Available online at http://scik.org
J. Math. Comput. Sci. 6 (2016), No. 2, 262-271
ISSN: 1927-5307

COMMUTATIVE GROUPOID ALGEBRA

AHMED ABD EL-MONSEF ALLAM, NABILA NASSIIEF MIKHAEEL AND HUDA HAMDAN MERDACH*

Assiut University, Egypt

Copyright © 2016 Allam, Mikhaeel and Merdach. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract: In this paper we introduce of commutative groupoid algebra which is an equivalent definition of lattice commutative groupoid algebra and Futher we prove that it is regular autometrized algebra. Futher we remark that the binary operation Θ on commutative groupoid algebra can never be associative.

Keywords: commutative groupoid algebra; lattice commutative groupoid algebra; autometrized algebras; autometrized algebras.

2010 AMS Subject Classification: 13A99.

1. Introduction

In this paper we have two sections. In the first section we introduce the concept of commutative groupoid algebra with the binary operation Θ and obtain certain properties. Futher we prove that commutative groupoid algebra is equipped with a structure of a bounded lattice and also is lattice commutative groupoid algebra. It is also observed that the binary operation Θ can never be associative. In the second section we introduce two more binary operations "+", "-" on commutative groupoid algebra and obtain certain properties with these operations. Futher we prove that any commutative groupoid algebra is a "metric space". Also we prove that every commutative groupoid algebra can be made into a regular autometrized algebra.

2. Commutative groupoid algebra

In this section introduce a concept of commutative groupoid algebra and obtain some properties. Futher we prove that commutative groupoid algebra is equipped with a structure of a bounded lattice and also is lattice commutative groupoid algebra.

---

*Corresponding author

Received October 21, 2015

Definition 2.1 Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice with order reversing involution “ $\neg$ ” and a binary operation “$\odot$” satisfying the following axioms: $\forall$ a, b, c $\in$ L.

(I_1) $a \odot (b \odot \neg c) = b \odot (a \odot \neg c)$.

(I_2) $a \odot \neg a = 0$.

(I_3) $a \odot b = b \odot a$.

(I_4) $a \odot \neg b = b \odot \neg a = 0 \Rightarrow a = b$.

(I_5) $a \odot \neg(\neg b \odot a) = b \odot \neg(\neg a \odot b)$.

(L_1) $\neg((a \vee b) \odot \neg c) = \neg(a \odot \neg c) \wedge \neg(b \odot \neg c)$.

(L_2) $\neg((a \wedge b) \odot \neg c) = \neg(a \odot \neg c) \vee \neg(b \odot \neg c)$.

If $(L, \wedge, \vee, 0, 1)$ satisfying (I_1)-(I_5) then $(L, \wedge, \vee, 0, 1)$ is said to be quasi- lattice commutative groupoid algebra.

Definition 2.2 An algebra $(L, \odot, \neg, 0, 1)$ of type (2, 1, 0, 0) is called a commutative groupoid algebra. If it satisfies the following conditions $\forall$ x, y, z $\in$ L.:

(G1) $x \odot (y \odot \neg z) = y \odot (x \odot \neg z)$

(G2) $1 \odot x = x$

(G3) $x \odot 0 = 0$

(G4) $x \odot y = y \odot x$

(G5) $x \odot \neg(\neg y \odot x) = y \odot \neg(\neg x \odot y)$.

(G6) $\neg 0 = 1$.

Lemma 2.3 Let L be commutative a groupoid algebra then

(1) $x \odot \neg x = 0$ for all $x \in L$.

(2) $\neg 1 = 0$.

Proof

(1) $x \odot \neg x = x \odot \neg(x \odot 1) = x \odot \neg(x \odot \neg 0) = \neg(0 \odot \neg x) \odot 0 = 0$.

(2) $\neg 1 = 1 \odot \neg 1 = 0$

Define a relation $\leq$ on L: $x \leq y \Leftrightarrow x \odot \neg y = 0$.

