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OPTIMAL PORTFOLIO AND CONSUMPTION WITH STOCHASTIC SALARY AND INFLATION HEDGING STRATEGY FOR DEFINED CONTRIBUTORY PENSION SCHEME

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Abstract. This paper consider the optimal portfolio and consumption with stochastic salary and inflation protection strategy for a defined contributory pension scheme. It was assume that a Pension Plan Member (PPM) made a stochastic cash inflows, which are invested into a risk-free asset (cash account), stocks and inflation-linked bonds. Due to high risk of inflation and diminishing value of pension benefits, Pension Fund Administrators (PFAs) starts investing the contributions of the PPM in inflation-linked bonds. The paper presents the value of the PPM's wealth at time t. A solution technique was constructed to provide analytical solution to our resulting Hamilton-Jacobi-Bellman (HJB) equation. The optimal consumption and variational form of Merton portfolio demand for stocks, inflation-linked bonds (indexed bonds) and cash account were obtained. The optimal portfolio values for inflation-linked bonds includes an inter-temporal hedging term that offset any shock to the stochastic contribution of the PPM. It was found that the terminal consumption depend on the initial wealth, the present value of future contributions, coefficients of relative risk aversion and the discount (preference) rate which captures the PPM's preference over time.

Keywords: optimal portfolio, consumption, stochastic salary, defined contributory, pension plan member, pension fund administrator, inflation hedging.

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1. Introduction

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The inter-temporal consumption optimization problem with a stochastic income stream is considered. It is assumed that the PPM with initial capital is at work before retirement age, T and consumption is continuous over time. This work is inspired by works of Samuelson [19] and Merton ([15], [16], [17]) which show that time variation in investment strategy imply optimal portfolio strategies for multi-period investors which may be different from those of single-period investors. Multi-period investors give value to assets that are have short-term risk-return and have the ability to hedge consumption against adverse shifts in future investment opportunities. Furthermore, they found that investors preferred risky assets that reflect inter-temporal hedging process. Inter-temporal hedging terms as concept is detailed in the works of [12], [1], [2], [5], [6], [12] and [11]. [8] considered optimal consumption and investment problem of investors. [14], [13] and [9], provide an approximate solution and analytical results to the inter-temporal consumption problem related to Labor-Income stream. [9], assumed that the asset return is non-stochastic. [23] considered a tractable model of precautionary savings in continuous time and assumed that the uncertainty is about the timing of the income loss in addition to the assumption of non-stochastic asset return. [4] considered labor supply flexibility and portfolio choice of individual life cycle. They determined the objective of maximizing the expected discounted lifetime utility and assumed that the utility function has two argument (consumption and labor/leisure). [3] used the quadratic utility function that has the characterization of linear marginal utility. This utility function is not attractive in describing the behavior of individual towards risk as it implies increasing absolute risk aversion. [18], [17], [19], [24] investigated the continuous-time consumption model with stochastic asset returns and stochastic labor income using Martingale approach. But, [19], [20], and [21] considered continuous case adopting dynamics programming approach. They assumed that the utility function is a linear combination of two Constant Relative Risk Aversion (CRRA) utility functions with respect to consumption and labor supply. They derived analytically a closed-form solution for the consumption, labor-supply and portfolio.

[4] concluded that labor income induces the individual to invest an additional amount of wealth to the risky asset. They shown that labor income and investment choices are related, while they failed to analyzed the optimal consumption process. [24] analyzed the

optimal consumption process and treated consumption and leisure as a 'composite' good. They assumed that people work for their whole lifetime which is unrealistic. In this paper, we assume that people work up to a certain age, T and continue to enjoy the the labor income and investment returns throughout their lifetime. The above authors considered investment of the individual labor income stream into a risk-free and a risky assets. In this paper, the investment of individual labor income stream into a risk-free asset and risky assets is considered. The aim is to find the optimal values of wealth, investment portfolios and consumption of the PPM up to terminal time.

The remainder of this paper is organized as follows. In section 2, financial market models are presented. In section 3, modeling of wealth and contribution dynamics of a PPM are presented. In section 4, the stochastic salary and contribution process of a PPM was presented. The wealth, portfolio and consumption processes of a PPM were presented in section 5. In section 6, the construction of the present value of a PPMs contributions was presented. In section 7, expected utility of wealth, optimal portfolio and consumption processes for a PPM were presented. Finally, section 8 concludes the presentation in the paper.

2. The Financial Market Models

The Brownian motions $W^S(t) = (W_1^S(t), \ldots, W_n^S(t))$, $t \in [0,T]$ and $W^Q(t) = (W_1^Q(t), \ldots, W_m^Q(t))$, $t \in [0,T]$ are n-dimensional and m-dimensional processes respectively, defined on a given probability space $(\Omega, \mathbb{F}, \{\mathbb{F}_t^S\}_{t\geq 0}, \{\mathbb{F}_t^Q\}_{t\geq 0}, \mathbb{P})$, $t \in [0,T]$, where $\mathbb{P}$ is the real world probability measure, t is the time period, T the terminal time and $\sigma_{i,j}^S$ and $\sigma_{k,j}^Q$ are the volatility of stock i and volatility of the indexed bond k with respect to changes in $W_j^S(t)$ and $W_j^Q(t)$ respectively. $\mu := (\mu_1, \ldots, \mu_n)'$ and $\alpha := (\alpha_1, \ldots, \alpha_m)'$ are the appreciation rate vectors for stocks and indexed bonds respectively, $g_Q > 0$ is the expected rate of inflation and r is the short term interest rate. Moreover, $\Sigma^S = diag(\sigma_{i,j}^S)_{i,j}^n$ and $\Sigma^Q = diag(\sigma_{k,j}^Q)_{k,j}^m$ are the volatility matrices for the stocks and indexed bonds respectively referred to as the coefficients of the markets and are progressively measurable with respect to the filtration $\mathbb{F}$ such that $\mathbb{F}_t^S \in \mathbb{F}^S$ and $\mathbb{F}_t^Q \in \mathbb{F}^Q$ such that $\mathbb{F}^S \cup \mathbb{F}^Q \in \mathbb{F}$ and $\mathbb{F}^S \cap \mathbb{F}^Q = \phi$.

We assume that the financial market M^{n+m+1} is arbitrage-free, complete and continuously open between time 0 and T, i.e, the processes θ^S and θ^Q satisfies respectively

$$\theta := \begin{pmatrix} \theta^Q \\ \theta^S \end{pmatrix} = \begin{pmatrix} (\Sigma^Q)^{-1}(\alpha + \pi - r1_m) \\ (\Sigma^S)^{-1}(\mu - r1_n) \end{pmatrix}. \tag{1}$$

where, 1_n is $n \times n$ identity matrix, 1_m is $m \times m$ identity matrix, θ^Q is the price of inflation risk at time t and θ^S is the market price of risk of stocks at time t. In this paper, we assume that the PFA faces a market that is characterized by a risk-free asset (cash account) and two risky assets, all of whom are trade-able. Therefore, the dynamics of the underlying assets are given in (2) to (4)

$$\begin{cases} dB(t) = rB(t)dt, \\ B(0) = 1; \end{cases} \tag{2}$$

$$\begin{cases} dS_i(t) = S_i(t)\left(\mu_i dt + \sum_{j=1}^n \sigma_{i,j}^S dW_j^S(t)\right), \\ S_i(0) = s_i > 0, i = 1, 2, \ldots, n; \end{cases} \tag{3}$$

$$\begin{cases} dI_k(t, Q(t)) = \left(r1_m + \Sigma^Q \theta^Q) I_k(t, Q(t)\right) dt + \sum_{j=1}^m \sigma_{k,j}^Q I_k(t, Q(t)) dW^Q(t), \\ I_k(0) > 0, k = 1, 2, \ldots, m, \end{cases} \tag{4}$$

where,

- $W^S(t)$ and $W^Q(t)$ are independent,
- $B(t)$ is the price process of the cash account at time t,
- $S(t)$ is stock price process at time t.
- $Q(t)$ is the inflation index at time t and has the dynamics:

$$dQ(t) = g_Q Q(t)dt + \sigma_Q Q(t) dW^Q(t),$$

where π is the expected rate of inflation, which is the difference between nominal interest rate, r and real interest rate R (i.e. $g_Q = r - R$).

- $I(t, Q(t))$ is the inflation-linked bond price process at time t.
- $1_m = (\underbrace{1, 1, \ldots, 1}_{\text{m times}})'$ and $1_n = (\underbrace{1, 1, \ldots, 1}_{\text{n times}})'$.
- $\|\theta^S\| = \sqrt{\sum_{i=1}^n (\theta_i^S)^2}$.
- $\|\theta^Q\| = \sqrt{\sum_{k=1}^m (\theta_k^Q)^2}$.

The exponential process

$$Z(t) := \exp\left(-\theta' \cdot W(t) - \frac{1}{2}\|\theta\|^2 t\right), 0 \le t \le T, \tag{5}$$

where, $W(t) = (W^Q(t), W^S(t))'$ is assumed to be a martingale.

