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A WAVELET OPERATIONAL MATRIX METHOD FOR SOLVING INITIAL - BOUNDARY VALUE PROBLEMS FOR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. A fractional partial differential equation (FPDE) is a partial differential equation which involves fractional calculus operators. In this paper, the numerical solutions of Initial - Boundary value problems for FPDEs have been approximated using Haar wavelet operational matrix method. The FPDEs are reduced into simple algebraic equations which can be solved easily by computer aided techniques. The simplicity and effectiveness of the proposed method are illustrated by providing several examples with numerical simulations.

Keywords: Fractional calculus; Fractional partial differential equation; Wavelet analysis; Haar wavelets; Operational matrix.

2010 AMS Subject Classification: 42C40, 26A33, 34A08, 34K37.

1. Introduction

In recent years, FPDEs like fractional convection - diffusion equations, Space - time fractional advection - dispersion equation etc. have attracted the attentions of many researchers and are

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becoming increasingly popular due to their applications in various fields of science and engineering [10]. The FPDEs have shown to be adequate models for various physical phenomena in areas such as damping laws, diffusion processes etc. Other applications include arterial science, electromagnetism, electro - chemistry, the theory of Ultra - slow processes and finance.

Further it is worthwhile to mention that, a small number of algorithms for the numerical solution of FPDEs have been suggested. The need to exploit some efficient and reliable numerical scheme is a problem of fundamental interest. In the literature, small numbers of numerical methods have been proposed for obtaining approximate solutions to FPDEs [1, 4, 5]. The aim of all these studies has been to obtain effective algorithms that are suitable for the digital computer. But all these methods have their own advantages, disadvantages and limitations in finding the solution of FPDEs.

Recently, orthogonal wavelet bases are becoming more popular for numerical solutions of partial differential equations. It is because of their excellent properties such as, ability to detect singularities, flexibility to represent a function at different levels of resolution and compact support. In recent years, there has been a growing interest in developing Wavelet based numerical algorithms for solutions of FPDEs.

The Haar wavelet is the simplest example of orthogonal wavelets, compactly supported on the interval [0, 1). Historically Chen and Hsiao [2], first proposed a Haar operational matrix for the integration of Haar function vectors and used it for solving differential equations. Recently, there has been some considerable developments in the application of Wavelet basis to solve differential equations of fractional order [3, 9, 11].

In this paper, our purpose is to provide a new numerical scheme based on the operational matrices of orthogonal functions to solve the initial and boundary value problems for FPDEs. The proposed scheme transforms the problem into a simple linear algebraic system. Some examples solved by the proposed method are demonstrated. Some of the advantages of the proposed operational method are; the computation is simple and computer oriented; the scope of application is wide and results obtained are very accurate and reliable.

2. Preliminaries and Basic concepts

2.1 Fractional derivative and integral. The caputo fractional derivative of order $\alpha > 0$, is given as [7],

$$D_t^{\alpha} f(t) = \begin{cases} \frac{d^m}{dt^m} f(t) & \alpha = m \in \mathbb{N} \\ \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^m(z)}{(t-z)^{\alpha-m+1}} dz, & 0 \leq m-1 < \alpha < m \end{cases}$$

Partial Caputo fractional derivative of $y(x,t) \in C^n\big([0,1]\times[0,1]\big)$ is defined as,

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} y(x,t) = \begin{cases} I_t^{n-\alpha} \frac{\partial^n}{\partial t^n} y(x,t), & n-1 < \alpha \leq n \\ \frac{\partial^n}{\partial t^n} y(x,t), & \alpha = n \in \mathbb{N} \end{cases}$$

where I_t^{α} is the Riemann - Liouville fractional integral given as,

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-z)^{\alpha-1} f(z) dz$$

with $I_t^0 f(t) = f(t)$.

