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A REGULARIZATION ITERATIVE PROCESS FOR SOLVING GENERALIZED VARIATIONAL INEQUALITIES IN BANACH SPACES

S.Y. CHO[1], X. QIN[2,3,*]

[1]Department of Mathematics, Gyeongsang National University, Jinju 660-701, Korea

[2]Department of Mathematics, Wuhan University of Technology, Wuhan 430000, China

[3]Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia

Abstract. In this article, we investigate a generalized variational inequality based on a regularization iterative process. Strong convergence theorems of solutions are established in a 2-uniformly smooth and uniformly convex Banach space.

Keywords. Accretive operator; Banach spaces; Fixed point; Variational inequality.

1. Introduction-preliminaries

Variational inequality theory, which was introduced in sixties, has emerged as an interesting and fascinating branch of applicable mathematics with a wide range of applications in finance, economics, optimization, engineering, and medicine. Variational inequality theory is dynamic and experiencing an explosive growth in both theory and applications. Recently, fixed-point methods have been extensively investigated for solving variational inequalities; see [1-13] and the references therein. Among the fixed-point algorithms, Mann-like iterative algorithms are efficient for solving several nonlinear problems. However, Mann-like iterative algorithms are only weakly convergent even in Hilbert spaces. In many disciplines, including economics [14],

E-mail address: ooly61@hotmail.com

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image recovery [15], quantum physics [16], and control theory [17], problems arises in infinite dimension spaces. In such problems, strong convergence is often much more desirable than weak convergence, for it translates the physically tangible property that the energy $\|x_n - x\|$ of the error between the iterate x_n and the solution x eventually becomes arbitrarily small. Recently, Moudafi [18] introduced a viscosity method for solving fixed points of nonlinear operators in the framework of Hilbert spaces. He showed that the convergence point is not only a fixed point of nonlinear operators but a unique solution to some monotone variational inequality; see [18] for more details and the references therein.

Let C be a nonempty closed and convex subset of a Banach space E. Let E^* be the dual space of E and $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$$

for all $x \in E$. In particular, $J = J_2$ is called the normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \in E$. We denote by j the single normalized duality mapping. Further, we have the following properties of the generalized duality mapping J_q:

(a) $J_q(tx) = t^{q-1}J_q(x)$ for all $x \in E$ and $t \in [0, \infty)$;

(b) $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \in E$ with $x \neq 0$;

(c) $J_q(-x) = -J_q(x)$ for all $x \in E$.

Let $U = \{x \in X : \|x\| = 1\}$. A Banach space E is said to uniformly convex if, for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geq \varepsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

A Banach space E is said to be smooth if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. The modulus of smoothness of E is defined by

$$\rho(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau\},$$

where $\rho : [0,\infty) \to [0,\infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0}\frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \le 2$. A Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \le c\tau^q$ for all $\tau > 0$.

Note that typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p,2\}$-uniformly smooth for every $p > 1$. Note also that no Banach space is q-uniformly smooth for $q > 2$; see [19] for more details.

Let C be a nonempty closed convex subset of E. Recall that an operator A of C into E is said to be accretive iff

$$\langle Ax - Ay, j(x-y)\rangle \ge 0, \quad \forall x,y \in C,$$

where $j(x-y) \in J(x-y)$.

For $\alpha > 0$, recall that an operator A of C into E is said to be α-inverse-strongly accretive if

$$\langle Ax - Ay, j(x-y)\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x,y \in C,$$

where $j(x-y) \in J(x-y)$.

Let D be a subset of C and Q be a mapping of C into D. Then Q is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \ge 0$. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

The following result describes a characterization of sunny nonexpansive retractions on a s-mooth Banach space.

Proposition 1.1. [20] *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*

(1) *Q is sunny and nonexpansive;*

(2) $\|Qx - Qy\|^2 \le \langle x - y, J(Qx - Qy)\rangle, \quad \forall x,y \in E;$

(3) $\langle x - Qx, J(y - Qx)\rangle \le 0, \quad \forall x \in E, y \in C.$

Proposition 1.2. [11] *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

Recently, Aoyama *et al.* [7] considered the following generalized variational inequality problem:

Let E be a smooth Banach space and C a nonempty closed convex subset of E and A an accretive operator of C into E. Find a point $u \in C$ such that

$$\langle Au, J(v-u)\rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

Next, we use $VI(C,A)$ to denote the set of solutions of generalized variational inequality problem (1.1). In Hilbert spaces, generalized variational inequality reduces to the classical monotone variational inequality.