Lemma 2.4 In any commutative groupoid algebra, the following conditions hold $\forall$ x, y, z $\in$ L:

(1) $x \odot \neg y = 0 = \neg x \odot y \Leftrightarrow x = y$.

(2) $x \odot \neg y = 0 = y \odot \neg z$ then $x \odot \neg z = 0$.

(3) $x \leq y \Leftrightarrow z \odot \neg y \leq z \odot \neg x$ and $x \odot \neg z \leq y \odot \neg z$.

(4) $\neg(\neg(x \odot \neg y) \odot \neg y) \odot \neg y = x \odot \neg y$.

(5) $((x \Theta \neg z) \Theta \neg(y \Theta \neg z)) \leq (x \Theta \neg y)$.

Proof

(1) $x \Theta \neg y = 0$ implies to $x \leq y$, also $\neg x \Theta y = 0$ implies to $y \Theta \neg x = 0$ and then $y \leq x$. therefore $x = y$. Conversely, if $x = y \Rightarrow x \Theta \neg y = x \Theta \neg x = 0$. Also $\neg x \Theta y = 0$.

(2) We have $x \Theta \neg z = (x \Theta 1) \Theta \neg z = (x \Theta \neg(x \Theta \neg y)) \Theta \neg z = \neg z \Theta (\neg(\neg x \Theta y) \Theta y) = \neg(\neg x \Theta y) \Theta (y \Theta \neg z) = \neg(\neg x \Theta y) \Theta 0 = 0$.

(3) Suppose $x \leq y$ then $x \Theta \neg y = 0$.

Now, consider $(z \Theta \neg y) \Theta \neg(z \Theta \neg x) = \neg(z \Theta \neg x) \Theta (\neg y \Theta z) = \neg y \Theta (z \Theta \neg(z \Theta \neg x)) = \neg y \Theta (\neg(\neg z \Theta x) \Theta x) = \neg(\neg z \Theta x) \Theta (\neg y \Theta x) = \neg(\neg z \Theta x) \Theta 0 = 0$.

Also, $(x \Theta \neg z) \Theta \neg(y \Theta \neg z) = (\neg z \Theta x) \Theta \neg(y \Theta \neg z) = x \Theta (\neg z \Theta \neg(y \Theta \neg z)) = x \Theta (\neg(z \Theta \neg y) \Theta \neg y) = \neg(z \Theta \neg y) \Theta (x \Theta \neg y) = \neg(z \Theta \neg y) \Theta 0 = 0$.

Conversely, suppose that $x \Theta \neg z \leq y \Theta \neg z$ and take $z = 0$ we get $x \leq y$.

(4) $\neg(\neg(x \Theta \neg y) \Theta \neg y) \Theta \neg y = \neg y \Theta \neg(\neg y \Theta \neg(x \Theta \neg y)) = (x \Theta \neg y) \Theta \neg(y \Theta (x \Theta \neg y)) = (x \Theta \neg y) \Theta \neg(x \Theta (y \Theta \neg y)) = (x \Theta \neg y) \Theta \neg 0 = x \Theta \neg y$.

(5) $[(x \Theta \neg z) \Theta \neg(y \Theta \neg z)] \Theta \neg(x \Theta \neg y) = \neg(x \Theta \neg y) \Theta [\neg(y \Theta \neg z) \Theta (x \Theta \neg z)] = \neg(x \Theta \neg y) \Theta [x \Theta (\neg(y \Theta \neg z) \Theta \neg z)] = \neg(x \Theta \neg y) \Theta [x \Theta (\neg(z \Theta \neg y) \Theta \neg y)] = \neg(x \Theta \neg y) \Theta [\neg(z \Theta \neg y) \Theta (x \Theta \neg y)] = \neg(z \Theta \neg y) \Theta [\neg(x \Theta \neg y) \Theta (x \Theta \neg y)] = (\neg z \Theta \neg y) \Theta 0 = 0$. Thus $(x \Theta \neg z) \Theta \neg(y \Theta \neg z) \leq (x \Theta \neg y)$.