Fractional differential and integral operators satisfy following basic properties,

(P_1) $I_t^{\alpha} I_t^{\beta} f(t) = I_t^{\alpha+\beta} f(t)$

(P_2) $\frac{\partial^{\alpha}}{\partial t^{\alpha}} I_t^{\beta} y(x,t) = I_t^{\beta-\alpha} y(x,t)$

(P_3) $I_t^{\alpha} \frac{\partial^{\alpha}}{\partial t^{\alpha}} y(x,t) = y(x,t) - \sum_{i=0}^{n-1} \frac{t^i}{i!} \left(\frac{\partial^i}{\partial t^i} y(x,t) \right)_{t=0}$

2.2 Haar wavelets and Function approximation. Haar mother wavelet is defined as

$$h(x) = \emptyset(2x) - \emptyset(2x-1)$$

where $\emptyset(x)$ is a scaling function defined as

$$\emptyset(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & otherwise \end{cases}$$

Now define

$$h_{j,k}(x) = 2^{j/2} h(2^j x - k), \ \ 0 \leq k < 2^j, \ \ j = 0,1,2,---$$

then for the Hilbert space $L_2[0,1]$, the Haar system $\{\emptyset, h_{j,k} : 0 \leq j, 0 \leq k < 2^j\}$ forms an orthonormal basis.

Hence for some fixed J we may consider the inner product expansion of $f \in L_2[0,1]$ as,

$$f(x) = \langle f, \emptyset \rangle \emptyset(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle f, h_{j,k} \rangle h_{j,k}(x) = K^T H(x) \tag{1}$$

where K is 1×2^J coefficient vector and $H(x)$ is the Haar function vector given by,

$$H(x) = [\emptyset, h, h_{1,0}, h_{1,1}, h_{2,0}, ---, h_{2,3}, ---, h_{J-1,0}, ---, h_{J-1,2^J-1}]^T$$

For our convenience let us denote $\emptyset = h_0,\ h = h_1 = h_{0,0}$ and $h_i = h_{j,k}$ where the index i can be obtained from the relation $i = 2^j + k, (i = 0, 1, 2, ---, m-1 : m = 2^J)$. Hence (1) reduced into

$$f(x) = \sum_{i=0}^{m-1} k_i h_i(x) = K^T H(x)$$

Now on using Haar wavelets, a function $y(x,t) \in L_2[0,1] \times [0,1]$ can be approximated as,

$$y(x,t) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} c_{i,j} h_i(x) h_j(t) = H^T(x) C H(t)$$

where C is $2^J \times 2^J$ coefficient matrix determined by the inner product $c_{i,j} = \langle h_i(x), \langle y(x,t), h_j(t) \rangle \rangle$.

Now on using Haar wavelets, the fractional integral of a function $f \in L_2[0,1]$ can be given as,

$$\begin{aligned}
I_x^\alpha f(x) &= C^T I_x^\alpha H(x) = C^T \left[I_x^\alpha h_0(x), I_x^\alpha h_1(x), ---, I_x^\alpha h_{m-1}(x) \right]^T \\
&= C^T \Big[\sum_{i=0}^{m-1} A_{0,i} h_i(x), \sum_{i=0}^{m-1} A_{1,i} h_i(x), ---, \sum_{i=0}^{m-1} A_{m-1,i} h_i(x) \Big]^T \\
&= C^T \begin{pmatrix} A_{0,0}^\alpha & A_{0,1}^\alpha & --- & A_{0,m-1}^\alpha \\ A_{1,0}^\alpha & A_{1,1}^\alpha & --- & A_{1,m-1}^\alpha \\ - & - & --- & - \\ - & - & --- & - \\ A_{m-1,0}^\alpha & A_{m-1,1}^\alpha & --- & A_{m-1,m-1}^\alpha \end{pmatrix} \begin{pmatrix} h_0(x) \\ h_1(x) \\ - \\ - \\ h_{m-1}(x) \end{pmatrix} \\
&= C^T A_\alpha H(x)
\end{aligned}$$

where A_α is the operational matrix of fractional order integration for the Haar function vector $H(x)$.