Now the state-price density function is defined by

$$\Lambda(t) = \frac{Z(t)}{B(t)} = \exp\left(-(r + \frac{1}{2}\|\theta\|^2)t - \theta' \cdot W(t)\right), 0 \leq t \leq T. \tag{6}$$

Using *Itô Lemma* on (3) and (4), the following solutions

$$B(t) = \exp(rt); \tag{7}$$

$$I_k(t, Q(t)) = I_k(0) \exp\left(\left(r1_m + \Sigma^Q \theta^Q - \frac{1}{2}\sum_{j=1}^{m}(\sigma_{k,j}^Q)^2\right) t + \sum_{j=1}^{m} \sigma_{k,j}^Q W_j^Q(t)\right); \tag{8}$$

$$S_i(t) = s_i \exp\left(\left(\mu_i - \frac{1}{2}\sum_{j=1}^{n}(\sigma_{i,j}^S)^2\right) t + \sum_{j=1}^{n} \sigma_{i,j}^S W_j^S(t)\right); \tag{9}$$

are obtained.

3. Modeling of Wealth and Contribution Dynamics of a PPM

In this section, the wealth dynamics of a PPM and contribution process of the member in pension funds is modeled.

Definition 1. *Let $\Delta = \left(\Delta_k^Q(t), \Delta_i^S(t)\right)$ consist of portfolios $\Delta_k^Q(t)$ which is the proportion of PPM's wealth invested in the inflation-linked bond k at time t, $\Delta_i^S(t)$ the proportion of PPM's wealth invested in stock i at time t. Again, let $C(t)$ be the consumption process, then that the pair (Δ, C) is self-financing if the corresponding wealth process $X^{\Delta,C}(t), t \in [0,T]$, satisfies*

$$\begin{aligned} dX^{\Delta,C}(t) &= \sum_{i=1}^{n} \Delta_i^S(t) X^{\Delta,C}(t) \frac{dS_i(t)}{S_i(t)} + \sum_{k=1}^{m} \Delta_k^Q(t) X^{\Delta,C}(t) \frac{dI_k(t,Q(t))}{I_k(t,Q(t))} \\ &+ \left(1 - \sum_{i=1}^{n} \Delta_i^S(t) - \sum_{k=1}^{m} \Delta_k^Q(t)\right) X^{\Delta,C}(t) \frac{dB(t)}{B(t)} - C(t)dt, \end{aligned} \tag{10}$$

Therefore, the overall sum of the proportion of PPM's wealth invested in stocks is given as $\sum_{i=1}^{n} \Delta_i^S(t)$, the overall sum of the proportion of PPM's wealth invested in inflation-linked bonds is given as $\sum_{k=1}^{m} \Delta_k^Q(t)$ and the overall sum of the proportion of PPM's wealth invested in cash account is given as $1 - \sum_{i=1}^{n} \Delta_i^S(t) - \sum_{k=1}^{m} \Delta_k^Q(t)$. The requirement of being self-financing implies that the change in wealth must equal the difference between the capital gains (gross gains) and infinitesimal consumption.

4. The Stochastic Salary and Contribution Process of a PPM

In this subsection, the process of deriving the PPMs contribution from the salary is considered. It is that the 'effective salary' of the PPM is driven by a geometric Brownian motion. Further assumptions are that the expected growth rate of a PPMs salary and the volatility of the salary driven by the source of uncertainty of inflation are deterministic. Then, a wealth process of the PPM with an initial value whose dynamics is a combination of the gross wealth and contributions of the PPM at time t is derived. The present value of the expected future contributions process by following Zhang, 2007 approach is also derived. The wealth dynamics of the PPM at time t is obtained as well. The dynamics of the PPM effective salary is given by

$$\begin{cases} dD(t) = D(t)\left(\beta dt + \sigma_D' \cdot dW^Q(t)\right), \\ D(0) = d > 0, \end{cases} \tag{11}$$

where $D(t)$ is the salary of the PPM at time t, β is the expected growth rate of salary of PPM and $\sigma_D = \sigma_D e$ is the volatility of salary which is driven by the source of uncertainty of inflation, $W^Q(t)$, where $e = (\underbrace{1, 0, \ldots, 0}_{\text{m times}})'$. It is assumed in this paper that β and σ_D are constants. It is further assumed that from now on $\Sigma^S(t)$, $\Sigma^Q(t)$, $\mu(t)$, $\alpha(t)$, $g_Q(t)$ and $r(t)$ are constants in time. Using *Itôs Lemma* on (11), we have (12) to be the solution to the stochastic differential equation (SDE)(11):

$$D(t) = d\exp\left(\left(\beta - \frac{1}{2}\|\sigma_D\|^2\right)t + \sigma_D' \cdot W^Q(t)\right). \tag{12}$$

5. The Wealth, Portfolio and Consumption Processes of a PPM

In this subsection, the dynamics of a PPM who with flows of contributions into the pension funds is presented.

If the PPM contributes continuously to his defined contribution pension fund with a fixed contribution rate of $c > 0$ then, the PPMs corresponding wealth process with an initial value of $x, (0 < x < \infty)$, which we denoted by $X^{\Delta,D,C}(t)$, is governed by the following

SDE:

$$(13)\quad \begin{cases} dX^{\Delta,D,C}(t) = \sum_{i=1}^{n} \Delta_i^S(t) X^{\Delta,D,C}(t) \frac{dS_i(t)}{S_i(t)} + \sum_{k=1}^{m} \Delta_k^Q(t) X^{\Delta,D,C}(t) \frac{dI_k(t,Q(t))}{I_k(t,Q(t))} + \\ \left(1 - \sum_{i=1}^{n} \Delta_i^S(t) - \sum_{k=1}^{m} \Delta_k^Q(t)\right) X^{\Delta,D,C}(t) \frac{dB(t)}{B(t)} + (cD(t) - C(t))\, dt \\ X^{\Delta,D,C}(0) = x, \end{cases}$$

where $cD(t)$ is the amount of money contributed into the pension fund at time t. It is assumed that the contributions are invested continuously over time. The contribution at time t, $cD(t)$, can be viewed as the rate of a random endowment and is strictly positive. Substituting the assets dynamics defined by (2) to (4), we obtain the following

$$(14)\quad \begin{aligned} dX^{\Delta,D,C}(t) &= \sum_{i=1}^{n} \Delta_i^S(t) X^{\Delta,D,C}(t) \left(\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{i,j}^S(t) dW_j^S(t)\right) \\ &+ \sum_{k=1}^{m} \Delta_k^Q(t) X^{\Delta,D,C}(t) \left((r(t)1_m + \Sigma^Q(t)\theta^Q(t)\right) dt + \sum_{j=1}^{m} X^{\Delta,D,C}(t) \sigma_{k,j}^Q dW_j^Q(t) \\ &+ \left(1 - \sum_{i=1}^{n} \Delta_i^S(t) - \sum_{k=1}^{m} \Delta_k^Q(t)\right) X^{\Delta,D,C}(t) r(t) dt + (cD(t) - C(t))\, dt, \end{aligned}$$

$$(15)\quad \begin{aligned} dX^{\Delta,D,C}(t) &= \{r(t) X^{\Delta,D,C}(t) + \sum_{i=1}^{n} \Delta_i^S(t) X^{\Delta,D,C}(t) \left(\mu_i(t) - r(t)\right) \\ &+ \sum_{k=1}^{m} \Delta_k^Q(t) X^{\Delta,D,C}(t) \left((r(t)1_m + \Sigma^Q(t)\theta^Q(t) - r(t)\right) + cD(t) - C(t)\} dt \\ &+ \sum_{j=1}^{m} \Delta_k^Q(t) X^{\Delta,D,C}(t) \sigma_{k,j}^Q dW_j^Q(t) + \sum_{j=1}^{n} \Delta_i^S(t) X^{\Delta,D,C}(t) \sigma_{i,j}^S(t) dW_j^S(t), \end{aligned}$$

$$(16)\quad \begin{cases} dX^{\Delta,D,C}(t) = X^{\Delta,D,C}(t)([r(t) + \Delta_S(t)\left(\mu(t) - r(t)1_n\right) \\ +\Delta_Q(t) \cdot (\Sigma^Q(t)\theta^Q(t))' + cD(t) - C(t)]dt \\ + \left[\Sigma^{S_1}(t)\Delta_S'(t) + \Sigma^Q \Delta_Q'(t)\right]' \cdot dW^Q(t) + (\Sigma^{S_2}(t)\Delta_S'(t))' \cdot dW^S(t)), \\ X^{\Delta,D,C}(0) = x, \end{cases}$$

where, $\Delta_S(t) = (\Delta_1^S(t), \Delta_2^S(t), \ldots, \Delta_n^S(t))$ and $\Delta_Q(t) = (\Delta_1^Q(t), \Delta_2^Q(t), \ldots, \Delta_m^Q(t))$.