Aoyama *et al.* [7] proved that generalized variational inequality (1.1) is equivalent to a fixed point problem. The element $u \in C$ is a solution of generalized variational inequality (1.1) if and only if $u \in C$ satisfies equation

$$u = Q_C(u - \lambda Au), \tag{2.2}$$

where $\lambda > 0$ is a constant and Q_C is a sunny nonexpansive retraction from E onto C.

For solving solutions of monotone variational inequalities, Iiduka *et al.* [8] proved the following theorem.

Theorem ITT. *Let C be a nonempty closed convex subset of a real Hilbert space H and let A be an α-inverse strongly monotone operator of H into H with $VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in C$ and*

$$x_{n+1} = P_C(\alpha_n x_n + (1-\alpha_n)P_C(x_n - \lambda_n Ax_n))$$

for every $n = 1, 2, \ldots,$ where C is the metric projection from H onto C, $\{\alpha_n\}$ is a sequence in $[-1,1]$, and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\alpha_n\} \in [a,b]$ for some a,b with $-1 < a < b < 1$ and $\{\lambda_n\} \in [c,d]$ for some c,d with $0 < c < d < 2(1+a)\alpha$, then $\{x_n\}$ converges weakly to some element of $VI(C,A)$.

Recently, Aoyama, Iiduka and Takahashi [7] obtained a weak Theorem in a uniformly convex and 2-uniformly smooth Banach space. To be more precise, they proved the following result.

Theorem AIT. *Let E be a uniformly convex and 2-uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\alpha > 0$ and A be an α-inverse strongly-accretive operator of C into E with $S(C,A) \neq \emptyset$, where*

$$S(C,A) = \{x^* \in C : \langle Ax^*, J(x - x^*)\rangle \geq 0,\ x \in C\}.$$

If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen such that $\lambda_n \in [a, \frac{\alpha}{K^2}]$ for some $a > 0$ and $\alpha_n \in [b,c]$ for some b,c with $0 < b < c < 1$, then the sequence $\{x_n\}$ defined by the following manners:

$$x_1 = x \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n) Q_C(x_n - \lambda_n A x_n) \tag{Υ'}$$

converges weakly to some element z of $S(C,A)$, where K is the 2-uniformly smoothness constant of E.

In this paper, motivated by research work going on this direction, we investigate generalized variational inequality (1.1) based on a regularization iterative process. Strong convergence theorems of solutions are established in a 2-uniformly smooth and uniformly convex Banach space.

In order to prove our main results, we need the following lemmas and definitions.

Lemma 1.3. [19] *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

Lemma 1.4. [21]. *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where γ_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 1.5. [7] *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then, for all $\lambda > 0$, $VI(C,A) = F(Q_C(I-\lambda A))$.*

Lemma 1.6. [22] *Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E and $T : K \to K$ a nonexpansive mapping. Then $I-T$ is demi-closed at zero.*

Lemma 1.7. [23] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space e and let $\{\beta_n\}$ be a sequence in $(0,1)$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \ge 0$ and $\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|\right) \le 0$. Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

2. Main results

Theorem 2.1. *Let E be a 2-uniformly smooth and uniformly convex Banach space with the best smooth constant K. Let C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C and let $A : C \to E$ be an α-inverse-strongly accretive mapping with $VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$, $y_n = Q_C(x_n - \lambda_n Ax_n)$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n$, $n \ge 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,\alpha/K^2)$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n+1}| = 0$. Then $\{x_n\}$ converges strongly to $\bar{x}$, where $\bar{x} = Q_{VI(C,A)}f\bar{x}$.*

Proof. Fixing $x^* \in VI(C,A)$, we see $x^* = Q_C(x^* - \lambda_n Ax^*)$. Using Lemma 1.3 and Lemma 1.5, we have

$$\begin{aligned}
\|x^* - y_n\|^2 &= \|Q_C(x^* - \lambda_n Ax^*) - Q_C(x_n - \lambda_n Ax_n)\|^2 \\
&\le \|\lambda_n(Ax_n - Ax^*) - (x_n - x^*)\|^2 \\
&\le \|x_n - x^*\|^2 + 2K^2\lambda_n^2\|Ax_n - Ax^*\|^2 - 2\lambda_n\langle Ax_n - Ax^*, J(x_n - x^*)\rangle \qquad (2.1) \\
&\le \|x_n - x^*\|^2 - 2\lambda_n(\alpha - \lambda_n K^2)\|Ax_n - Ax^*\|^2 \\
&\le \|x_n - x^*\|^2.
\end{aligned}$$