3. Proposed Method

In this section, the properties of Haar wavelets and the operational matrix are used to approximate the numerical solutions of Initial - Boundary value problem for FPDEs of the form,

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}y(x,t)-a(x)\frac{\partial^{\beta}}{\partial x^{\beta}}y(x,t)+b(x)\frac{\partial^{\gamma}}{\partial x^{\gamma}}y(x,t)+d(x)y(x,t)=f(x,t) \tag{2}$$

where $0<\alpha\leq 2,\ 1<\beta\leq 2,\ 0<\gamma\leq 1$ with initial conditions,

$$\left.\begin{array}{ll} (i)\ \ y(x,0)=p(x), & \left(\frac{\partial}{\partial t}y(x,t)\right)_{t=0}=q_1(x) \\ & \text{OR} \\ (ii)\ \ y(x,0)=p(x), & y(x,1)=q_2(x) \end{array}\right\} \tag{3}$$

and boundary conditions,

$$y(0,t)=\emptyset(t),\ y(1,t)=w(t) \tag{4}$$

Now using the properties of Haar wavelets, we have

$$\frac{\partial^{\beta}}{\partial x^{\beta}}y(x,t)=H^{T}(x)CH(t) \tag{5}$$

On using the property (P3), we have

$$y(x,t)=I_x^{\beta}H^{T}(x)CH(t)+g(t)x+u(t) \tag{6}$$

Now on using boundary conditions, we get $u(t)=\emptyset(t)$ and

$$g(t)=-\left(\int_0^1\frac{(1-z)^{\beta-1}}{\Gamma(\beta)}H^{T}(z)dz\right)CH(t)+w(t)-\phi(t)$$

So (6) becomes,

$$y(x,t)=I_x^{\beta}H^{T}(x)CH(t)-x\left(\int_0^1\frac{(1-z)^{\beta-1}}{\Gamma(\beta)}H^{T}(z)dz\right)CH(t)+x\big(w(t)-\phi(t)\big)+\phi(t) \tag{7}$$

Now the Haar wavelets $\psi_{j,k};j=0,1,2,---,J;0\leq k<2^{j}$ are supported on dyadic intervals $I_{j,k}=\left[\frac{k}{2^j},\frac{k+1}{2^j}\right]$ of $[0,1]$. Hence

$$\int_0^1\frac{(1-z)^{\beta-1}}{\Gamma(\beta)}h_{j,k}(z)dz=\int_{I_{j,k}}\frac{(1-z)^{\beta-1}}{\Gamma(\beta)}h_{j,k}(z)dz$$

Now suppose, $I^{0-}_{j,k} = \left[\frac{k}{2^j}, \frac{2k+1}{2^{j+1}}\right]$ and $I^{0+}_{j,k} = \left[\frac{2k+1}{2^{j+1}}, \frac{k+1}{2^j}\right]$ are the left and right halves of $I_{j,k}$ So we have,

$$\int_0^1 \frac{(1-z)^{\beta-1}}{\Gamma(\beta)} h_{j,k}(z)dz = \int_{I^{0-}_{j,k}} \frac{(1-z)^{\beta-1}}{\Gamma(\beta)} h_{j,k}(z)dz - \int_{I^{0+}_{j,k}} \frac{(1-z)^{\beta-1}}{\Gamma(\beta)} h_{j,k}(z)dz$$

Now

$$u^{\beta}_{\emptyset} = \int_0^1 \frac{(1-z)^{\beta-1}}{\Gamma(\beta)} \phi(z)dz = \frac{1}{\Gamma(\beta+1)}$$

and

$$u^{\beta}_{j,k} = \int_0^1 \frac{(1-z)^{\beta-1}}{\Gamma(\beta)} h_{j,k}(z)dz = 2\left(1-\frac{2k+1}{2^{j+1}}\right)^{\beta} - \left(1-\frac{k}{2^j}\right)^{\beta} - \left(1-\frac{k+1}{2^j}\right)^{\beta}$$

So we have,

$$\begin{aligned}\int_0^1 \frac{(1-z)^{\beta-1}}{\Gamma(\beta)} H(z)dz &= \left[u^{\beta}_{\emptyset}, u^{\beta}_{0,0}, u^{\beta}_{1,0}, u^{\beta}_{1,1}, u^{\beta}_{2,0}, ---, u^{\beta}_{2,3}, ---, u^{\beta}_{j-1,0}, ---, u^{\beta}_{j-1,2^j-1}\right]^T \\ &= \left[u_0, u_1, ---, u_{m-1}\right]^T\end{aligned}$$