Definition 2. *A portfolio process Δ is said to be admissible if the corresponding wealth process $X^{\Delta,D,C}(t)$ satisfies*

$$(17)\quad \mathbb{P}\left(X^{\Delta,D,C}(t) + E_t\left[\int_t^T \frac{\Lambda(u)}{\Lambda(t)} cD(u)du\right] \geq 0\right) = 1, \forall t \in [0,T].$$

where, $E_t(\cdot|\mathbb{F}_t)$. The class of admissible portfolio processes is denoted by Π_D.

6. Construction of the Present Value of a PPM's Contributions

Definition 3. *The value of the PPM's expected future contribution process is defined as*

$$\Psi(t) = E_t\left[\int_t^T \frac{\Lambda(u)}{\Lambda(t)} cD(u)du\right], \tag{18}$$

where E_t is the conditional expectation with respect to the Brownian filtration $\{\mathbb{F}(t)\}_{t\geq 0}$ and

$$\Lambda(t) \equiv \exp(-rt)Z(t) \tag{19}$$

is the stochastic discount factor which adjusts for nominal interest rate and market price of risks for stocks and indexed bond.

By observing the Markovian structure of the expression on the right-hand side of (19), it should be noted that it is possible to express $\Psi(t)$ in terms of the instantaneous contribution $cD(t)$. The following Theorem establishes this fact which follows Zhang (2007).

Theorem 1. *Let $\Psi(t)$ be the value of the expected future contribution process at time t, then*

$$\Psi(t) = \frac{1}{\tilde{\alpha}}\left(\exp((\tilde{\alpha}(T-t)) - 1\right) cD(t), \forall t \in [0, T] \tag{20}$$

with $\tilde{\alpha} \equiv \beta - r - \sigma_D \cdot \theta$ and

$$\Psi(0) = \frac{1}{\tilde{\alpha}}\left(\exp(\tilde{\alpha}T) - 1\right) cy. \tag{21}$$

Proof: By definition 3, we have that

$$\begin{aligned} \Psi(t) &= E_t\left[\int_t^T \frac{\Lambda(u)}{\Lambda(t)} cD(u)du\right] &(22)\\ &= cD(t)E_t\left[\int_t^T \frac{\Lambda(u)D(u)}{\Lambda(t)D(t)} du\right] &(23)\end{aligned}$$

The process $\Lambda(.)$ and $D(.)$ are geometric Brownian motions and therefore follows that $\dfrac{\Lambda(u)D(u)}{\Lambda(t)D(t)}$ is independent of $\{\mathbb{F}(t), u \geq t\}$. Consequently, the conditional expectation collapses to an unconditional expectation and we obtain

$$\Psi(t) = cD(t)f(t, T), \tag{24}$$

with the deterministic function $f(t,T)$ defined by

$$f(t,T) \equiv E\left[\int_0^{T-t} \Lambda(\tau)\frac{D(\tau)}{D(0)}\right] d\tau \tag{25}$$

But,

$$\begin{aligned}\Lambda(\tau)\frac{D(\tau)}{D(0)} &= \exp\left[(-r-\frac{1}{2}\|\theta\|^2)\tau - \theta'\cdot W(\tau)\right]\exp\left[(\beta-\frac{1}{2}\|\sigma_D\|^2)\tau + \sigma_D'\cdot W^Q(\tau)\right] \\ &= \exp\left(\beta-r\right)\tau\exp\left(\sigma_D-\theta^Q\right)'\cdot W^Q(\tau)\exp\left(-\frac{1}{2}\|\theta\|^2\tau - \frac{1}{2}\|\sigma_D\|\tau - \theta'\cdot W(\tau)\right)\end{aligned}$$

Therefore,

(26)

$$\Lambda(\tau)\frac{D(\tau)}{D(0)} = \exp\left(\beta-r\right)\tau\exp\left(\sigma_D-\theta^Q\right)' W^Q(\tau)\exp\left(-\frac{1}{2}\left(\|\theta\|^2+\frac{1}{2}\|\sigma_D\|^2\right)\tau - \theta'\cdot W(\tau)\right)$$

where $W(\tau) = \left(W^Q(\tau), W^S(\tau)\right)'$. Taking the mathematical expectation of (26), we obtain

$$E\left[\Lambda(\tau)\frac{D(\tau)}{D(0)}\right] = \exp\left((\beta-r-\sigma_D\cdot\theta)\tau\right) \tag{27}$$

The last equality is obtained by the fact that an exponential Martingale has expectation of one. Integrating both sides of (27), we obtain

$$\begin{aligned}\int_0^{T-t} E\left[\Lambda(\tau)\frac{D(\tau)}{D(0)}\right] d\tau &= \int_0^{T-t}\exp\left((\beta-r-\sigma_D\cdot\theta)\tau\right)d\tau \\ &= \frac{1}{\beta-r-\sigma_D\cdot\theta}\left[\exp((\beta-r-\sigma_D\cdot\theta)(T-t))-1\right]\end{aligned}$$

Therefore,

$$\Psi(t) = \frac{1}{\beta-r-\sigma_D\cdot\theta}\left[\exp((\beta-r-\sigma_D\cdot\theta)(T-t))-1\right]cD(t) \tag{28}$$

At $t=0$, we have the present value of PPM's future contribution to be

$$\Psi(0) = \Psi = \frac{1}{\beta-r-\sigma_D\cdot\theta}\left[\exp((\beta-r-\sigma_D\cdot\theta)T)-1\right]cy \tag{29}$$

Theorem 1 gives the value of PPMs future contributions. This will enable the PPMs and PFAs to determine the value of their pension contributions up to terminal period.

Lemma 1. *Suppose that theorem 1 holds, then*

$$d\Psi(t) = \Psi(t)\left((r+\sigma_D\cdot\theta)dt + \sigma_D'\cdot dW^Q(t)\right) - cD(t)dt. \tag{30}$$

Proof: Taking differential of both sides of (30), we have

$$\begin{aligned} d\Psi(t) &= d(\frac{1}{\beta - r - \sigma_D\cdot\theta}[\exp((\beta - r - \sigma_D\cdot\theta)(T-t)) - 1]cD(t)) \\ &= \frac{1}{\beta - r - \sigma_D\cdot\theta}[\exp((\beta - r - \sigma_D\cdot\theta)(T-t)) - 1]cD(t) \\ &\quad + \frac{1}{\beta - r - \sigma_D\cdot\theta}cD(t)d[\exp((\beta - r - \sigma_D\cdot\theta)(T-t)) - 1] \\ &= \frac{c}{\beta - r - \sigma_D\cdot\theta}[\exp((\beta - r - \sigma_D\cdot\theta)(T-t)) - 1]D(t)(\beta dt + \sigma_D'\cdot dW^Q(t)) \\ &\quad - \frac{c}{\beta - r - \sigma_D\cdot\theta}D(t)(\beta - r - \sigma_D\cdot\theta)\exp((\beta - r - \sigma_D\cdot\theta)(T-t))dt \\ &= \frac{1}{\beta - r - \sigma_D\cdot\theta}[\exp((\beta - r - \sigma_D\cdot\theta)(T-t)) - 1]D(t)(r + \sigma_D\cdot\theta)dt \\ &\quad + \sigma_D'\cdot dW^Q(t)) - cD(t)dt \end{aligned}$$

Therefore,

$$d\Psi(t) = \Psi(t)((r + \sigma_D\cdot\theta)dt + \sigma_D'\cdot dW^Q(t)) - cD(t)dt. \tag{31}$$

Definition 4. *The value of the wealth of the PPM is given by the process*

$$V(t) = X^{\Delta,D,C}(t) + \Psi(t) \tag{32}$$

Proposition 1. *Let $V(t)$ be the wealth process of a PPM and $X^{\Delta,D,C}(t)$ and $\Psi(t)$ satisfies (31) and (11), then*

$$\begin{cases} dV(t) = [r(\Psi(t) + X^{\Delta,D,C}(t)) + \Psi(t)\sigma_D\cdot\theta + \Delta_S(t)X^{\Delta,D,C}(t)(\mu - r1_n) \\ +\Delta_Q(t)X^{\Delta,D,C}(t)\cdot(\Sigma^Q\theta^Q) - C(t)]dt + (X^{\Delta,D,C}(t)\Sigma^S\Delta_S'(t))'\cdot dW^S(t) + \\ (X^{\Delta,D,C}(t)\Sigma^Q\Delta_Q'(t) + \Psi(t)\sigma_D)'\cdot dW^Q(t) \\ V(0) = v = x + \Psi. \end{cases} \tag{33}$$

Proof: Taking the differential of both sides of (32) and substituting in (31) and (11), we obtain the desired result.