Lemma 2.5 Let L be a commutative groupoid algebra. Then $\neg(\neg x) = x$. $\forall x \in L$.

Now we define two binary operations $\vee$ and $\wedge$ on a commutative groupoid algebra L by

$x \wedge y = x \Theta \neg(x \Theta \neg y) = y \Theta \neg(y \Theta \neg x)$

$x \vee y = \neg[\neg(x \Theta \neg y) \Theta \neg y] = \neg[\neg(y \Theta \neg x) \Theta \neg x]$.

Theorem 2.6 In any lattice commutative groupoid algebra L. the following hold $\forall x, y \in L$.

(1) $\neg(x \vee y) = \neg x \wedge \neg y$.

(2) $\neg(x \wedge y) = \neg x \vee \neg y$.

Proof:

(1) Since $\neg(x \vee y) \Theta \neg(\neg x \wedge \neg y) = \{\neg(x \Theta \neg y) \Theta \neg y\} \Theta \neg\{\neg x \Theta \neg(\neg x \Theta y)\} = \{\neg(y \Theta \neg x) \Theta \neg x\} \Theta \neg\{\neg(y \Theta \neg x) \Theta \neg x)\} = 0$. Thus $\neg(x \vee y) \leq \neg x \wedge \neg y$. Also, $(\neg x \wedge \neg y) \Theta \neg(\neg(x \vee y)) = 0$.

(2) From (1) we have $\neg(\neg x \vee \neg y) = \neg(\neg x) \wedge \neg(\neg y) = x \wedge y$. Thus $\neg(x \wedge y) = \neg x \vee \neg y$.

Theorem 2.7 In any commutative groupoid algebra L. the following hold $\forall x, y \in L$:

(1) $x \wedge y \leq x, y \leq x \vee y$.

(2) $x \vee y$ is the least upper bound of $\{x, y\}$.

(3) $x \wedge y$ is the greatest lower bound of $\{x, y\}$.

proof:

(1) Since $(x \wedge y)\ \Theta\ \neg x = (y\ \Theta\ \neg(y\ \Theta\ \neg x))\ \Theta\ \neg x = \neg x\ \Theta\ (\neg(y\ \Theta\ \neg x)\ \Theta\ y) = \neg(y\ \Theta\ \neg x)\ \Theta\ (y\ \Theta\ \neg x) = 0$. Also, $(x \wedge y)\ \Theta\ \neg y = (x\ \Theta\ \neg(x\ \Theta\ \neg y))\ \Theta\ \neg y = (x\ \Theta\ \neg y)\ \Theta\ \neg(x\ \Theta\ \neg y) = 0$. Also, $x\ \Theta\ \neg(x \vee y) = x\ \Theta\ [\neg(x\ \Theta\ \neg y)\ \Theta\ \neg y] = \neg(x\ \Theta\ \neg y)\ \Theta\ (x\ \Theta\ \neg y) = 0$. $y\ \Theta\ \neg(x \vee y) = y\ \Theta\ [\neg(y\ \Theta\ \neg x)\ \Theta\ \neg x)] = \neg(y\ \Theta\ \neg x)\ \Theta\ (y\ \Theta\ \neg x) = 0$.

(2) From 1, it can be observed that $x \vee y$ is an upper bound for $\{x, y\}$. Suppose that r be any upper bound for x, y. this implies that $x\ \Theta\ \neg r = 0 = y\ \Theta\ \neg r$.

Now we shall prove that $x \vee y \leq r$.