Now suppose, $\nu : [0,1] \to \mathbb{R}$ is a continuous function. At collocation points $x_i = \frac{2i+1}{2^J+1}, i = 0,1,---,m-1$ define a matrix $B_{\beta,\upsilon}$ as given in [8],

$$B_{\beta,\upsilon} = \begin{pmatrix} \nu(x_0)u_0 & \nu(x_1)u_0 & --- & \nu(x_{m-1})u_0 \\ \nu(x_0)u_1 & \nu(x_1)u_1 & --- & \nu(x_{m-1})u_1 \\ - & - & --- & - \\ - & - & --- & - \\ - & - & --- & - \\ \nu(x_0)u_{m-1} & \nu(x_1)u_{m-1} & --- & \nu(x_{m-1})u_{m-1} \end{pmatrix}$$

Hence by (7),

$$y(x,t) = H^T(x)A^T_{\beta}CH(t) - B^T_{\beta,\nu}CH(t) + x\big(w(t)-\phi(t)\big) + \phi(t) \tag{8}$$

where $\upsilon(x) = x$ and A_{β} is the Haar wavelet operational matrix of fractional integration of order $\beta > 0$. Now on applying Caputo operator $\frac{\partial^{\gamma}}{\partial x^{\gamma}}$ on (6) and using the property of Caputo derivative and property (P2), we have

$$\frac{\partial^\gamma}{\partial x^\gamma}y(x,t) = I_x^{\beta-\gamma}H^T(x)CH(t) - \left(\int_0^1 \frac{x^{1-\gamma}(1-z)^{\beta-1}}{\Gamma(\beta)\Gamma(2-\gamma)}H^T(z)dz\right)CH(t) + \frac{x^{1-\gamma}\big(w(t)-\phi(t)\big)}{\Gamma(2-\gamma)} \tag{9}$$

Now on substituting (5), (7) and (9) in (2), we have

$$\begin{aligned}\frac{\partial^\alpha}{\partial t^\alpha}y(x,t) &= a(x)H^T(x)CH(t) - b(x)H^T(x)A^T_{\beta-\gamma}CH(t) - \frac{\big(w(t)-\phi(t)\big)b(x)x^{1-\gamma}}{\Gamma(2-\gamma)} \\ &\quad - d(x)\phi(t) + \Big[\int_0^1 \frac{b(x)x^{1-\gamma}(1-z)^{\beta-1}}{\Gamma(\beta)\Gamma(2-\gamma)}H^T(z)dz\Big]CH(t) - xd(x)\big(w(t)-\phi(t)\big) \\ &\quad - d(x)H^T(x)A^T_\beta CH(t) + \Big[\int_0^1 \frac{xd(x)(1-z)^{\beta-1}}{\Gamma(\beta)}H^T(z)dz\Big]CH(t)\end{aligned}$$

Now on using the following notations,

$$\tau(x) = \frac{b(x)x^{1-\gamma}}{\Gamma(2-\gamma)},\ \theta(x) = xd(x),\ r(x,t) = x\big(w(t)-\phi(t)\big)+\phi(t)+q_1(x)t+p(x)$$

$$e(x,t) = \frac{\big(w(t)-\phi(t)\big)b(x)}{\Gamma(2-\gamma)}x^{1-\gamma} + xd(x)\big(w(t)-\phi(t)\big) + d(x)\phi(t)$$

