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SOME MAPPINGS ON PRODUCTS OF GENERALIZED G-METRIC SPACES

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Abstract. In this paper, we introduce a new class of mappings in general metric spaces and study some of their properties.

Keywords: metric space; G-metric space; G-Cauchy and G-convergent sequences; fixed point and G-contractive mappings.

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1. Introduction and notations

Metric spaces are playing an increasing role in mathematics and the applied sciences. Over the past two decades the development of fixed point theory in metric spaces has attracted considerable attention due to numerous applications in areas such as variational and linear inequalities, optimization, and approximation theory. Different generalizations of the notion of a metric space have been proposed by Gahler and by Dhage. The notion of D-metric space is a generalization of usual metric spaces and it is introduced by Dhage [8, 9]. Recently, Mustafa and Sims [10, 11, 12] have shown that most of the results concerning Dhage's D-metric spaces are invalid.

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In [10, 13, 14, 15], they introduced a improved version of the generalized metric space structure which they called G-metric spaces. For more results on G-metric spaces and fixed point results, one can refer to the papers [1,2,3,4,5,6,7,8,9,13,14,15,16,18].

Before giving our main result we recall some of the basic concepts and results for G-metric spaces.

Definition 1.1. ([11]) *Let X be an non-empty set and let $G : X \times X \times X \longrightarrow \mathbb{R}_+$ be a function satisfying the following conditions*

(1) $G(x,y,z) = 0$ *if* $x = y = z$

(2) $0 < G(x,x,y)$ *for all* $x,y \in X$ *with* $x \neq y$

(3) $G(x,x,y) \leq G(x,y,z)$ *for all* $x,y,z \in X$ *with* $y \neq z$.

(4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \ldots$ *(symmetry in all three variables)*

(5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ *(for all* $x,y,z,a \in X$. *(rectangle inequality)*

Then the function G is called a generalized metric or,more specifically,a G-metric on X and the pair (X,G) *is called a G-metric space.*

Remark 1.1. *Axiom* 4. *asserts that the value of* $G(x,y,z)$ *is independent of the order of* x,y *and z, and is usually known as the symmetry of G in them.*

Definition 1.2. ([11]) *Let* (X,G) *be a G-metric space, and let* (x_n) *be a sequence of points of* X.

(1) We say that (x_n) *is G-convergent to* $x \in X$ *if* $\lim_{n,m \longrightarrow \infty} G(x_n,x_m,x) = 0$; *that is, for any* $\varepsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $G(x,x_n,x_m) < \varepsilon$, *for all* $n;m \geq n_0$ *.We call x the limit of the sequence* x_n *and write* $x_n \longrightarrow x$ *or* $\lim_{n \longrightarrow \infty} x_n = x$.

(2) The sequence x_n *is said to be a G-Cauchy sequence if, for every* $\varepsilon > 0$, *there is a positive integer* n_0 *such that* $G(x_n,x_m,x_l) < \varepsilon$, *for all* $n,m,l \geq n_0$ *that is, if* $G(x_n,x_m,x_l) \longrightarrow 0$, *as* $n,m,l \longrightarrow 0$.

(3) (X,G) *is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in* (X,G) *is G-convergent in* X.

Definition 1.3. *([11]) Let* (X,G) *and* (X',G') *be two G-metric spaces, and let* $f : (X,G) \longrightarrow$?(X',G') *be a function. Then* f *is said to be G-continuous at a point* $a \in X$ *if and only if given*

$\varepsilon > 0$, there exists δ such that $x, y \in X$; and $G(a,x,y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is Gcontinuous on X if and only if it is G-continuous at all $a \in X$.

Proposition 1.1. ([11] *The self-map T on a G-metric space (X,G) is G-continuous at $x \in X$ if and only if for every sequence $(x_n)_{n\geq 1} \in X$ with $x_n \longrightarrow x$, we have*

$$Tx_n \longrightarrow Tx.$$

Proposition 1.2. ([11] *Let (X,G) ba a G-metric space and let $(x_n)_n$ be a sequence of X. If $x_n \longrightarrow x$ in (X,G) and $x_n \longrightarrow y$ in (X,G), then $x = y$.*

In [1] S. Gahler introduced the following definition of a 2-normed space:

Definition 1.4. *Let X be a real linear space of dimension greater than 1 and let $\|.,\ .\|$ be a real valued function on $X \times X \longrightarrow \mathbb{R}$ satisfying the following four properties:*

(G_1) $\|x\ ,y\| \geq 0$ *and* $\|x\ ,y\|$ *= 0 if and only if if the vectors x and y are linearly dependent;*

(G_2) $\|x\ ,y\| = \|y,\ x\|$

(G_3) $\|x,\ \alpha y\| = |\alpha|\|x\ ,y\|$ *for every real number* α*;*

(G_4) $\|x,\ y+z\| \leq \|x\ ,y\| + \|x\ ,z\|$ *for every* $x, y, z \in X$.