7. Expected Utility of Wealth, Optimal Portfolio and Consumption Processes for a PPM

In this section, we study the expected utility of wealth, optimal portfolio and consumption strategies of a PPM in a contributory phase of a DC pension fund. We define the general value function

$$U(t,v) = E[J(V(t)|X^{\Delta,D,C}(t) = x, \Psi(t) = \Psi]$$

where $U(t,v)$ is the path of $V(t, X^{\Delta,D,C}(t), \Psi(t))$ given the portfolio strategy $\Delta(t) = (\Delta'_S(t), \Delta'_Q(t))$. Define Π_V to be the set of all admissible portfolio strategy that are $\mathbb{F}_V$-progressively measurable, that satisfy the integrability conditions

$$E[\int_t^T \Delta_S(u)\Delta'_S(u)du] < \infty, E[\int_t^T \Delta_Q(u)\Delta'_Q(u)du] < \infty. \tag{34}$$

and let $J(t,v)$ be a concave function in $V(t)$ such that $J(t,v)$ satisfies the HJB equation

$$J(t,v) = \sup_{\Delta\in\Pi_V} E[J(V(t)|X^{\Delta,D,C}(t) = x, \Psi(t) = \Psi] = \sup_{\Delta\in\Pi_V} U(t,v) \tag{35}$$

$$J_t(V(t)) + \sup\nolimits_{\Delta\in\Pi_V} H^V(t) = 0, \tag{36}$$

where,

$$\begin{aligned} H^V(t) &= rxJ_x + \Psi\sigma_D \cdot \theta J_\Psi + \Delta_S(t)x(\mu - r1_n)J_x \\ &+\Delta_Q(t)x\cdot(\Sigma^Q\theta^Q)J_x - CJ_x + \exp(-\rho t)J(C) + \frac{1}{2}(\Sigma^S\Delta'_S(t))'x^2(\Sigma^S\Delta'_S(t))J_{xx} + \\ &\frac{1}{2}(\Sigma^Q\Delta'_Q(t))'x^2(\Sigma^Q\Delta'_Q(t))J_{xx} + \frac{1}{2}\Psi^2\sigma'_D\sigma_D J_{\Psi\Psi} + x\Psi\sigma'_D\Sigma^Q\Delta'_Q(t)J_{x\Psi} \end{aligned} \tag{37}$$

Let $J(V(t))$ be the solution of the HJB equation (36). Since the utility function is concave and the value function is smooth i.e., $J(V(t)) \in C^{1,2}(\Re \times [0,T])$, then (36) is well-defined. Hence, we have the following:

$$\Delta'^*_S(t) = \frac{-((\Sigma^S)^2)^{-1}(\mu - r1_n)J_x}{xJ_{xx}} \tag{38}$$

$$\Delta'^*_Q(t) = \frac{-((\Sigma^Q)^2)^{-1}(\Sigma^Q\theta^Q J_x - \Sigma^Q\Psi(t)\sigma_D J_{x\Psi})}{xJ_{xx}} \tag{39}$$

From (37) the consumption part can be expressed as

$$\begin{aligned} \frac{\partial H^V(t)}{\partial C^*(t)} &= -J_x + \exp(-\rho t)J'(C^*(t)) = 0 \\ \exp(-\rho t)J'(C^*(t)) &= (J_x) \\ C^*(t) &= J^{-1}\left(\frac{(J_x)}{\exp(-\rho t)}\right) \end{aligned} \tag{40}$$

Therefore,

$$
\begin{aligned}
C^*(t) &= \left(\frac{dJ(C(t))}{dC(t)}\right)^{-1}(J_x)\exp(\rho t) \\
C^*(t) &= K((J_x)\exp(\rho t))
\end{aligned} \tag{41}
$$

where

$$
\left(\frac{dJ(C(t))}{dC(t)}\right)^{-1} = K. \tag{42}
$$

Substituting (38), (39) and (41) into (36), we obtain the following: where $\{\Delta^*, C^*\}$ is optimal strategy.

Thus, the HJB equation of (36) becomes

(43)

$$
\begin{aligned}
&J_t(V(t)) + r\Psi J_\Psi(V(t)) + \frac{1}{2}\sigma_D'\sigma_D\Psi^2 J_{\Psi\Psi}(V(t)) + \sigma_D\cdot\theta\Psi J_\Psi(V(t)) + rxJ_x(V(t)) - \\
&\quad \frac{1}{2}\frac{(\theta^S)'\theta^S J_x(V(t))^2}{J_{xx}(V(t))} - \frac{1}{2}\frac{\sigma_D'\sigma_D\Psi^2 J_{x\Psi}(V(t))^2}{J_{xx}(V(t))} - \frac{1}{2}\frac{(\theta^Q)'\theta^Q J_x(V(t))^2}{J_{xx}(V(t))^2} - \\
&\qquad \frac{\sigma_D\cdot\theta\Psi J_{x\Psi}(V(t))J_x(V(t))}{J_{xx}(V(t))} - K(J_x(V(t))\exp(\rho t))J_x(V(t)) \\
&\qquad\qquad + \exp(-\rho t)(JKJ_x(V(t))\exp(\rho t)) = 0
\end{aligned}
$$

Proposition 2 gives the solution of the HJB equation (43) under CRRA utility function.

Proposition 2. *The solution to the HJB equation (43) with*

$$
\begin{cases}
Q(t) = \exp[r(T-t)] \\
Q(T) = 1
\end{cases} \tag{44}
$$

$$
\begin{cases}
P(t) = \left(\begin{array}{l}
\dfrac{2\gamma^2(\gamma-1)\exp\left[-\dfrac{\rho t}{\gamma} + \dfrac{r(\gamma-1)(T-t)}{\gamma}\right]}{(1-\gamma)(\theta'\theta - \gamma\theta'\theta - 2\gamma\rho + 2\gamma r - 2\gamma^2 r)} + \\
\dfrac{\exp\left[\frac{(1-\gamma)}{2\gamma^2}\theta'\theta(T-t) - \frac{\rho T}{\gamma}\right]}{\gamma\theta'\theta - \theta'\theta + 2\gamma\rho - 2\gamma r + 2\gamma^2 r} \times \\
\left\{\exp\left[\frac{\rho T}{\gamma}\right](2\gamma^2 r - 2\gamma r + 2\gamma\rho + \gamma\theta'\theta - \theta'\theta) - 2\gamma^2\right\}
\end{array}\right)^{\frac{\gamma}{1-\gamma}} \\
P(T) = 1
\end{cases}
$$

is given by

$$\begin{cases} J^{\Delta,C}(t,x,\Psi) &= \left(\dfrac{[x+\Psi]^{1-\gamma}}{1-\gamma} - \dfrac{C(t)^{1-\gamma}}{1-\gamma}\right)\dfrac{\exp[r(1-\gamma)(T-t)]}{\phi_f^{\gamma}}\times \\ & \left\{\begin{array}{l} 2\gamma^2\exp\left(-\dfrac{\rho t}{\gamma}+\dfrac{r(\gamma-1)(T-t)}{\gamma}\right)+ \\ \exp\left(-\dfrac{\theta'\theta(\gamma-1)}{2\gamma^2}(T-t)-\dfrac{\rho T}{\gamma}\right)\times \\ \left(\phi_f\exp\left(\dfrac{\rho T}{\gamma}\right)-2\gamma^2\right) \end{array}\right\}^{\gamma} \\ J^{\Delta,C}(T,x,\Psi) &= \dfrac{[x+\Psi]^{1-\gamma}}{1-\gamma} - \dfrac{C(t)^{1-\gamma}}{1-\gamma}. \end{cases}$$

where,

$\phi_f = \gamma\theta'\theta - \theta'\theta + 2\gamma\rho - 2\gamma r + 2\gamma^2 r.$

Proof: Let

$$J^{\Delta,C}(t,x,\Psi) = \left(\frac{(x+\Psi)^{1-\gamma}}{1-\gamma} - \frac{C^{1-\gamma}}{1-\gamma}\right)(Q(t)P(t))^{1-\gamma},$$

$\gamma > 0,\ \gamma \neq 1$ is given by

$$J_t(t,x,\Psi) = [x+\Psi]^{1-\gamma}(Q(t)P(t))^{-\gamma}(Q(t)P'(t)+Q'(t)P(t)) \tag{45}$$

$$J_x(t,x,\Psi) = [x+\Psi]^{-\gamma}(Q(t)P(t))^{1-\gamma} \tag{46}$$

$$J_\Psi(t,x,\Psi) = [x+\Psi]^{-\gamma}(Q(t)P(t))^{1-\gamma} \tag{47}$$

$$J_{xx}(t,x,\Psi) = -\gamma[x+\Psi]^{-1-\gamma}(Q(t)P(t))^{1-\gamma} \tag{48}$$

$$J_{x\Psi}(t,x,\Psi) = -\gamma[x+\Psi]^{-1-\gamma}(Q(t)P(t))^{1-\gamma} \tag{49}$$

$$J_{\Psi\Psi}(t,x,\Psi) = -\gamma[x+\Psi]^{-1-\gamma}(Q(t)P(t))^{1-\gamma} \tag{50}$$