$$s(x,t) = -x\big(w(t)-\phi(t)\big) - \phi(t) + \big(q_2(x)-p(x)\big)t + p(x)$$

We have,

$$\begin{aligned}\frac{\partial^\alpha}{\partial t^\alpha}y(x,t) &= a(x)H^T(x)CH(t) - b(x)H^T(x)A^T_{\beta-\gamma}CH(t) \\ &\quad + \Big[\int_0^1 \frac{\tau(x)(1-z)^{\beta-1}}{\Gamma(\beta)}H^T(z)dz\Big]CH(t) - d(x)H^T(x)A^T_\beta CH(t) \\ &\quad + \Big[\int_0^1 \frac{\theta(x)(1-z)^{\beta-1}}{\Gamma(\beta)}H^T(z)dz\Big]CH(t) - e(x,t) \\ &= a(x)H^T(x)CH(t) - b(x)H^T(x)A^T_{\beta-\gamma}CH(t) + B^T_{\beta,\tau}CH(t) \\ &\quad - d(x)H^T(x)A^T_\beta CH(t) + B^T_{\beta,\theta}CH(t) + e(x,t)\end{aligned} \tag{10}$$

Operating on both sides of (10) by I_t^{α} and using (P3) along with two dimensional Haar wavelet series for $(x,t) = H^T(x)EH(t)$, we have

$$\begin{aligned} y(x,t) = &\Big[a(x)H^T(x) - b(x)H^T(x)A^T_{\beta-\gamma} + B^T_{\beta,\tau} - d(x)H^T(x)A^T_{\beta} + B^T_{\beta,\theta}\Big]CI_t^{\alpha}H(t) \\ &+ \sigma(x)t + \varepsilon(x) + H^T(x)EI_t^{\alpha}H(t) \end{aligned} \tag{11}$$

Now since $I_t^{\alpha}H(t) = A_{\alpha}H(t)$ and using initial conditions (i) of (3), we have, $\varepsilon(x) = p(x)$ and $\sigma(x) = q_1(x)$. Using this in (11), we get

$$\begin{aligned} y(x,t) = &\Big[a(x)H^T(x) - b(x)H^T(x)A^T_{\beta-\gamma} + B^T_{\beta,\tau} - d(x)H^T(x)A^T_{\beta} + B^T_{\beta,\theta}\Big]CA_{\alpha}H(t) \\ &+ q_1(x)t + p(x) + H^T(x)EA_{\alpha}H(t) \end{aligned} \tag{12}$$

Now in (12) on using the boundary conditions (ii) of (3), we get $\varepsilon(x) = p(x)$ and

$$\begin{aligned} \sigma(x) = q_2(x) - p(x) - &\Big[\big\{a(x)H^T(x) - b(x)H^T(x)A^T_{\beta-\gamma} + B^T_{\beta,\tau} - d(x)H^T(x)A^T_{\beta} \\ &+ B^T_{\beta,\theta}\big\}C + H^T(x)E\Big]\Big[\int_0^1 \frac{(1-z)^{\alpha-1}}{\Gamma(\alpha)}H(z)dz\Big] \end{aligned} \tag{13}$$

On substituting the values of $\sigma(x)$ and $\varepsilon(x)$ in (11), we have

$$\begin{aligned} y(x,t) = [q_2(x) - p(x)]t - &\Big[\big\{a(x)H^T(x) - b(x)H^T(x)A^T_{\beta-\gamma} + B^T_{\beta,\tau} - d(x)H^T(x)A^T_{\beta} \\ &+ B^T_{\beta,\theta}\big\}C + H^T(x)E\Big]\Big(A_{\alpha}H(t) - B_{\alpha,\nu}\Big) + p(x) \end{aligned} \tag{14}$$

Now define following diagonal matrices,

$$L=\begin{pmatrix} a(x_1) & 0 & --- & 0 \\ 0 & a(x_2) & --- & 0 \\ - & - & --- & - \\ - & - & --- & - \\ 0 & 0 & --- & a(x_m) \end{pmatrix}, M=\begin{pmatrix} b(x_1) & 0 & --- & 0 \\ 0 & b(x_2) & --- & 0 \\ - & - & --- & - \\ - & - & --- & - \\ 0 & 0 & --- & b(x_m) \end{pmatrix},$$

$$N=\begin{pmatrix} d(x_1) & 0 & --- & 0 \\ 0 & d(x_2) & --- & 0 \\ - & - & --- & - \\ - & - & --- & - \\ 0 & 0 & --- & d(x_m) \end{pmatrix}$$

where $x_i = x\left(\frac{2i-1}{2m}\right)$, $i = 1,2,---,m$. Now from (8) and (12), we have

$$(15) \qquad EA_\alpha + R = \left(A_\alpha^T - HB_{\beta,\nu}^T\right)C - H\left(LH^T - MH^T A_{\beta-\gamma}^T - NH^T A_\beta^T + B_{\beta,\tau}^T + B_{\beta,\theta}^T\right)CA_\alpha$$