The function $\|.\ ,.\|$ will be called a 2-norm on X and the pair $(X, \|.\ ,.\|)$ a linear 2-normed space.The theory of 2-normed spaces was first introduced by Gähler in 1960 [12]. Since then, various notions in normed spaces have been extended to 2-normed spaces by many authors.

A sequence (x_k) in a 2-normed space $(X, \|.,\ .\|)$ is said to converge to some $a \in X$ in the 2-norm if

$$\lim_{k\longrightarrow\infty} \|x_k - a, u_1\| = 0, \text{ for every } u_1 \in X.$$

A sequence (x_k) in a 2-normed space $(X, \|.,\ .\|)$ is said to be Cauchy with respect to the 2-norm if

$$\lim_{k,l\longrightarrow\infty} \|x_k - x_l, u_1\| = 0, \text{ for every } u_1 \in X.$$

If every Cauchy sequence in X converges to some $a \in X$, then X is said to be complete with respect to the 2-norm. Any complete 2-normed space is said to be 2-Banach space.

Definition 1.5. ([19])*Let X and Y be 2-normed spaces, $z_0 \in X$ and $f : X \longrightarrow Y$ be a mapping. Then f is said to be 2- continuous at z_0 if for every $\varepsilon > 0$, there exists positive real number β such that*

$$\|x - z_0,\ y - z_0\|_X < \beta \ \textit{ implies } \ \|f(x) - f(z_0),\ f(y) - f(z_0)\|_Y < \varepsilon.$$

And f is said to be 2-continuous (on X) if f is 2-continuous at x for all $x \in X$.

Theorem 1.1. *Let X be a linear real space*

(1) If $(X, \|.\|)$ is a 2-normed space, then the map $G : X \times X \times X \longrightarrow \mathbb{R}$ defined by

$$G(x,y,x) = \|x - z, y - z\| \ \textit{ for all} x,y,z \in X$$

is a G-metric.

(2) If (X, G) is a G-metric space, then the map $\|.\,,\,.\| : X \times X \longrightarrow \mathbb{R}$ defined by

$$\|x,\ y\| = G(x,y,0); \ \textit{ for all } \ x,y \in X$$

is a 2-norm.

2. G-METRIC IN PRODUCT OF GENERALIZED METRIC SPACES

Definition 2.1. *Let (X_i, G_i) be an G-metric space for $i = 1,2,...,d$ and let $X = \prod_{1 \le i \le d} X_i$ and $G : X \times X \times X \longrightarrow \mathbb{R}_+$ be a function defined as*

$$G(x,y,z) = \sum_{1 \le i \le d} G_i(x_i, y_i, z_i), \textit{ for } x = (x_k)_{1 \le k \le d}, y = (y_k)_{1 \le k \le d} \textit{ and } z = (z_k)_{1 \le k \le d} \in X.$$

Proposition 2.1. *Let (X_i, G_i) be an G_i-metric space for $i = 1,2,...,d$ and let $X = \prod_{1 \le i \le d} X_i$ and $G : X \times X \times X \longrightarrow \mathbb{R}_+$ be a function such that*

$$G(x,y,z) = \sum_{1 \le i \le d} G_i(x_i, y_i, z_i), \textit{ for } x = (x_i)_{1 \le k \le d}, y = (y_i)_{1 \le k \le d} \textit{ and } z = (z_i)_{1 \le k \le d} \in X.$$

Then (X, G)-is a G-metric space.

Proof. (1) Assume that $x=y=z$ i.e.; $x_k=y_k=z_k$ for $k=1,2,...,d$ and we have

$$G(x,x,x)=\sum_{1\leq k\leq d} G_k(x_k,x_k,x_k)=0 \ \big(\text{since } G_k \text{ is a } G_k\text{-metric for } k=1,2,...,d\big).$$

Thus, $G(x,y,z)=0$ if $x=y=z$.