Since $(x \vee y)\ \Theta\ \neg r = \neg(\neg(x\ \Theta\ \neg y)\ \Theta\ \neg y)\ \Theta\ \neg r = \neg(\neg(x\ \Theta\ \neg y)\ \Theta\ \neg y)\ \Theta\ (\neg r\ \Theta 1) = \neg(\neg(x\ \Theta\ \neg y)\ \Theta\ \neg y)\ \Theta\ (\neg r\ \Theta\ \neg 0) = \neg(\neg(x\ \Theta\ \neg y)\ \Theta\ \neg y)\ \Theta\ (\neg r\ \Theta\ \neg(y\ \Theta\ \neg r)) = \neg(\neg(x\ \Theta\ \neg y)\ \Theta\ \neg y)\ \Theta\ (\neg y\ \Theta\ \neg(\neg y\ \Theta\ r)) = \neg(\neg(x\ \Theta\ \neg y)\ \Theta\ \neg y)\ \Theta\ (\neg(\neg y\ \Theta\ r)\ \Theta\ \neg y) = \neg(\neg y\ \Theta\ r)\ \Theta\ (\neg(\neg(x\ \Theta\ \neg y)\ \Theta\ \neg y)\ \Theta\ \neg y) = \neg(r\ \Theta\ \neg y)\ \Theta\ (x\ \Theta\ \neg y)$ (by lemma 2.4 (4)). $= \neg(r\ \Theta\ \neg y)\ \Theta\ (x\ \Theta\ \neg y) = (x\ \Theta\ \neg y)\ \Theta\ \neg(r\ \Theta\ \neg y). \leq x\ \Theta\ \neg r = 0$. (by lemma 2.4(5)). So $r \geq x \vee y$. Therefor $x \vee y =$ l. u. b $\{x, y\}$

(3) From (1) it can be observed that $x \wedge y$ is a lower bound for $\{x, y\}$. Suppose that r is any lower bound for $\{x, y\}$ then $r \leq x$ and $r \leq y$. this implies that $r\ \Theta\ \neg x = 0 = r\ \Theta\ \neg y$.

Since $r\ \Theta\ \neg(x \wedge y) = (r\ \Theta\ 1)\ \Theta\ \neg(x\ \Theta\ \neg(x\ \Theta\ \neg y)) = (r\ \Theta\ \neg 0)\ \Theta\ \neg(x\ \Theta\ \neg(x\ \Theta\ \neg y)) = (r\ \Theta\ \neg(r\ \Theta\ \neg x)\ \Theta\ \neg(x\ \Theta\ \neg(x\ \Theta\ \neg y) = (x\ \Theta\ \neg(\neg r\ \Theta\ x))\ \Theta\ \neg(x\ \Theta\ \neg(x\ \Theta\ \neg y)) \leq \neg(\neg r\ \Theta\ x)\ \Theta\ (x\ \Theta\ \neg y)$ (By lemma 2.4 (6)) $= (\neg y\ \Theta\ x)\ \Theta\ \neg(\neg r\ \Theta\ x) \leq \neg y\ \Theta\ r = 0$ (By lemma 2.4 (6)). Therefor $r \leq x \wedge y$ and hence $x \wedge y =$ g. l. b$\{x, y\}$

Remark 2.8

1) Let $(L, \Theta, \leq)$ be a commutative groupoid then L is a partially ordered set “Poset” from lemma 2.3(1) and lemma 2.4(1, 2). It is clear that L is a partially ordered set “Poset”. Since for $x \in L$ we have $x\ \Theta\ \neg 1 = 0 \Rightarrow x \leq 1$. Also, $0\ \Theta\ \neg x = 0 \Rightarrow 0 = x$. L is a **bounded poset**.

2) From lemma 2.7 we have that every two elements in a commutative groupoid algebra has supremum and infumum. Hence $(L, \leq, \vee, \wedge, 0, 1)$ is a bounded lattice with bounds 0 and 1.

Now we have the following corollaries 2.9 and 2.10 as consequence of lemma 2.7.

Corollary 2.9 In any commutative groupoid algebra L the following hold $\forall\ x, y \in L$:

1) $x \leq y,\ x \leq z \Rightarrow x \leq y \wedge z$.