Substituting (45)-(50) into (43), we have we obtain the following:

$$
\begin{gathered}
(x+\Psi)^{1-\gamma}(Q(t)P(t))^{-\gamma}(Q(t)P'(t)+Q'(t)P(t))+r\Psi(x+\Psi)^{1-\gamma}(Q(t)P(t))^{1-\gamma}- \\
\frac{1}{2}\sigma_D'\sigma_D\Psi^2\gamma(x+\Psi)^{-1-\gamma}(Q(t)P(t))^{1-\gamma}+\sigma_D\Psi\theta^Q(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}- \\
K((\exp(\rho t))(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma})+rx(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}+ \\
\frac{1}{2}\frac{(\theta^S)'\theta^S(x+\Psi)^{-2\gamma}(Q(t)P(t))^{2(1-\gamma)}}{\gamma(x+\Psi)^{-1-\gamma}(Q(t)P(t))^{1-\gamma}}+\frac{1}{2}\frac{\sigma_D'\sigma_D\Psi^2\gamma^2(x+\Psi)^{-2\gamma-2}(Q(t)P(t))^{2(1-\gamma)}}{\gamma(x+\Psi)^{-1-\gamma}(Q(t)P(t))^{1-\gamma}} \\
-\frac{\sigma_D\Psi\gamma\theta^Q(x+\Psi)^{-2\gamma-1}(Q(t)P(t))^{2(1-\gamma)}}{\gamma(x+\Psi)^{-1-\gamma}(Q(t)P(t))^{1-\gamma}}+\frac{1}{2}\frac{(\theta^Q)'\theta^Q(x+\Psi)^{-2\gamma}(Q(t)P(t))^{2(1-\gamma)}}{\gamma(x+\Psi)^{-1-\gamma}(Q(t)P(t))^{1-\gamma}}
\end{gathered}
$$

$$
+\exp(-\rho t)J(K(\exp(\rho t)(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}))=0,
$$

$$
\begin{gathered}
(x+\Psi)^{1-\gamma}(Q(t)P(t))^{-\gamma}Q(t)P'(t)+(x+\Psi)^{1-\gamma}(Q(t)P(t))^{-\gamma}Q'(t)P(t) \\
+rx(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}-K((\exp(\rho t))(x+\Psi)^{-\gamma}(Q(t)P(t))^{-\gamma})+ \\
(x+\Psi)^{1-\gamma}(Q(t)P(t))^{1-\gamma}(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}+ \\
rx(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}+\frac{1}{2}\frac{(\theta^S)'\theta^S(x+\Psi)^{1-\gamma}(Q(t)P(t))^{1-\gamma}}{\gamma}+ \\
\frac{1}{2}\frac{(\theta^Q)'\theta^Q(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}}{\gamma}+\exp(-\rho t)J(K(\exp(\rho t)(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}))=0,
\end{gathered}
$$

$$
\begin{gathered}
\Longrightarrow (x+\Psi)^{1-\gamma}(Q(t)P(t))^{-\gamma}Q'(t)P(t)+r\Psi(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}+ \\
rx(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}=0
\end{gathered} \tag{51}
$$

$$
\begin{gathered}
\Longrightarrow (x+\Psi)^{1-\gamma}(Q(t)P(t))^{-\gamma}Q(t)P'(t)- \\
K((\exp(\rho t))(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma})(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma} \\
+\frac{1}{2}(\frac{(\theta^S)'\theta^S(x+\Psi)^{1-\gamma}(Q(t)P(t))^{1-\gamma}}{\gamma}+\frac{(\theta^Q)'\theta^Q(x+\Psi)^{1-\gamma}(Q(t)P(t))^{1-\gamma}}{\gamma}) \\
+\exp(-\rho t)J(K(\exp(\rho t)(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}))=0,
\end{gathered} \tag{52}
$$

From (51), we have that

$$
Q'(t)+rQ(t)=0 \tag{53}
$$

Solving (53), we have

$$
\begin{cases} Q(t)=\exp[r(T-t)] \\ Q(T)=1 \end{cases} \tag{54}
$$

From (52), we have

(55)
$$(x+\Psi)^{1-\gamma}(Q(t)P(t))^{-\gamma}Q(t)P'(t) - K((\exp(\rho t))(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma})(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}$$
$$+\frac{1}{2}\theta'\theta(x+\Psi)^{1-\gamma}(Q(t)P(t))^{1-\gamma} + \exp(-\rho t)J(K(\exp(\rho t)(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma})) = 0.$$

But, $J(C(t)) = \dfrac{[C(t)]^{1-\gamma}}{1-\gamma}$ and $\left(\dfrac{dJ(C)}{dC}\right)^{-1} = K.$
Then (55) becomes

$$(x+\Psi)^{1-\gamma}(Q(t)P(t))^{-\gamma}Q(t)P'(t) - ((\exp(\rho t))(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma})^{-\frac{1}{\gamma}}(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}$$
$$+\frac{1}{2\gamma}\theta'\theta(x+\Psi)^{1-\gamma}(Q(t)P(t))^{1-\gamma} + \exp(-\rho t)\frac{1}{1-\gamma}(\exp(\rho t)(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma})^{-\frac{1}{\gamma}(1-\gamma)} = 0$$

Simplifying, we have

$$\begin{aligned}&(x+\Psi)^{1-\gamma}(Q(t)P(t))^{-\gamma}Q(t)P'(t) - \left(\exp\left[-\frac{\rho t}{\gamma}\right]\right)\\&\times(x+\Psi)(Q(t)P(t))^{\frac{\gamma-1}{\gamma}}(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}+\\&\frac{1}{2\gamma}\theta'\theta(x+\Psi)^{1-\gamma}(Q(t)P(t))^{1-\gamma} + \exp(-\rho t)\frac{1}{1-\gamma}\\&\times\left(\exp\left[\rho t\left(\tfrac{\gamma-1}{\gamma}\right)\right](x+\Psi)^{1-\gamma}(Q(t)P(t))^{\frac{(1-\gamma)(\gamma-1)}{\gamma}}\right) = 0\end{aligned}$$

Further simplification, we have the following

$$\begin{aligned}&(x+\Psi)^{1-\gamma}(Q(t)P(t))^{-\gamma}Q(t)P'(t) - \left(\exp\left[-\frac{\rho t}{\gamma}\right]\right)\\&\times(x+\Psi)(Q(t)P(t))^{\frac{\gamma-1}{\gamma}}(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}+\\&\frac{1}{2\gamma}\theta'\theta(x+\Psi)^{1-\gamma}(Q(t)P(t))^{1-\gamma} + \exp(-\rho t)\frac{1}{1-\gamma}\exp\left[-\frac{\rho t(\gamma-1)}{\gamma}\right]\\&\times(x+\Psi)^{1-\gamma}(Q(t)P(t))^{\frac{(1-\gamma)(\gamma-1)}{\gamma}} = 0\end{aligned}$$

$$\begin{aligned}&Q(t)P'(t) - \left(\exp\left[-\frac{\rho t}{\gamma}\right]\right)(Q(t)P(t))^{\frac{2\gamma-1}{\gamma}}+\\&\tfrac{1}{2\gamma}\theta'\theta(Q(t)P(t)) + \frac{1}{1-\gamma}\exp\left[-\frac{\rho t}{\gamma}\right](Q(t)P(t))^{\frac{2\gamma-1}{\gamma}} = 0\end{aligned}$$

$$Q(t)P'(t) + \frac{1}{2\gamma}\theta'\theta Q(t)P(t) + \left(\frac{1}{1-\gamma} - 1\right)\times\ \exp\left[-\frac{\rho t}{\gamma}\right](Q(t)P(t))^{\frac{2\gamma-1}{\gamma}} = 0$$

$$Q(t)P'(t) + \frac{1}{2\gamma}\theta'\theta Q(t)P(t) + \frac{\gamma}{1-\gamma}\exp\left[-\frac{\rho t}{\gamma}\right](Q(t)P(t))^{\frac{2\gamma-1}{\gamma}} = 0$$

$$P'(t) + \frac{1}{2\gamma}\theta'\theta P(t) + \frac{\gamma}{1-\gamma}\exp\left[-\frac{\rho t}{\gamma}\right]Q(t)^{\frac{\gamma-1}{\gamma}}(P(t))^{\frac{2\gamma-1}{\gamma}} = 0$$

Substituting in the value of $Q(t) = \exp[r(T-t)]$, we have

$$P'(t) + \frac{1}{2\gamma}\theta'\theta P(t) + \frac{\gamma}{1-\gamma}\exp\left[-\frac{\rho t}{\gamma}\right]\exp\left[r(T-t)\frac{\gamma-1}{\gamma}\right](P(t))^{\frac{2\gamma-1}{\gamma}} = 0$$

$$P'(t)+\frac{1}{2\gamma}\theta'\theta P(t)+\frac{\gamma}{1-\gamma}\exp\left[r(T-t)\frac{\gamma-1}{\gamma}-\frac{\rho t}{\gamma}\right](P(t))^{\frac{2\gamma-1}{\gamma}}=0$$

$$P'(t)+\frac{1}{2\gamma}\theta'\theta P(t)+\frac{\gamma}{1-\gamma}\exp\left[-\frac{1}{\gamma}(\rho t-r(T-t)(\gamma-1))\right](P(t))^{\frac{2\gamma-1}{\gamma}}=0 \tag{56}$$