(2) Let $x=(x_k)_{1\leq k\leq d}, y=(y_k)_{1\leq k\leq d}$ with $x\neq y$ we have that

$$G(x,x,y)=\sum_{1\leq k\leq d} G_k(x_k,x_k,y_k)\geq 0 \quad \text{since each } G_k \text{ is a } G_k\text{-metric.}$$

(3) Let $x=(x_k)_{1\leq k\leq d}, y=(y_k)_{1\leq k\leq d}$ and $z=(z_k)_{1\leq k\leq d}\in X$ with $y\neq z$, we have that

$$G(x,x,y)=\sum_{1\leq k\leq d} G_k(x_k,x_k,y_k)\leq \sum_{1\leq k\leq d} G_k(x_k,y_k,z_k)=G(x,y,z).$$

(4) Let $x=(x_k)_{1\leq k\leq d}, y=(y_k)_{1\leq k\leq d}$ and $z=(z_k)_{1\leq k\leq d}\in X$. Since for $k=1,2,...,d$

$$G_k(x_k,y_k,z_k)=G_k(x_k,z_k,y_k)=G_k(y_k,x_k,z_k)=G_k(y_k,z_k,x_k)=G_k(z_k,y_k,x_k)=G_k(z_k,x_k,y_k)$$

we have that

$$G(x,y,z)=G(x,z,y)=G(y,z,x)=... \ \text{(symmetry in all three variables)}$$

.

(5) Let $x=(x_k)_{1\leq k\leq d}, y=(y_k)_{1\leq k\leq d},\ z=(z_k)_{1\leq k\leq d}$ and $a=(a_k)_{1\leq k\leq d}\in X$

$$\begin{aligned}
G(x,y,z) &= \sum_{1\leq k\leq d} G_k(x_k,y_k,z_k)\\
&\leq \sum_{1\leq k\leq d}\Big(G_k(x_k,a_k,a_k)+G(a_k,y_k,z_k)\Big)\\
&= \sum_{1\leq k\leq d} G_k(x_k,a_k,a_k)+\sum_{1\leq k\leq d} G_k(a_k,y_k,z_k)\\
&\leq G(x,a,a)+G(a,y,z) \ \text{(rectangle inequality).}
\end{aligned}$$

Then the function G is called a generalized metric or,more specifically,a G-metric on X and the pair (X,G) is called a G-metric space.

□

The following are some examples of G-metric spaces.

Example 2.1. *Let X_k be a metric space with the metric d_k for $k = 1,2,...,d$. For all $x = (x_k)_{1\le k\le d}, y = (y_k)_{1\le k\le d}$ and $z = (z_k)_{1\le k\le d} \in X = \prod_{1\le k\le d} X_k$. Define the quantities G_s and G_m by*

$$G_s(x,y,z) = \sum_{1\le k\le d} G_k^s(x_k,y_k,z_k),$$

where

$$G_k^s(x_k,y_k,z_k) = \Big(d_k(x_k,y_k) + d_k(y_k,z_k) + d_k(z_k,x_k)\Big),\ k = 1,2,...,d$$

and

$$G_m(x,y,z) = \sum_{1\le k\le d} G_k^m(x_k,y_k,z_k),$$

where

$$G_k^m(x_k,y_k,z_k) := \max\Big(d_k(x_k,y_k), d_k(y_k,z_k), d_k(z_k,x_k)\Big)\ \textit{for}\ k = 1.2...,d.$$

A simple computation shows that G_m and G_s are G-metrics on X.

Example 2.2. *Let $X = \mathbb{R}^d$ and consider the map $G : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}_+$ define for all $x = (x_k)_{1\le k\le d}, y = (y_k)_{1\le k\le d}$ and $z = (z_k)_{1\le k\le d} \in \mathbb{R}^d$ by*

$$G(x,y,z) = \begin{cases} 0, & \textit{if}\ x = y = z \\ \\ \sum_{1\le k\le d} \big(|x_k| + |y_k| + |z_k|\big) & \textit{otherwise} \end{cases}$$

then it is readily checked that (X,G) is a G-metric space.

Theorem 2.1. *Let (X_k,G_k) be an complete G_k-metric space for $k = 1,2,...,d$. Then the Space (X,G) where $X = \prod_{1\le k\le d} X_k$ and $G = \sum_{1\le k\le d} G_k$ is a complete G-metric space.*

Proof. By Proposition 2.1 ,(X,G) is a G-metric space. To prove that is a G-complete, let $\big(x_n = (x_{1,n}, x_{2,n}, ..., x_{d,n})\big)_{n\ge 1}$ be a G-Cauchy sequence in X i.e.;

$$G(x_n,x_m,x_l) \longrightarrow 0 \quad \text{as} \quad n,m,l \longrightarrow \infty.$$

Consider the sequence $(x_{k,n})_{n\ge 1} \subset X_k$ for a fixed $k : 1 \le k \le d$.