Solving this eq. (15) we can obtain C. Now on using (8) or (12), the approximate solutions of $y(x,t)$ can be obtained at the collocation points.

4. Numerical examples

In this section, we apply the proposed Haar wavelet operational matrix method to approximate the numerical solutions of some FPDEs.

Example 1. Consider the initial - boundary value problem of FPDE of order $\alpha, (0 < \alpha \leq 1)$ [6]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + x\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = 2t^\alpha + 2x^2 + 2, \;\; 0 < t < 1, \;\; 0 < x < 1$$

where initial - boundary conditions are given as, $u(x,0) = x^2$, $u(x,1) = x^2 + 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}$ and $u(0,t) = 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}$, $u(1,t) = 1 + 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}$

Taking $J = 4$, $J = 5$, $J = 6$ and using proposed method, we obtain the numerical results of $u(x,t)$ for $\alpha = 0.5$ which are shown in figures 1 - 3. When $\alpha = 0.5$ the exact solution of the above problem is $u(x,t) = x^2 + 2\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}$. Figure 4 shows the exact solution for $\alpha = 0.5$. From figure 1 - 4, we can see that with J increasing, the numerical solution more and more close to the exact solution. Table 1, shows the absolute errors for $\alpha = 0.5$, $t = 0.5$ and different values of J for example 1.