Using Mathematica 6.0 to solve (56), we have the following result:

$$\begin{cases} P(t) &= \left(\begin{array}{l} \dfrac{2\gamma^2(\gamma-1)\exp\left[-\dfrac{\rho t}{\gamma}+\dfrac{r(\gamma-1)(T-t)}{\gamma}\right]}{(1-\gamma)(\theta'\theta-\gamma\theta'\theta-2\gamma\rho+2\gamma r-2\gamma^2 r)}+ \\ \dfrac{\exp\left[\frac{(1-\gamma)}{2\gamma^2}\theta'\theta(T-t)-\frac{\rho T}{\gamma}\right]}{\gamma\theta'\theta-\theta'\theta+2\gamma\rho-2\gamma r+2\gamma^2 r}\times \\ \left\{\exp\left[\frac{\rho T}{\gamma}\right](2\gamma^2 r-2\gamma r+2\gamma\rho+\gamma\theta'\theta-\theta'\theta)-2\gamma^2\right\} \end{array}\right)^{\frac{\gamma}{1-\gamma}} \\ P(T) &= 1 \end{cases}$$

Hence, the solution to the HJB equation (43) is obtained as follows:

$$\begin{cases} J^{\Delta,C}(t,x,\Psi) &= \left(\dfrac{[x+\Psi]^{1-\gamma}}{1-\gamma}-\dfrac{C(t)^{1-\gamma}}{1-\gamma}\right)\dfrac{\exp[r(1-\gamma)(T-t)]}{\phi_f^{\gamma}}\times \\ & \quad \left\{\begin{array}{l} 2\gamma^2\exp\left(-\dfrac{\rho t}{\gamma}+\dfrac{r(\gamma-1)(T-t)}{\gamma}\right) \\ \exp\left(-\dfrac{\theta'\theta(\gamma-1)}{2\gamma^2}(T-t)-\dfrac{\rho T}{\gamma}\right)\left(\phi_f\exp\left(\dfrac{\rho T}{\gamma}\right)-2\gamma^2\right) \end{array}\right\}^{\gamma} \\ J^{\Delta,C}(T,x,\Psi) &= \dfrac{[x+\Psi]^{1-\gamma}}{1-\gamma}-\dfrac{C(T)^{1-\gamma}}{1-\gamma}. \end{cases}$$

where,

$\phi_f=\gamma\theta'\theta-\theta'\theta+2\gamma\rho-2\gamma r+2\gamma^2 r.$

From the solution above, the following:

$$\frac{\exp[r(1-\gamma)(T-t)]}{\phi_f^{\gamma}}\exp\left[\frac{\theta'\theta(1-\gamma)}{2\gamma}(T-t)\right]$$

$$\times\left\{2\gamma^2\exp\left(-\frac{\rho t}{\gamma}+\frac{r(\gamma-1)(T-t)}{\gamma}-\frac{\theta'\theta(\gamma-1)}{2\gamma^2}(T-t)\right)+\exp\left(-\frac{\rho T}{\gamma}\right)\left(\phi_f\exp\left(\frac{\rho T}{\gamma}\right)-2\gamma^2\right)\right\}^{\gamma}$$

can be extracted and hence,

$$\exp\left[r(1-\gamma)(T-t)+\frac{\theta'\theta(1-\gamma)}{2\gamma}(T-t)\right]\frac{1}{\phi_f^{\gamma}}\times$$
$$\left\{2\gamma^2\exp\left(-\frac{\rho t}{\gamma}+\frac{r(\gamma-1)(T-t)}{\gamma}-\frac{\theta'\theta(\gamma-1)}{2\gamma^2}(T-t)\right)+\exp\left(-\frac{\rho T}{\gamma}\right)\left(\phi_f\exp\left(\frac{\rho T}{\gamma}\right)-2\gamma^2\right)\right\}^{\gamma}$$

$$\exp\left[r(1-\gamma)(T-t)+\frac{\theta'\theta(1-\gamma)}{2\gamma}(T-t)\right]\frac{1}{\phi_f^{\gamma}}\exp[\rho T]$$
$$\times\left\{\begin{array}{l} 2\gamma^2\exp\left(-\frac{\rho(T-t)}{\gamma}+\frac{r(\gamma-1)(T-t)}{\gamma}-\frac{\theta'\theta(\gamma-1)}{2\gamma^2}(T-t)\right) \\ +\exp(-1)\left(\phi_f\exp\left(\frac{\rho T}{\gamma}\right)-2\gamma^2\right)\end{array}\right\}^{\gamma}$$

$$\exp\left[r(1-\gamma)(T-t)+\frac{\theta'\theta(1-\gamma)}{2\gamma}(T-t)\right]\frac{1}{\phi_f^{\gamma}}\exp[\rho T]$$
$$\times\left\{\begin{array}{l} 2\gamma^2\exp\left(-\frac{\rho(T-t)}{\gamma}-\frac{r(1-\gamma)(T-t)}{\gamma}+\frac{\theta'\theta(1-\gamma)}{2\gamma^2}(T-t)\right) \\ +\exp(-1)\left(\phi_f\exp\left(\frac{\rho T}{\gamma}\right)-2\gamma^2\right)\end{array}\right\}^{\gamma}$$

Intuitively, we have that the growth rate from the PPM's contribution is $r(1-\gamma)(T-t)+\frac{\theta'\theta(1-\gamma)}{2\gamma}(T-t)$
which was discussed above. Now there is additional term as a result of the presence of consumption process, which is

$$\frac{1}{\phi_f^{\gamma}}\exp[\rho T]\left\{2\gamma^2\exp\left(-\frac{\rho(T-t)}{\gamma}-\frac{r(1-\gamma)(T-t)}{\gamma}+\frac{\theta'\theta(1-\gamma)}{2\gamma^2}(T-t)\right)\right.$$
$$\left.+\exp(-1)\left(\phi_f\exp\left(\frac{\rho T}{\gamma}\right)-2\gamma^2\right)\right\}^{\gamma}.$$

Since PPM will like to invest more if the stock markets are doing well, the only thing of interest in the above expression are terms which contain the sharp ratio (i.e. $\theta'\theta$) while for all other parameters remain fixed, that is,

$$-\frac{\rho}{\gamma}-\frac{r(1-\gamma)}{\gamma}+\frac{\theta'\theta(1-\gamma)}{2\gamma^2}=\frac{1}{\gamma}\left(-\rho+\frac{(1-\gamma)}{\gamma}\left(\frac{\theta'\theta}{2}-r\right)\right).$$

PPM will invest more into the pension fund if $\rho<\frac{(1-\gamma)}{\gamma}\left(\frac{\theta'\theta}{2}-r\right)$.
This implies that if the discount rate of the PPM's preference over time, is less than the value $\frac{(1-\gamma)}{\gamma}\left(\frac{\theta'\theta}{2}-r\right)$ the PPM will like to invest more into the pension fund and may not invest more if $\rho\geq\frac{(1-\gamma)}{\gamma}\left(\frac{\theta'\theta}{2}-r\right)$.
Proposition 3, determined the optimal portfolio and consumption processes of the PPM in the pension scheme.

Proposition 3. *Suppose that the function*

$$J(V(t)) = \left(\frac{(x+\Psi)^{1-\gamma}}{1-\gamma} - \frac{C^{1-\gamma}}{1-\gamma}\right)(Q(t)P(t))^{1-\gamma}$$

is the solution of the HJB equation (36), then

(57)
$$C^*(t) = \frac{(x+\Psi)\phi_f \exp\left[\frac{1}{\gamma}\left(\rho - \left\{\frac{(1-\gamma)}{2\gamma^2}\theta'\theta + r(1-\gamma)\right\}\right)(T-t)\right]}{2\gamma^2\left(\exp\left[\frac{1}{\gamma}\left(\rho - \left\{\frac{(1-\gamma)}{2\gamma}\theta'\theta + r(1-\gamma)\right\}\right)(T-t)\right] - 1\right) + \phi_f \exp\left[\frac{\rho T}{\gamma}\right]}$$

$$\Delta_S'^*(t) = \frac{((\Sigma^S)^2)^{-1}(\mu - r1_n)}{\gamma} + \frac{((\Sigma^S)^2)^{-1}(\mu - r1_n)\Psi(t)}{X(t)\gamma}, \tag{58}$$

$$\Delta_Q'^*(t) = \frac{(\Sigma^Q)^{-1}\theta^Q}{\gamma} + \frac{(\Sigma^Q)^{-1}\theta^Q\Psi(t)}{X(t)\gamma} - \frac{(\Sigma^Q)^{-1}\sigma_D\Psi(t)}{X(t)}. \tag{59}$$

(60)
$$\Delta_0'^*(t) = 1 + \frac{(\Sigma^Q)^{-1}\sigma_D\Psi(t)}{X(t)} - \frac{1}{\gamma}\left(\begin{array}{l} \frac{((\Sigma^S)^2)^{-1}(\mu - r1_n)}{X(t)} + \frac{((\Sigma^S)^2)^{-1}(\mu - r1_n)\Psi(t)}{X(t)} \\ +(\Sigma^Q)^{-1}\theta^Q + \frac{(\Sigma^Q)^{-1}\theta^Q\Psi(t)}{X(t)} \end{array}\right),$$