Since

$$G_k(x_{k,n},x_{k,m},x_{k,l}) \le G(x_n,x_m,x_l),$$

it follows that $(x_{k,n})_n$ is a G_k-Cauchy sequence in X_k, and therefore tends to a limit $x_k \in X_k$ (since X_k is G-complete) i.e.;

$$G_k(x_{k,n}, x_{k,m}, x_k) \longrightarrow 0 \text{ as } n,m \longrightarrow \infty.$$

It therefore suffices to confirm that

$$x_n \longrightarrow x := (x_1, x_2, ..., x_d) \in X = \prod_{1\leq k\leq d} X_k.$$

It is know that there exists a positive integer n_k such that

$$G_k(x_{k,n}, x_{k,m}, x_k) < \frac{\varepsilon}{d} \text{ for } n,m \geq n_k \text{ and for some } \varepsilon > 0.$$

We deduce that for $n,m \geq \max\limits_{1\leq k\leq d}(n_k)$

$$G(x_n, x_m, x) = \sum_{1\leq k\leq d} G_k(x_{k,n}, x_{k,m}, x_k) < \sum_{1\leq k\leq d} \frac{\varepsilon}{d} = \varepsilon.$$

$$x_n \longrightarrow_G x$$

$\square$

Definition 2.2. *Let (X,G) be an G-metric space. An map $T : X \longrightarrow X$ is said to be*

(1) G-isometric if

$$G(Tx, Ty, Tz) = G(x,y,z) \textit{ for all } x,y,z \in X.$$

(2) G-contractive if there is a constant β with the choice $0 \leq \beta < 1$ such that

$$G(Tx, Ty, Tz) \leq \beta G(x,y,z) \textit{ for all } x,y,z \in X.$$

(3) G-expansive if here is a constant α with the choice $\alpha > 1$ such that

$$G(Tx, Ty, Tz) \geq \alpha G(x,y,z) \textit{ for all } x,y,z \in X.$$

Theorem 2.2. *Let (X_k, G_k) be a complete G_k-metric space and $T_k : X_k \longrightarrow X_k$ be a map for $k = 1.2...,d$. Consider the G-metric space $X = \prod\limits_{1\leq k\leq d} X_k$ with $G = \sum\limits_{\leq k\leq d} G_k$ and the map $T = T_1 \times T_2 \times ... \times T_d : X \longrightarrow X$ defined by*

$$T(x) = T(x_1, x_2, ..., x_d) := (T_1x_1, T_2x_2, ..., T_dx_d).$$

If each T_k is is a contractive mapping for $k = 1,2...,d$, then T has a unique fixed point.

Proof. From Theorem 2.1 it is know that (X,G)-is a complete G-metric space. Since each T_k is contractive mapping, there exists a constant β_k $(0<\beta_k<1)$ such that

$$G_k(T_kx_k,T_ky_k,T_kz_k)\le\beta_kG_k(x_k,y_k,z_k)\ \text{ for all } x_k,y_k,z_k\in X_k,\ \ k=1,2,...,d.$$

We have for $x=(x_k)_{1\le k\le d}, y=(y_k)_{1\le k\le d}$ and $z=(z_k)_{1\le k\le d}\in X$ that

$$\begin{aligned} G(Tx,Ty,Tz) &= \sum_{1\le k\le d} G_k(T_kx_k,T_ky_k,T_kz_k) \\ &\le \sum_{1\le k\le d}\beta_kG_k(x_k,y_k,z_k) \\ &\le \max_{1\le k\le d}(\beta_k)\sum_{1\le k\le d}G_k(x_k,y_k,z_k) \\ &\le \max_{1\le k\le d}(\beta_k)G(x,y,z) \end{aligned}$$

Hence T is a contractive mapping on X. So Theorem 2.2 in [18] confirms that T has a unique fixed point. □

Theorem 2.3. *Let (X_k,G_k) be a complete G_k-metric space and $T_k:X_k\longrightarrow X_k$ be a map for $k=1.2...,d$. Consider the G-metric space $X=\prod_{1\le k\le d}X_k$ with $G=\sum_{\le k\le d}G_k$ and the map $T=T_1\times T_2\times...\times T_d:X\longrightarrow X$ defined by*

$$T(x)=T(x_1,x_2,...,x_d):=(T_1x_1,T_2x_2,...,T_dx_d).$$

If each T_k is is a surjective expansive mapping for $k=1,2...,d$, then T has a unique fixed point.

Proof. From Theorem 2.1 it is know that (X,G)-is a complete G-metric space. Since each T_k is expansive mapping, there exists a constant α_k $(\alpha_k>1)$ such that

$$G_k(T_kx_k,T_ky_k,T_kz_k)\ge\alpha_kG_k(x_k,y_k,z_k)\ \text{ for all } x_k,y_k,z_k\in X_k,\ \ k=1,2,...,d.$$