2) $y \leq x,\ z \leq x \Rightarrow y \vee z \leq x$.

proof:

1) Let $x \leq y$, $x \leq z$ then by lemma 2.4(3) $z \,\Theta\, \neg y \leq z \,\Theta\, \neg x$, and $x \,\Theta\, \neg z \leq y \,\Theta\, \neg z$, and $x \,\Theta\, \neg z = 0$. Since $x \,\Theta\, \neg(y \wedge z) = x \,\Theta\, \neg[z \,\Theta\, \neg(z \,\Theta\, \neg y)] \leq x \,\Theta\, \neg[z \,\Theta\, \neg(z \,\Theta\, \neg x)] = x \,\Theta\, \neg[x \,\Theta\, \neg(x \,\Theta\, \neg z)] = x \,\Theta\, \neg(x \,\Theta\, \neg 0) = x \,\Theta\, \neg x = 0$. Therefor $x \leq y \wedge z$.

2) Let $y \leq x$, $z \leq x$ then by lemma (2.4) we have $y \,\Theta\, \neg z \leq x \,\Theta\, \neg z$, and $z \,\Theta\, \neg x = 0$. $(y \vee z) \,\Theta\, \neg x = \neg[\neg(y \,\Theta\, \neg z) \,\Theta\, \neg z] \,\Theta\, \neg x \leq \neg[\neg(x \,\Theta\, \neg z) \,\Theta\, \neg z] \,\Theta\, \neg x = \neg[\neg x \,\Theta\, \neg(z \,\Theta\, \neg x)] \,\Theta\, \neg x = \neg(\neg x \,\Theta\, \neg 0) \,\Theta\, \neg x = \neg(\neg x) \,\Theta\, \neg x = 0$. Therefore $y \vee z \leq x$.

Corollary 2.10 In any lattice commutative groupoid algebra L the following hold $\forall\ x, y, z \in L$:

(1) $(x \vee y) \,\Theta\, \neg z \leq x \,\Theta\, \neg z$ and $(x \vee y) \,\Theta\, \neg z \leq y \,\Theta\, \neg z$.

(2) $x \,\Theta\, \neg z \leq (x \wedge y) \,\Theta\, \neg z$ and $\neg y \,\Theta\, z \leq (x \wedge y) \,\Theta\, \neg z$.

Proof: Clear by using 2.9 and lemma (2.4)(3).

Theorem 2.11 In any commutative groupoid algebra L the following hold $\forall\ x, y, z \in L$:

(1) $\neg[(x \vee y) \,\Theta\, \neg z] = \neg(x \,\Theta\, \neg z) \wedge \neg(y \,\Theta\, \neg z)$

(2) $\neg[(x \wedge y) \,\Theta\, z] = \neg(x \,\Theta\, \neg z) \vee \neg(y \,\Theta\, \neg z)$

Proof:

(1) By corollaries 2.9 and 2.10 we get

$\neg [(x \vee y) \,\Theta\, \neg z] \leq \neg (x \,\Theta\, \neg z) \wedge \neg (y \,\Theta\, \neg z)$

Now consider

$\{\neg (x \,\Theta\, \neg z) \wedge \neg(y \,\Theta\, \neg z)\} \,\Theta\, ((x \vee y) \,\Theta\, \neg z)$

$= \{\neg(x \,\Theta\, \neg z) \,\Theta\, \neg [\neg(x \,\Theta\, \neg z) \,\Theta\, (y \,\Theta\, \neg z)]\} \,\Theta\, \{\neg(\neg(x \,\Theta\, \neg y) \,\Theta\, \neg y) \,\Theta\, \neg z\}$

$= [\neg(\neg(x \,\Theta\, \neg y) \,\Theta\, \neg y)] \,\Theta\, \{\{\neg(x \,\Theta\, \neg z) \,\Theta\, \neg[\neg(x \,\Theta\, \neg z) \,\Theta\, (y \,\Theta\, \neg z)]\} \,\Theta\, \neg z\}$