Proof: From (41), we have that

$$\begin{aligned} C^*(t) &= K[J_x \exp(\rho t)] \\ &= K[(x+\Psi)^{-\gamma}(Q(t)P(t))^{1-\gamma}\exp(\rho t)]^{\frac{1}{\gamma}} \\ &= (x+\Psi)(Q(t)P(t))^{\frac{\gamma-1}{\gamma}}\exp\left(-\frac{\rho t}{\gamma}\right) \end{aligned}$$

$$= \frac{(x+\Psi)\phi_f \exp\left[r\frac{(\gamma-1)}{\gamma}(T-t) - \frac{\rho t}{\gamma}\right]}{2\gamma^2\exp\left[r\frac{\rho-1}{\gamma}(T-t) - \frac{\rho t}{\gamma}\right] + \exp\left[\frac{(1-\gamma)}{2\gamma^2}\theta'\theta(T-t) - \frac{\rho T}{\gamma}\right] \times \left\{\exp\left[\frac{\rho T}{\gamma}\right]\phi_f - 2\gamma^2\right\}}$$

$$= \frac{(x+\Psi)\phi_f \exp\left[r\frac{(\gamma-1)}{\gamma}(T-t) - \frac{\rho t}{\gamma} + \frac{\rho T}{\gamma} - \frac{(1-\gamma)}{2\gamma^2}\theta'\theta(T-t)\right]}{2\gamma^2\exp\left[r\frac{(\gamma-1)}{\gamma}(T-t) - \frac{\rho t}{\gamma} + \frac{\rho T}{\gamma} - \frac{(1-\gamma)}{2\gamma^2}\theta'\theta(T-t)\right] - 2\gamma^2 + \phi_f\exp\left[\frac{\rho T}{\gamma}\right]}$$

$$= \frac{(x+\Psi)\phi_f \exp\left[r\frac{(\gamma-1)}{\gamma}(T-t) - \frac{\rho t}{\gamma} + \frac{\rho T}{\gamma} - \frac{(1-\gamma)}{2\gamma^2}\theta'\theta(T-t)\right]}{2\gamma^2\left(\exp\left[r\frac{(\gamma-1)}{\gamma}(T-t) - \frac{\rho t}{\gamma} + \frac{\rho T}{\gamma} - \frac{(1-\gamma)}{2\gamma^2}\theta'\theta(T-t)\right] - 1\right) + \phi_f\exp\left[\frac{\rho T}{\gamma}\right]}$$

$$= \frac{(x+\Psi)\phi_f \exp\left[\frac{\rho(T-t)}{\gamma} - \left\{\frac{(1-\gamma)}{2\gamma^2}\theta'\theta + r\frac{(1-\gamma)}{\gamma}\right\}(T-t)\right]}{2\gamma^2\left(\exp\left[\frac{\rho(T-t)}{\gamma} - \left\{\frac{(1-\gamma)}{2\gamma^2}\theta'\theta + r\frac{(1-\gamma)}{\gamma}\right\}(T-t))\right] - 1\right) + \phi_f\exp\left[\frac{\rho T}{\gamma}\right]}$$

$$= \frac{(x+\Psi)\phi_f \exp\left[\frac{1}{\gamma}\left(\rho - \left\{\frac{(1-\gamma)}{2\gamma^2}\theta'\theta + r(1-\gamma)\right\}\right)(T-t)\right]}{2\gamma^2\left(\exp\left[\frac{1}{\gamma}\left(\rho - \left\{\frac{(1-\gamma)}{2\gamma}\theta'\theta + r(1-\gamma)\right\}\right)(T-t)\right] - 1\right) + \phi_f \exp\left[\frac{\rho T}{\gamma}\right]}$$

Intuitively, the expected growth rate (GR) of PPM optimal consumption is equals

$$GR = \frac{1}{\gamma}\left(\rho - \left\{\frac{(1-\gamma)}{2\gamma}\theta'\theta + r(1-\gamma)\right\}\right) \tag{61}$$

This is referred to as the Euler's equation for the inter-temporal maximization under uncertainty. The coefficient $\frac{1}{\gamma}$ is referred to as the elasticity of substitution of consumption in economics. The positive term $\theta'\theta$ captures the uncertainty of the financial markets. When the market become risky, it induces the PPM not make more contributions into the pension fund. From (61), we have that for constant γ, if $\rho > \left\{\frac{(1-\gamma)}{2\gamma}\theta'\theta + r(1-\gamma)\right\}$ the growth rate of the expected consumption is strictly positive, it is strictly negative if $\rho < \left\{\frac{(1-\gamma)}{2\gamma}\theta'\theta + r(1-\gamma)\right\}$ and constant if $\rho = \left\{\frac{(1-\gamma)}{2\gamma}\theta'\theta + r(1-\gamma)\right\}$.
Intuitively, as the discount rate captures the PPM's preference over time, if ρ is greater than $\frac{(1-\gamma)}{2\gamma}\theta'\theta + r(1-\gamma)$, it implies that PPM will like to consume more since the markets are risky to invest into. If ρ is less $\frac{(1-\gamma)}{2\gamma}\theta'\theta + r(1-\gamma)$, it implies that PPM will like to consume less and invest more into the pension fund. Finally, if ρ is equal to $\frac{(1-\gamma)}{2\gamma}\theta'\theta + r(1-\gamma)$, then PPM will be at the critical position to determine whether to consume more and invest less or to invest more and to consume less.

We obtain our desire results for $\Delta_S'^*(t)$ and $\Delta_Q'^*(t)$ by substituting (46)-(50) into the optimal strategies in (38) and (39).

The optimal rules in (58)-(60), we referred to as the variational form of classical Merton rule, tells us that under the CRRA utility function, the proportion of wealth invested in the stock markets at time t is equal to $\frac{((\Sigma^S)^2)^{-1}(\mu - r1_n)}{\gamma} + \frac{((\Sigma^S)^2)^{-1}(\mu - r1_n)\Psi(t)}{X(t)\gamma}$. We have two terms. The first term is the classical optimal portfolio rule in stock markets. The second term is proportional to the ratio of the present value of expected future contribution to the optimal portfolio value-to-date in relation to stocks. It is the additional returns into the portfolio value as a result of the inclusion of present value of the expected future contribution in the pension fund.

Again, the proportion of wealth in the indexed bond at time t is $\dfrac{(\Sigma^Q)^{-1}\theta^Q}{\gamma}+\dfrac{(\Sigma^Q)^{-1}\theta^Q\Psi(t)}{X(t)\gamma}-\dfrac{(\Sigma^Q)^{-1}\sigma_D\Psi(t)}{X(t)}$. The optimal investment strategy above is made up of three parts: The first two terms is the variational form of classical optimal portfolio rule. The second term of the first two terms is proportional to the ratio of the present value of expected future contribution to the optimal portfolio value-to-date in relation to the indexed bond. The term $\dfrac{(\Sigma^Q)^{-1}\sigma_D\Psi(t)}{X(t)}$ is an inter-temporal hedging term that offset any shock (inflation risk) to the contribution of the PPM at time t.

From (57), we have at $t=T$, the terminal optimal consumption to be

$$C^*(T)=(x+\Psi)\exp\left(\frac{\gamma}{\rho T}\right).$$

This shows that the optimal terminal consumption is a function of the initial wealth x, the present value of future contributions, Ψ, γ and ρ. Again, at t=0, the optimal consumption is

$$C^*(0)=\frac{(x+\Psi)\,\phi_f\exp\left[\frac{1}{\gamma}\left(\rho-\left\{\frac{1-\gamma}{2\gamma^2}\theta'\theta+r(1-\gamma)\right\}\right)T\right]}{2\gamma^2\left(\exp\left[\frac{1}{\gamma}\left(\rho-\left\{\frac{1-\gamma}{2\gamma^2}\theta'\theta+r(1-\gamma)\right\}\right)T\right]-1\right)+\phi_f\exp\left(\frac{\rho T}{\gamma}\right)}.$$