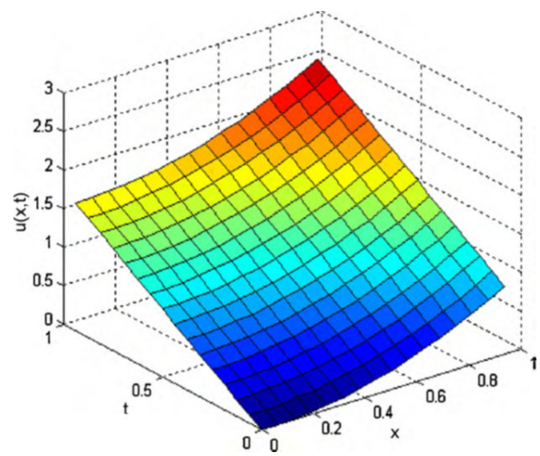

Figure 1. Numerical solution for J = 4

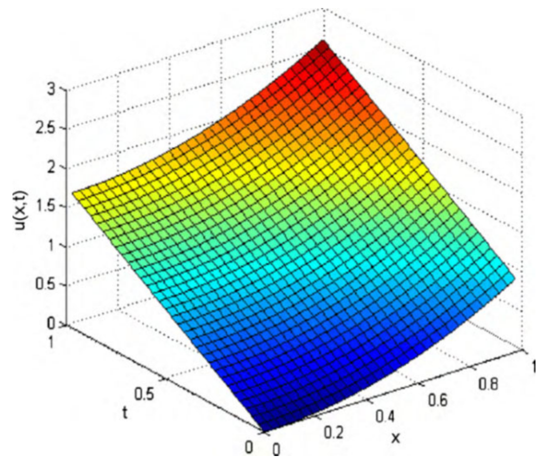

Figure 2. Numerical solution for J = 5

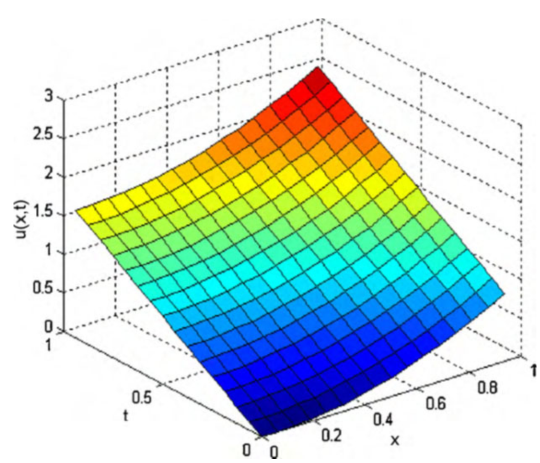

Figure 3. Numerical solution for J = 6

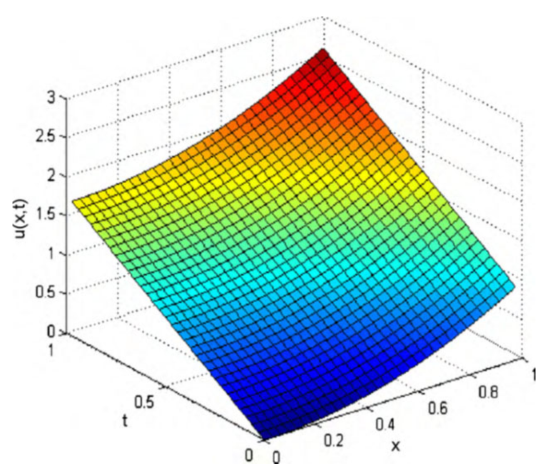

Figure 4. Exact solution for α = 0.5

Table 1. Absolute error for $\alpha = 0.5$, $t = 0.5$ and different values of J for example 1.

x	$J=4$	$J=5$	$J=6$
0.1	6.039×10^{-3}	6.037×10^{-3}	1.141×10^{-3}
0.2	4.181×10^{-3}	4.179×10^{-3}	1.245×10^{-3}
0.3	2.826×10^{-2}	2.527×10^{-2}	1.579×10^{-3}
0.4	1.787×10^{-2}	1.243×10^{-2}	4.327×10^{-3}
0.5	1.000×10^{-6}	1.000×10^{-6}	1.000×10^{-6}
0.6	5.854×10^{-2}	3.276×10^{-2}	6.385×10^{-3}
0.7	2.723×10^{-2}	2.349×10^{-2}	1.834×10^{-3}
0.8	4.854×10^{-2}	3.834×10^{-2}	3.672×10^{-3}
0.9	5.283×10^{-2}	3.754×10^{-2}	1.047×10^{-2}

Example 2. Consider the space - time fractional convection - diffusion equation,

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -b(x)\frac{\partial u}{\partial x} + a(x)\frac{\partial^\beta u}{\partial x^\beta} + q(x,t), \ \ 0 \le x \le 1, \ \ 0 < t \le 1$$

where $a(x) = \Gamma(2.8)\frac{x}{2}$, $b(x) = x^{0.8}$, $q(x,t) = \frac{2x^2(1-x)t^{1.2}}{\Gamma(2.2)} + 0.2x^{1.8}(1+t^2)$ and initial - boundary conditions are $u(x,0) = x^2(1-x)$, $u(x,1) = 2x^2(1-x)$, $u(0,t) = 0$, $u(1,t) = 0$ Now we apply proposed method to obtain numerical solutions of the given problem. The absolute error for $\alpha = 0.8$, $\beta = 1.5$, $t = 0.2$ and $J = 4$, $J = 5$, $J = 6$ are shown in Table 2. When $\alpha = 0.8$, $\beta = 1.5$ the exact solution of the above problem is $u(x,t) = x^2(1-x)(1+t^2)$. From Table 2, we see that we can achieve a good approximation with the exact solution by using the proposed method and with J increasing, the absolute error became more and more small. From Figure 5 we can find the numerical solutions which are in good agreement with exact solutions.