$= \neg(\neg(x \,\Theta\, \neg y) \,\Theta\, \neg y) \,\Theta\, \{\{\neg(x \,\Theta\, \neg z) \,\Theta\, \neg z\} \,\Theta\, \neg[\neg(x \,\Theta\, \neg z) \,\Theta\, (y \,\Theta\, \neg z)]\}$

$= \neg(\neg(x \,\Theta\, \neg y) \,\Theta\, \neg y) \,\Theta\, \{\{\neg(x \,\Theta\, \neg z) \,\Theta\, \neg z\} \,\Theta\, \neg[y \,\Theta\, \{\neg(x \,\Theta\, \neg z) \,\Theta\, \neg z\}]\}$

$= \neg(\neg(x \,\Theta\, \neg y) \,\Theta\, \neg y) \,\Theta \{\neg[\neg(\neg(x \,\Theta\, \neg z) \,\Theta\, \neg z) \,\Theta\, \neg y] \,\Theta\, \neg y\}$

$= \neg[\neg(\neg(x \,\Theta\, \neg z) \,\Theta\, \neg z) \,\Theta\, \neg y] \,\Theta\, [\neg(\neg(x \,\Theta\, \neg y) \,\Theta\, \neg y) \,\Theta\, \neg y]$

By lemma (2.4) (4)

$= \neg[\neg(\neg(x \,\Theta\, \neg z) \,\Theta\, \neg z) \,\Theta\, \neg y] \,\Theta (x \,\Theta\, \neg y)$ (By lemma 2.4 (4))

$\geq (\neg(x \,\Theta\, \neg z) \,\Theta\, \neg z) \,\Theta\, x$. (By lemma 2.4 (5))

$= x \Theta (\neg(x \Theta \neg z) \Theta \neg z) = \neg(x \Theta \neg z) \Theta (x \Theta \neg z) = 0$

(2) Similar to the proof (1).

From remark 2.8 and theorem 2.11 we have the following

Theorem 2.12 let (L, ϴ, ¬, 0, 1) be a commutative groupoid algebra, then (L, ∨, ∧, 0, 1) is a lattice commutative groupoid algebra.

Remark 2.13 Let (L, ϴ, ¬, 0, 1) be a commutative groupoid then ϴ can never be associative as the following example.

Example 2.14

Let L = {0, a, b, c, d, e, f, 1} be a chain defined 0 < a < b < c < d < e < f < 1. Define ¬ and ϴ as follows

x	**0**	**a**	**b**	**c**	**d**	**e**	**f**	**1**
¬x	1	f	e	d	c	b	a	0

ϴ	**0**	**a**	**b**	**c**	**d**	**e**	**f**	**1**
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	a	b
c	0	0	0	0	0	a	b	c
d	0	0	0	0	a	c	c	d
e	0	0	0	a	c	d	d	e
f	0	0	a	b	c	d	e	f
1	0	a	b	c	d	e	f	1

Clearly ϴ is not associative since (d ϴ e) ϴ f = c ϴ f = b ≠ d ϴ (e ϴ f) = d ϴ d = a. Thus (d ϴ e) ϴ f ≠ d ϴ (e ϴ f).

3. Autometrization on commutative groupoid algebra

In this section we introduce two binary operations on a commutative groupoid algebra namely + and – and we obtain a few results concerning their operations defined. Also, we obtain some geometric properties of commutative groupoid algebra. Also we prove any commutative groupoid algebra is a metric space. Futher we prove that every commutative groupoid algebra can be made into regular autometrized algebra.

We begin with the following

Let L be commutative groupoid algebra. Define + and – on L as follows.

$x + y = \neg(\neg x \Theta \neg y)$, $x - y = x \Theta \neg y$. $\forall\, x, y \in L$

Then we obtain the following

Lemma 3.1 Let L be commutative groupoid algebra, then (L, +, 0) is a commutative monoid.

