpha = 0. \tag{3.13}$$

Therefore, equation (3.13), (α) curve is an geodesic line of surface if and only if

$$\frac{k_1^{\alpha'}}{k_2^{\alpha'}} = \cot\Sigma_\alpha \quad . \tag{3.14}$$

Similarly, differentiating $-k_1^\alpha \sin\Sigma_\alpha + k_2^\alpha \cos\Sigma_\alpha - k_1^\beta \sin\Sigma_\beta + k_2^\beta \cos\Sigma_\beta = 0$ with respect to v ,then (β) curve is an geodesic line of surface if and only if

$$\frac{k_1^{\beta'}}{k_2^{\beta'}} = \cot\Sigma_\beta . \tag{3.15}$$

Conflict of Interests

The author declares that there is no conflict of interests.

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