We have for $x=(x_k)_{1\le k\le d}, y=(y_k)_{1\le k\le d}$ and $z=(z_k)_{1\le k\le d}\in X$ that

$$\begin{aligned} G(Tx,Ty,Tz) &= \sum_{1\le k\le d} G_k(T_kx_k,T_ky_k,T_kz_k) \\ &\ge \sum_{1\le k\le d}\alpha_kG_k(x_k,y_k,z_k) \\ &\ge \min_{1\le k\le d}(\alpha_k)\sum_{1\le k\le d}G_k(x_k,y_k,z_k) \\ &\ge \min_{1\le k\le d}(\alpha_k)G(x,y,z) \end{aligned}$$

Hence T is a contractive mapping on X. Let $y=(y_k)_{1\leq k\leq d}\in X=\prod_{1\leq k\leq d}X_k$, we have $y_k\in X_k$ and it follows that there exists $x_k\in X_k$ such that $y_k=T_kx_k$,sincce T_k is surjectve. This implise that $y=(T_1x_1,T_2x_2,...,T_dx_d)=Tx$ and hence T is surjective.So Theorem 2.1 in [16] confirms that T has a unique fixed point. □

3. MAPPINGS IN PRODUCT OF GENERALIZED METRIC SPACES

Let (X,d_X) and (Y,d_Y) are metrics spaces. A map $T:X\longrightarrow Y$ is called an isometric if and only if

$$d_Y(Tx,Ty)=d_X(x,y) \text{ for all } x,y\in X.$$

Let $(X,\|.\,,\|_X)$ and $(Y,\|.\,.\,\|_Y)$ are 2-normed spaces . A map $f:X\longrightarrow Y$ is called an isometric if and only if

$$\|x\,,\,y\|_X=\|f(x)\,,\,f(y)\|_Y \text{ for all } x,y\in X.$$

Definition 3.1. *(See [4].) Let X and Y be 2-normed spaces and $f:X\longrightarrow Y$ be a mapping. Then f is called a 2-isometry if*

$$\|x-z,\,y-z\|_X=\|f(x)-f(z),\,f(y)-f(z)\| \ \textit{for all};\ x,y,z\in X.$$

Remark 1.3. Let X and Y be 2-normed spaces and $f:X\longrightarrow Y$ be a mapping. Then f is a 2-isometry if and only if f satisfies the following property:

$$\big|\|x-z,\,y-z\|_X-\|x'-z',\,y'-z'\|_X\big|=\big|\|f(x)-f(z),\,f(y)-f(z)\|_Y-\|f(x')-f(z'),\,f(y')-f(z')\|_Y\big|,$$

for all $x,y,z,x',y',z'\in X$.

Definition 3.2. *Let (X,G) be an G-metric space and let $T:X\longrightarrow X$ be a mapping. We say that T is a G-2-isometric mapping if*

$$G(T^2x,T^2y,T^2z)-2G(Tx,Ty,Tz)+G(x,y,z)=0 \ \textit{for all } x,y,z\in X.$$

Remark 3.1. *Every G-isometric mapping is an G-2-isometric.*

Example 3.1. *Let $T : (\mathbb{R}, G) \longrightarrow (\mathbb{R}, G)$ defined by $Tx = x+1$, where*

$$G(x,y,z) = |x-y| + |y-z| + |z-x|.$$

A simple computation shows that

$$\begin{aligned} & G(T^2x, T^2y, T^2z) - 2G(Tx, Ty, Tz) + G(x,y,z) \\ = \ & |x-y| + |y-z| + |z-x| - 2(|x-y| + |y-z| + |z-x|) + |x-y| + |y-z| + |z-x| \\ = \ & 0. \end{aligned}$$

Thus ,T is an G-2-isometry.

Proposition 3.1. *Let (X,G) be a G-metric space and $T : X \longrightarrow X$ be an G-2-isometry, then the following properties hold:*

(1) $G(Tx,Ty,Tz) \geq \dfrac{n-1}{n} G(x,y,z)$, $n \geq 1$, $x,y,z \in X$.

(2) $G(Tx,Ty,Tz) \geq G(x,y,z)$ *for all* $x,y,z \in X$.

(3) $G(T^nx,T^ny,T^nz) + (n-1)G(x,y,z) = nG(Tx,Ty,Tz)$ *for all* , $x,y,z \in X$, $n = 0,1,2,...$

(4) $\lim\limits_{n \longrightarrow \infty} \Big(G(T^nx.T^ny,T^nz) \Big)^{\frac{1}{n}} = 1$, *for* $x,y,z \in X$, $x,y,z \neq 0$.