Proof: It is sufficient to prove that + is associative and 0 is the identity element with respect to +. Let $x, y, z \in L$. Then $(x + y) + z = \neg(\neg x \Theta \neg y) + z = \neg[(\neg x \Theta \neg y) \Theta \neg z] = \neg[\neg z \Theta (\neg x \Theta \neg y)] = \neg[\neg x \Theta (\neg z \Theta \neg y))] = \neg[\neg x \Theta \neg(y + z)] = x + (y + z)$.

Also, $x + 0 = \neg(\neg x \Theta \neg 0) = \neg(\neg x \Theta 1) = \neg(\neg x) = x$.

Lemma 3.2 For a, b, c, x and y in a commutative groupoid algebra L, the following conditions hold:

(1) $a - a = 0$

(2) $a - 0 = a$.

(3) $(a - b) \vee 0 = a - b$

(4) $a - b = 0 \Leftrightarrow a \leq b$.

(5) $\neg[(a \wedge b) - c] = \neg(a - c) \vee \neg(b - c)$

(6) $a \vee b = (a - b) + b$

(7) $a \wedge b = b - (b - a)$

(8) $x \leq a + b \Leftrightarrow x - a \leq b$.

(9) $0 - a = 0$.

(10) $a - (b + c) = (a - b) - c$.

(11) $(a - b) + (b - c) \geq a - c$.

(12) $a = (a \vee 0) + (a \wedge 0)$

(13) $a - (b + c) = (a - c) - b$.

(14) $a \geq b \Rightarrow (a - b) + b = a$.

(15) $[a - (x \wedge y)] + b = [(a - x) + b] \vee [(a - y) + b]$

Now, we are in a position to introduce the concept of a metric on a commutative groupoid algebra.

Definition 3.3 let L be a commutative groupoid algebra. Define a map $*: L \times L \to L$ by $a * b = (a - b) + (b - a)$

Lemma 3.4 In a commutative groupoid algebra L, we have

$\neg(a \odot \neg b) \odot \neg(b \odot \neg a) = 1 \Rightarrow a = b$.

Proof: $a \odot \neg b = (a \odot \neg b) \odot 1 = (a \odot \neg b) \odot [\neg(b \odot \neg a) \odot \neg(a \odot \neg b)] = \neg(b \odot \neg a) \odot [(a \odot \neg b) \odot \neg(a \odot \neg b)] = \neg(b \odot \neg a) \odot 0 = 0$.

Now, $b \odot \neg a = (b \odot \neg a) \odot 1 = (b \odot \neg a) \odot [\neg(a \odot \neg b) \odot \neg(b \odot \neg a)] = \neg(a \odot \neg b) \odot [(b \odot \neg a) \odot \neg(b \odot \neg a)] = \neg(a \odot \neg b) \odot 0 = 0$. Therefor $a = b$.

Theorem 3.5 let L be a commutative groupoid algebra. then for all a, b, c in L, we have

$$a \leq b \Rightarrow a + c \leq b + c$$

Proof: Let $a \leq b$, then $a \odot \neg b = 0$,

$(a + c) \odot \neg(b + c) = \neg(\neg a \odot \neg c) \odot (\neg b \odot \neg c) = \neg b \odot [\neg(\neg a \odot \neg c) \odot \neg c] = \neg b \odot [\neg c \odot \neg(\neg a \odot \neg c)] = \neg b \odot [a \odot \neg(a \odot c)] = \neg(a \odot c) \odot (a \odot \neg b) = \neg(a \odot c) \odot 0 = 0$.

Corollary 3.6 In any commutative groupoid algebra L, we have

$$a \leq b \text{ and } c \leq d \Rightarrow a + c \leq b + d \text{ for all } a, b, c, d \in L$$

Theorem 3.7 $(L, *)$ is a metric space where $a * b = (a - b) + (b - a)$

Proof: Let $a, b, c \in L$

$a * b = (a - b) + (b - a) \geq a - a = 0$.(by lemma 3.2 (11). Also, $a * b = b * a$. Let $a * b = 0$. Thus $(a - b) + (b - a) = 0$ this implies that $\neg(a \odot \neg b) \odot \neg(b \odot \neg a) = 1 \Rightarrow$ (by lemma 3.4) $a = b$. Conversely, if $a = b$, then $a * a = 0$, $a - a = 0$. Finally, $(a * b) + (b * c) = \{(a - b) + (b - a)\} + \{(b - c) + (c - b)\} = \{(a - b) + (b - c)\} + \{(c - b) + (b - a)\} \geq (a - c) + (c - a) = a * c$. Thus * is a metric on L.