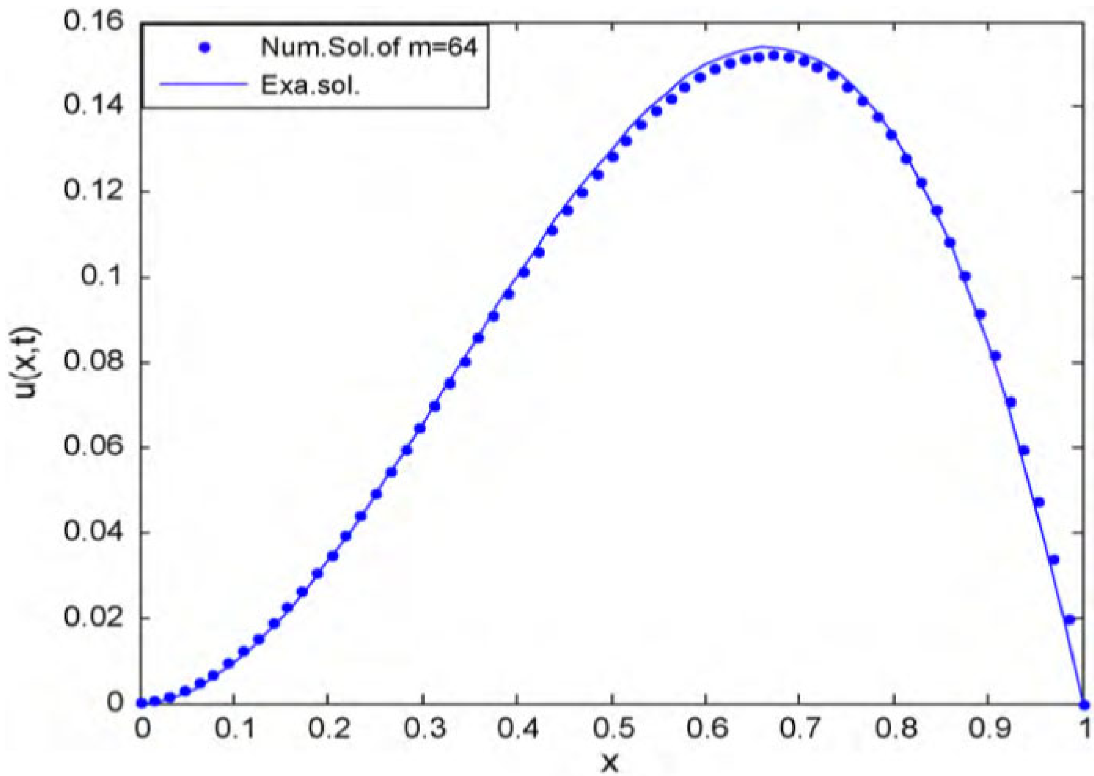

Figure 5. Comparison of numerical and exact solution for $t = 0.2$ and $J = 6$.

Table 2. Exact solution at $t = 0.2$ and absolute error for different values of J for example 2.

x	$J = 4$	$J = 5$	$J = 6$	Exact Solution
0.1	5.234×10^{-3}	1.032×10^{-4}	1.000×10^{-4}	0.0094
0.2	3.273×10^{-3}	1.971×10^{-4}	1.045×10^{-4}	0.0333
0.3	1.023×10^{-2}	8.527×10^{-3}	3.273×10^{-4}	0.0655
0.4	1.003×10^{-2}	7.253×10^{-3}	2.473×10^{-3}	0.0998
0.5	1.200×10^{-3}	1.020×10^{-3}	1.000×10^{-3}	0.1300
0.6	6.527×10^{-3}	3.952×10^{-3}	1.287×10^{-3}	0.1498
0.7	2.323×10^{-3}	1.619×10^{-3}	5.925×10^{-4}	0.1529
0.8	1.874×10^{-2}	1.352×10^{-2}	3.529×10^{-3}	0.1331
0.9	4.583×10^{-2}	2.034×10^{-2}	7.145×10^{-3}	0.0842

5. Conclusion

This paper presents a numerical scheme based on Haar wavelets and operational matrices of fractional order integration for solving numerically FPDEs with variable coefficients supplemented with initial and boundary conditions. The numerical examples show that the proposed method is very effective, accurate and easy to apply because it is computer oriented.

Conflicts of Interests

The authors declare that there is no conflict of interests.

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