Proof. Since the map T is a G- 2-isometry, we get

$$G(T^2x,T^2y,T^2z) - G(Tx,Ty,Tz) = G(Tx,Ty,Tz) + G(x,y,z) \text{ for all } x,y,z \in X. \tag{3.1}$$

Substituting T^kx, T^ky and T^kz for x,y and z respectively in formula (3.1) we have that

$$\begin{aligned} & G(T^{k+2}x, T^{k+2}y, T^{k+2}z) - G(T^{k+1}x, T^{k+1}y, T^{k+1}z) \\ = \ & G(T^{k+1}x.T^{k+1}y, T^{k+1}z) - G(T^kx, T^ky, T^kz), \ k \geq 0. \end{aligned}$$

We deduce that

$$\begin{aligned} 0 \leq G(T^nxT^ny, T^nz) &= \sum_{1 \leq k \leq n} \Big(G(T^kx, T^ky, T^kz) - G(T^{k-1}x, T^{k-1}y, T^{k-1}z) \Big) + G(x,y,z) \\ &= n \Big(G(Tx,Ty,Tz) - G(x,y,z) \Big) + G(x,y,z) \\ &= nG(Tx.Ty,Tz) + (1-n)G(x,y,z). \end{aligned}$$

From which 1. and 3.hold.

Letting $n \longrightarrow \infty$ in part 1. yields to part 2.

4. Take $x,y,z \in X, x \neq y \neq z$. It follows from part 3. that $\limsup\limits_{n\longrightarrow\infty} \big(G(T^n x, T^n y, T^n z)\big)^{\frac{1}{n}} \leq 1$. However, according to part 2., the sequence $\big(G(T^n x, T^n y, T^n z)\big)_{n\in\mathbb{N}}$ is monotonically increasing, so

$$\liminf_{n\longrightarrow\infty} \big(G(T^n x, T^n y, T^n z)\big)^{\frac{1}{n}} \geq \lim_{n\longrightarrow\infty} \big(G(x,y,z)\big)^{\frac{1}{n}} = 1,$$

which completes the proof. $\square$

Remark 3.2. *From part 3. of Proposition 3.1, it follows that for T is G-2-isometric map*

$$\begin{aligned} G(T^{2n}x, T^{2n}y, T^{2n}z) &= nG(T^{n+1}x, T^{n+1}y, T^{n+1}z) - n(n-1)G(Tx,Ty,Tz) \\ &\quad + (n-1)^2 G(x,y,z) \; n \geq 1, \; x,y,z \in X. \end{aligned}$$

Lemma 3.1. *Let (X,G) be an G-metric space and Let $T : X \longrightarrow X$ be an G-2-isometric map, then for all integer $k \geq 2$ and $x,y,z \in X$, we have*

$$G\big(T^k x, T^k y, T^k z\big) - G\big(T^{k-1}x, T^{k-1}y, T^{k-1}z\big) = G\big(Tx,Ty,Tz\big) - G\big(x,y,z\big)$$

Proof. We prove the assertion by induction on k. Since T is an G-2-isometric map the result is true for $k = 2$. Now assume that the result is true for k i.e.; for all $x,y,z \in X$,

$$G\big(T^k x, T^k y, T^k z\big) - G\big(T^{k-1}x, T^{k-1}y, T^{k-1}z\big) = G\big(Tx,Ty,Tz\big) - G\big(x,y,z\big), \tag{3.2}$$

and let us prove it of $k+1$. From (3.2) we obtain the following iqualities

$$\begin{aligned} G\big(T^{k+1}x, T^{k+1}y, T^{k+1}z\big) - G\big(T^k x, T^k y, T^k z\big) &= G\big(T^2x, T^2y, T^2z\big) - G\big(Tx,Ty,Tz\big) \\ &= G\big(Tx,Ty,Tz\big) - G\big(x,y,z\big). \end{aligned}$$

$\square$

Theorem 3.1. *Let (X,G) be an G-meric space and Let $T : X \longrightarrow X$ be an G-2-isometric map. Then for any positive integer n , T^n is an G-2-isometric map.*

Proof. We will induct on n, the result obviously holds for $n = 1$. Suppose then the assertion holds for $n \geq 2$, i.e

$$G\big(T^{2n}x, T^{2n}y, T^{2n}z\big) - G\big(T^{n}x, T^{n}y, T^{n}z\big) + G\big(x,y,z)\big) = 0,\ \forall x,y,z \in X.$$