Definition 3.8 [3] A system $A = (A, +, \leq, *)$ is called an "Autometrized Algebra" if and only if

1.1 $(A, +)$ is a binary commutative algebra with a distinguished element Zero: "0"

1.2 $\leq$ is an anti-symmetric, reflexive ordering on A and

1.3 $*: A \times A$ is a mapping satisfying the formal properties of a distance function namely:

1) $a * b \geq 0$ with equality $\Leftrightarrow a = b$

2) $a * b = b * a$ and

3) $a * c \leq a * b + b * c$

Definition 3.9 [3] An Autometrized Algebra L is said to be *regular* if $a * 0 = a$, $\forall a \in L$.

Theorem 3.10 Let L be a commutative groupoid algebra, then $a * 0 = a$, $\forall a \in L$.

Proof: $a * 0 = (a - 0) + (0 - a)$ [(by lemma 3.2(2) $a - 0 = a$, $0 – a = 0 \Theta \neg a = 0$].

$= a + 0 = \neg(\neg a \Theta \neg 0) = \neg(\neg a \Theta 1) = \neg(\neg a) = a$

By theorem 3.7 and theorem 3.8 we get the following

Theorem 3.10 Any commutative groupoid algebra L is a regular autometrized algebra.

We end this section by looking at the following example:

Example 3.10 Let $L = \{0, a, b, c, d, e, f, 1\}$ be a chain defined $0 < a < b < c < d < e < f < 1$. In example (2. 14) we defined $\neg$, Θ and in this example we define $*$ as follows

*****	**0**	**a**	**b**	**c**	**d**	**e**	**f**	**1**
0	0	a	b	c	d	e	f	1
a	a	0	a	b	c	b	e	f
b	b	a	0	a	c	d	d	e
c	c	b	a	0	a	c	c	d
d	1	c	b	a	0	0	b	c
e	e	d	d	c	0	0	a	b
f	f	e	d	c	b	a	0	a
1	1	f	e	d	c	b	a	0

Clearly 0 is the additive element, since $x * 0 = 0 * x = x$ for all x. Also every element which it is the inverse of itself since $x * x = 0$ for all x. Further it is observed that * is not associative. Since instance $(a * b) * c = a * c = b \neq a * (b * c) = a * a = 0$. Therefor * is not a group operation.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Jun,Y.B., Xu, Yang & Keyun Qin. Positive implication and Associative filters of lattice implication algebras. Bull. Korean Math. Soc., 35(1998), 53-61.

[2] Liu, J & Xu, Y. Filters and structure of lattice implication algebra. Chinese Science Bull., 42(1997), 1517-1520.

[3] Swamy, K. L. N. A General theory of Autometrized Algebras. Math Annallen, 157(1964), 65-74.

[4] Venkateswarlu Kolluru& and Berhanu Bekele. Implicative algebra. Mekelle University, 4 (2012):90-101.

[5] Xu, Pan. Y. Lattice implication ordered semigroups. Information Sci., 178(2008), 403-413.

[6] Xu, Y. Lattice implication algebras. J South West Jiaotong University, 28(1993), 20- 27.

[7] Zhu Yiquan & Tu Wenbiao. A Note on lattice implication algebras. Bull. Korean Math.Soc., 38 (2001), 191-195

[8] Zhu, H., Zhao, J.B & Xu, Y. The primary ideal of residuated lattice implication algebra. Journal of Zhengzhou University, 38(2006), 25-28.