Next, we determine how $C^*(0)$ changes as we vary the following parameters, x, y, γ, r, T, ρ, and σ_D. Hence,

$$\frac{\partial C^*(0)}{\partial x}=\frac{\phi_f\exp\left[\frac{1}{\gamma}\left(\rho-\left\{\frac{1-\gamma}{2\gamma^2}\theta'\theta+r(1-\gamma)\right\}\right)T\right]}{2\gamma^2\left(\exp\left[\frac{1}{\gamma}\left(\rho-\left\{\frac{1-\gamma}{2\gamma^2}\theta'\theta+r(1-\gamma)\right\}\right)T\right]-1\right)+\phi_f\exp\left(\frac{\rho T}{\gamma}\right)};$$

$$\frac{\partial C^*(0)}{\partial y}=\frac{c\phi_f\left(\exp(\bar{\alpha}T)-1\right)\exp\left(\dfrac{r(\gamma-1)-\dfrac{(1-\gamma)\theta'\theta}{2\gamma^2}+\rho}{\gamma}T\right)}{\bar{\alpha}\left[2\gamma^2\left(\exp\left(\dfrac{r(\gamma-1)-\dfrac{(1-\gamma)\theta'\theta}{2\gamma^2}+\rho}{\gamma}T\right)-1\right)+\phi_f\exp\left(\dfrac{\rho T}{\gamma}\right)\right]};$$

$$\frac{\partial C^*(0)}{\partial \gamma} = \frac{A_1(\gamma)\left(x\bar{\alpha} - cy(1-\exp(\bar{\alpha}T))\right)\exp\left(\frac{(\gamma-1)(2r\gamma^2+\theta'\theta)T}{2\gamma^3} + \frac{\rho T}{\gamma}\right)}{2\bar{\alpha}\gamma^4\left[\left(2r\gamma(\gamma-1) + 2\gamma^2\exp\left(\frac{(\gamma-1)(2r\gamma^2+\theta'\theta)T}{2\gamma^3}\right) + (\gamma-1)\theta'\theta + 2\gamma\rho\right)\exp\left(\frac{\rho T}{\gamma}\right) - 2\gamma^2\right]^2};$$

where,

$$\begin{aligned}A_1(\gamma) = {} & 4\gamma^5\left(2r\gamma - (\gamma-2)\theta'\theta - 2\gamma\rho\right)\exp\left(\frac{(\gamma-1)(2r\gamma^2+\theta'\theta)T}{2\gamma^3} + \frac{\rho T}{\gamma}\right) + \\ & (2r\gamma^2 + (3-2\gamma)\theta'\theta)\left((\gamma-1)(2r\gamma+\theta'\theta) + 2\gamma\rho\right)^2 T\exp\left(\frac{\rho T}{\gamma}\right) + \\ & 2\gamma^2(4r^2T(1-\gamma)\gamma^3 + 3T\theta^4 + 2T\gamma^2\theta'\theta(\theta'\theta+\rho) + 2\gamma^4(\theta'\theta+2\rho) - \\ & \gamma\theta'\theta T(5\theta'\theta+6\rho) + 2\gamma^3(2T\rho^2 + \theta'\theta(\rho T-2)) + 2r\gamma(3T\theta'\theta + \\ & \gamma(\gamma(T\theta'\theta - 2\gamma + 2T(\gamma-2)\rho) - 4T\theta'\theta)))\end{aligned};$$

$$\frac{\partial C^*(0)}{\partial T} = \frac{A_2(T)\phi_f\exp\left((\gamma(\theta'\theta + 2\gamma(\beta\gamma - r + \rho - \gamma\sigma_D\cdot\theta) - \theta'\theta)T\right)}{2\bar{\alpha}\gamma^3\left[2\gamma^2 - 2\gamma^2\left(\exp\left(\frac{(\gamma-1)(2r\gamma^2+\theta'\theta)}{2\gamma^3}T\right) + \phi_f\right)\exp\left(\frac{\rho T}{\gamma}\right)\right]^2};$$

where,

$$\begin{aligned}A_2(T) = {} & -4\bar{\alpha}cy\gamma^5\exp\left(\frac{(\gamma-1)(2r\gamma^2+\theta'\theta)}{2\gamma^3}T + \frac{\rho T}{\gamma}\right) - \\ & 2\gamma^2((\gamma-1)(2r\gamma^2+\theta'\theta) + 2\gamma^2\rho)(cy-\bar{\alpha}x)\exp(-\bar{\alpha}T) + \\ & 2cy\gamma^2\left(\gamma(\theta'\theta + 2\gamma(\beta\gamma - r + \rho - \gamma\sigma_D\cdot\theta)) - \theta'\theta\right) - \\ & cy\phi_f\left(2\beta\gamma^3 - 2r\gamma^2 + (\gamma-1)\theta'\theta - 2\gamma^3\sigma_D\cdot\theta\right)\exp\left(\frac{\rho T}{\gamma}\right) \\ & +\phi_f(\gamma-1)(2r\gamma^2+\theta'\theta)(cy-\bar{\alpha}x))\exp\left((\frac{\rho}{\gamma} - \bar{\alpha})T\right)\end{aligned},$$

$$\frac{\partial C^*(0)}{\partial \rho} = \frac{2A_3(\rho)\gamma\left(cy - (cy-\bar{\alpha}x)\exp(-\bar{\alpha}T)\right)\exp\left(\frac{\gamma(\theta'\theta + 2\gamma(\beta\gamma - r + \rho - \gamma\sigma_D\cdot\theta)) - \theta'\theta)T}{2\gamma^3}\right)}{\bar{\alpha}\left[2\gamma^2 - \left(2r\gamma(\gamma-1) + 2\gamma^2\exp\left(\frac{(\gamma-1)(2r\gamma^2+\theta'\theta)}{2\gamma^3}T\right) + (\gamma-1)\theta'\theta + 2\gamma\rho\right)\exp\left(\frac{\rho T}{\gamma}\right)\right]^2},$$

where,

$$A_3(\rho) = \left(2r\gamma(\gamma-1) - \theta'\theta + \gamma(\theta'\theta+2\rho)\right)T - 2\gamma^2\left(\exp\left(\frac{(\gamma-1)(2r\gamma^2+\theta'\theta)}{2\gamma^3} + \frac{\rho T}{\gamma}\right) - 1\right),$$

$$\frac{\partial C^*(0)}{\partial r} = \frac{A_4(r)\exp\left(\frac{(\gamma-1)(2r\gamma^2+\theta'\theta)T}{2\gamma^3}+\frac{\rho T}{\gamma}\right)\left((1-\bar{\alpha}T)\exp(\beta T)-\exp((r+\sigma_D\cdot\theta)T)\right)}{\left[-2\gamma^2+\exp\left(\frac{\rho T}{\gamma}\right)\left(2r\gamma(\gamma-1)+2\gamma^2\exp\left(\frac{(\gamma-1)(2r\gamma^2+\theta'\theta)T}{2\gamma^3}\right)+(\gamma-1)\theta'\theta+2\gamma\rho\right)\right]^2},$$

where,

$$\begin{aligned}
&A_4(r) = 2\gamma(\gamma-1)B_1(r)+\frac{T}{\gamma}(\gamma-1)\left((\gamma-1)(2r\gamma+\theta'\theta)+2\gamma\rho\right)B_1(r)+\\
&\frac{2\exp\left(\frac{\rho T}{\gamma}\right)}{\bar{\alpha}}\left(1+T\exp\left(\frac{(\gamma-1)(2r\gamma^2+\theta'\theta)T}{2\gamma^3}\right)\right)(\gamma-1)\gamma\times \\
&\left((\gamma-1)(2r\gamma+\theta'\theta)+2\gamma\rho\right)\left(cy(1-\exp(\bar{\alpha}T))-\bar{\alpha}x\right)-\\
&\frac{cy}{\bar{\alpha}}\exp\left(-(r+\sigma_D\cdot\theta)T\right)\left((\gamma-1)(2r\gamma+\theta'\theta)+2\gamma\rho\right)B_2(r)
\end{aligned},$$

$$B_1(r) = B_2(r)\left(x-\frac{cy(1-\exp(\bar{\alpha}T))}{\bar{\alpha}}\right),$$

$$B_2(r) = -2\gamma^2+\exp\left(\frac{\rho T}{\gamma}\right)\left(2r\gamma(\gamma-1)+2\gamma^2\exp\left(\frac{(\gamma-1)(2r\gamma^2+\theta'\theta)T}{2\gamma^3}\right)+(\gamma-1)\theta'\theta+2\gamma\rho\right);$$

$$\frac{\partial C^*(0)}{\partial\sigma_D} = \begin{aligned}&\frac{cy\theta^Q}{\bar{\alpha}^2B_2(r)}\exp\left(-\frac{(\theta'\theta+\gamma(-\theta'\theta+2\gamma(r-\rho+\gamma\sigma_D\cdot\theta)T}{2\gamma^3}\right)\times\\ &\left((\gamma-1)(2r\gamma+\theta'\theta)+2\gamma\rho\right)\left(\exp(\beta T)(1-\bar{\alpha}T)-\exp((r+\sigma_D\cdot\theta)T)\right)\end{aligned}.$$

8. Conclusion

The paper has presented the value of the PPM's wealth who made contributions continuously into the pension scheme and consume part of the wealth at time $t \leq T$. Analytical solution to the resulting Hamilton-Jacobi-Bellman (HJB) equation was obtained. The optimal consumption and variational form of Merton portfolio demand for stocks, inflation-linked bonds and cash account were obtained. The optimal portfolio values for inflation-linked bonds was found to includes an inter-temporal hedging term that offset any shock to the stochastic contribution of the PPM. It was discovered that the terminal consumption depend on the initial wealth, the present value of future contributions, coefficients of relative risk aversion and the discount rate which captures the PPM's preference over time.

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