Then

$$\begin{aligned}
& G\big(T^{2n+2}x, T^{2n+2}y, T^{2n+2}z\big) - 2G\big(T^{n+1}x, T^{n+1}y, T^{n+1}z\big) + G\big(x,y,z\big)^{p} \\
= \;& G\big(T^{2}T^{2n}x, T^{2}T^{2n}y, T^{2}T^{2n}z\big) - 2G\big(T^{n+1}x, T^{n+1}y, T^{n+1}z\big) + G\big(x,y,z\big) \\
= \;& 2G\big(T^{2n+1}x, T^{2n+1}y, T^{2n+1}z\big) - G\big(T^{2n}x, T^{2n}y, T^{2n}z\big) - 2G\big(T^{n+1}x, T^{n+1}y, T^{n+1}z\big) + G\big(x,y,z\big) \\
= \;& 2(2G\big(T^{n+1}x, T^{n+1}y, T^{n+1}z\big) - G\big(Tx,Ty,Tz\big)) - G\big(T^{2n}x, T^{2n}y, T^{2n}z\big) \\
& -2G\big(T^{n+1}x, T^{n+1}y, T^{n+1}z\big) + G\big(x,y,z\big) \\
= \;& 2G\big(T^{n+1}x, T^{n+1}y, T^{n+1}z\big) - G\big(T^{2n}x, T^{2n}y, T^{2n}z\big) - 2G\big(Tx,Ty,Tz\big) + G\big(x,y,z\big) \\
= \;& 2G\big(T^{n+1}x, T^{n+1}y, T^{n+1}z\big) - (2G\big(T^{n}x, T^{n}y, T^{n}z\big) - G\big(x,y,z\big)) - 2G\big(Tx,Ty,Tz\big) + G\big(x,y,z\big) \\
= \;& 2G\big(T^{n+1}x, T^{n+1}y, T^{n+1}z\big) - 2G\big(T^{n}x, T^{n}y, T^{n}z\big) - 2G\big(Tx,Ty,Tz\big) + 2G\big(x,y,z\big) \\
= \;& 2\big(G\big(Tx,Ty,Tz\big) - G\big(x,y,z\big)\big) - 2G\big(Tx,Ty,Tz\big) + 2G\big(x,y,z\big) \quad (\text{by } \textit{Lemma } 3.1). \\
= \;& 0.
\end{aligned}$$

Thus means that T^{n} is G-2-isometric map. □

Definition 3.3. *A map T on a G-metric space (X,G) is called power bounded if*

$$\sup\{\, G\big(T^{n}x, T^{n}y, T^{n}z\big),\ n = 1,2,\ldots\} < \infty \textit{ for all } x,y,z \in X.$$

Theorem 3.2. *Every power bounded G-2-isometric map is a G-isometric.*

Theorem 3.3. *Let $T : (X,G) \longrightarrow (X,G)$ be an bijective G-2-isometric map. Then T^{-1} is an G-2-isometric map.*

Proof. Since T is an G-2-isosometric, we have

$$G(T^{2}x, T^{2}y, T^{2}z) - 2G(Tx,Ty,Tz) + G(x,y,z) = 0 \text{ for all } x,y,z \in X.$$

Substituting $T^{-2}x, T^{-2}y$ and $T^{-2}z$ for x, y and z respectively in the above equation we obtain

$$G(x,y,z) - 2G(T^{-1}x, T^{-1}y, T^{-1}z) + G(T^{-2}x, T^{-2}y, T^{-2}z) = 0.$$

This implies that T^{-1} is a G-2-isometric. $\square$

Proposition 3.2. *Let (X, G_k) be a G-metric space and $T_k : X_k \longrightarrow X_k$ be an mapping. Let $X = \prod_{1 \le k \le d} X_k$ and $G = \sum_{1 \le k \le d} G_k$.Consider the map $T = T_1 \times T_2 \times ... \times T_d : (X, G) \longrightarrow (X, G)$ defined by*

$$Tx = (T_1x_1, T_2x_2, ..., T_dx_d).$$

If each T_k is a G-2-isometric map ,then T is a G-2-isometric map.

Proof. Let $x = (x_k)_{1 \le k \le d}, y = (y_k)_{1 \le k \le d}$ and $z = (z_k)_{1 \le k \le d} \in X$. We have that

$$\begin{aligned} & G(T^2x, T^2y, T^2z) - 2G(Tx, Ty, Tz) + G(x,y,z) \\ = & \sum_{1 \le k \le d} \Big(G_k(T_k^2x_k, T_k^2y_k, T_k^2z_k) - 2G_k(T_kx_k, T_ky_k, T_kz_k) + G_k(x_k, y_k, z_k) \Big) \\ = & \ 0. \end{aligned}$$

$\square$

Lemma 3.2. *If T is an G-2-isometric map, then for all $x, y, z \in X,\ k \in \mathbb{N}$*

$$G(T^kx, T^ky, T^kz) \le kG(Tx, Ty, Tz) - (k-1)G(x,y,z).$$

Theorem 3.4. *Let (X, G) be a complete G-metric space ant let $T : X \longrightarrow X$ be a G-continuous mapping. If T is an G-2-isometric mapping. Then T is injective and $\mathscr{R}(T)$ (the range of T) is closed in X.*

Proof. First, we prove that T is injective. Let $x, y \in X$ such that $Tx = Ty$. From part 2. of Proposition 3.1 we have that

$$G(Tx, Tx, Tz) \ge G(x, y, z) \ \text{ for all } \ z \in X.$$

Inparticular $G(Tx, Tx, Tx) \ge G(x, y, x)$,this implise that $G(x,y,x) = 0$. As $G(x,x,y) > 0$ for $x \ne y$,we deduce that $x = y$.

We prove that $\mathscr{R}(T)$ is closed. Let x_n be a sequence in X such that $Tx_n \longrightarrow y$ in (X,G). We have $T^2x_n \longrightarrow Ty$ in (X,G), since T is continuous. By Definition 3.1 we have

$$G(x_n,x_m,x_l) = -G(T^2x_n,T^2x_m,T^2x_l) + 2G(Tx_n,Tx_m,Tx_l) \text{ for all } n,m,l > 0.$$

It is well know that $(T^2x_n)_n$ and $(Tx_n)_n$ are Cauchy sequenses in (X,G) ,hence $(x_n)_n$ is a Cauchy sequence in (X,G). Due to the completeness of (X,G) ,there exists $x \in X$ such that $(x_n)_n$ is G-convergent to x. On the other hand,using the fact that T is G-continuous, we get that $Tx_n \longrightarrow Tx$.Therefore, by Proposition 1.1 $y = Tx$. Hence $\mathscr{R}(T)$ is G-closed. □

Conflict of Interests

The author declares that there is no conflict of interests.

References

[1] M. Abbas, T. Nazir, and Stojan Radenovic, Some periodic point results in generalized metric spaces, Appl. Math. Comput. 217 (2010) 4094–4099.

[2] H. Aydi, B. Damjanovic, B. Samet, W. Shatanawi, Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces, Math. Comput. Modelling 54 (2011) 2443–2450. 11.

[3] H. Aydi, W. Shatanawi and C. Vetro, On generalized weakly G-contraction mapping in G- metric spaces, Comput. Math. Appl. 62 (2011) 4222–4229.

[4] H. Aydi, A fixed point result involving a generalized weakly contractive condition in G-metric spaces, Bull. Math. Anal. Appl. 3 (2011) 180–188.

[5] H. Aydi, W. Shatanawi and M. Postolache, Coupled fixed point results for (ψ,ϕ)-weakly contractive mappings in ordered G-metric spaces, Comput. Math. Appl. 63 (2012) 298–309.

[6] H. Aydi, A common fixed point of integral type contraction in generalized metric spaces, J. Adv. Math. Stud. 5 (2012), 111-117.

[7] M. Bousselsal and Z.I. Al-Muhimeed, On a A new generalized weakly contraction mapping in G-metric spaces, Adv. Fixed Point Theory 3 (2013), 248–262.

[8] B.C. Dhage, Generalized metric space and mapping with fixed point, Bull. Calcutta Math. Soc. 84 (1992), 329–336.

[9] B.C. Dhage, Generalized metric spaces and topological structure I, Annalele Stinti
ce ale Universi- tatii Al.I. Cuza, vol.46, No.1 (2000), 3–24.

[10] Z. Mustafa, A new structure for generalized metric spaces with applications to fixed point theory, Ph.D. thesis, The University of Newcastle, Callaghan, Australia, (2005).

[11] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006) 289–297.

[12] Z. Mustafa, B. Sims, Some remarks concerning D-metric spaces, in: Proc. Int. Conf. on. Fixed Point Theory and Applications, Valencia, Spain, July 2003, pp. 189–198.

[13] Z. Mustafa and B. Sims, Fixed point theorems for contractive mappings in complete G- metric spaces, Fixed Point Theory Appl. 2009 (2009), Article ID 917175.

[14] Z. Mustafa, H. Obiedat and F. Awawdeh, Some fixed point theorem for mapping on complete G-metric spaces, Fixed Point Theory Appl. 2008 (2008), Article ID 189870.

[15] Z. Mustafa, W. Shatanawi and M. Bataineh, Existence of fixed point results in G-metric spaces, Int. J. Math. Math. Sci. 2009 (2009), Article ID 283028.

[16] Z.d Mustafa, F. Awawdeh and W. Shatanawi,Fixed Point Theorem for Expansive Mappings in G-Metric Spaces.Int. J. Contemp. Math. Sci. 5 (2010), 2463-2472.

[17] S. M. Patel, 2-Isometric Operators, Glasnik Math. 37 (2002), 143-147.

[18] T. Phaneendra and K. Kumara Swamy,Unique fixed ponit in G-metric space through greatest lower bound properties, Novi Sad J. Math. 43 (2013), 107–115.

[19] H.Y. Chu, S.H. Ku and D.Seung Kang, Characterizations on 2-isometries, J. Math. Anal. Appl. 340 (2008), 621-628.