It follows that

$$
\begin{aligned}
\|x_{n+1}-x^*\| &\le \alpha_n\|f(x_n)-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|Q_C(x_n-\lambda_nAx_n)-x^*\|\\
&\le \alpha_n\kappa\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|+\beta_n\|x_n-x^*\|+\gamma_n\|Q_C(x_n-\lambda_nAx_n)-x^*\|\\
&\le \big(1-\alpha_n(1-\kappa)\big)\|x_n-x^*\|+\alpha_n\|f(x^*)-x^*\|\\
&\le \max\{\|x_n-x^*\|+\frac{\|f(x^*)-x^*\|}{1-\kappa}\},
\end{aligned}
$$

which implies that sequence $\{x_n\}$ is bounded, so is $\{y_n\}$. Notice that

$$
\begin{aligned}
\|y_n-y_{n+1}\| &\le \|(x_{n+1}-\lambda_{n+1}Ax_{n+1})-(x_n-\lambda_nAx_n)\|\\
&\le \|(x_{n+1}-\lambda_{n+1}Ax_{n+1})-(x_n-\lambda_{n+1}Ax_n)\|+|\lambda_n-\lambda_{n+1}|\|Ax_n\|\\
&\le \|x_{n+1}-x_n\|-2\lambda_{n+1}\langle Ax_{n+1}-Ax_n,J(x_{n+1}-x_n)\rangle\\
&\quad+2K^2\lambda_{n+1}^2\|Ax_{n+1}-Ax_n\|^2+|\lambda_n-\lambda_{n+1}|\|Ax_n\| \qquad (2.2)\\
&\le \|x_{n+1}-x_n\|-2\lambda_{n+1}\alpha\|Ax_{n+1}-Ax_n\|^2\\
&\quad+2K^2\lambda_{n+1}^2\|Ax_{n+1}-Ax_n\|^2+|\lambda_n-\lambda_{n+1}|\|Ax_n\|\\
&\le \|x_{n+1}-x_n\|+|\lambda_n-\lambda_{n+1}|\|Ax_n\|.
\end{aligned}
$$

Let $x_{n+1}=(1-\beta_n)q_n+\beta_nx_n$. It follows that

$$
\begin{aligned}
\|q_{n+1}-q_n\| &= \|\frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}y_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_nf(x_n)+\gamma_ny_n}{1-\beta_n}\|\\
&= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1})+\frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}y_{n+1}\\
&\quad-\big(\frac{\alpha_n}{1-\gamma_n}f(x_n)+\frac{1-\alpha_n-\beta_n}{1-\beta_n}y_n\big)\|\\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-y_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-y_n\|+\|y_{n+1}-y_n\|.
\end{aligned}
$$

Using (2.2), one has

$$
\begin{aligned}
&\|q_{n+1}-q_n\|-\|x_{n+1}-x_n\|\\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-y_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-y_n\|+|\lambda_n-\lambda_{n+1}|\|Ax_n\|.
\end{aligned}
$$

From the restriction imposed on the control sequences, one has

$$
\limsup_{n\to\infty}\big(\|q_{n+1}-q_n\|-\|x_{n+1}-x_n\|\big)\le 0.
$$

This implies from Lemma 1.7 that $\lim_{n\to\infty}\|q_n - x_n\| = 0$. It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.3}$$

On the other hand, one has

$$\begin{aligned}\|x_n - y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|\\ &\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - y_n\| + \beta_n\|x_n - y_n\|.\end{aligned}$$

Using (2.3) and the fact that $\lim_{n\to\infty}\alpha_n = 0$, we obtain

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{2.4}$$

Next, we show that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, J(x_n - \bar{x})\rangle \le 0. \tag{2.5}$$

To show (2.5), we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, J(x_{n_i} - \bar{x})\rangle = \lim_{i\to\infty}\langle f(\bar{x}) - \bar{x}, J(x_{n_i} - \bar{x})\rangle. \tag{2.6}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. Next, we show that $w \in VI(C,A)$. From the assumption, we see that sequence $\{\lambda_{n_i}\}$ is bounded. So, there exists a subsequence $\{\lambda_{n_{i_j}}\}$ converges to λ_0. We may, without loss of generality, assume that $\lambda_{n_i} \to \lambda_0$. Observe that

$$\begin{aligned}\|x_{n_i} - Q_C(x_{n_i} - \lambda_0 Ax_{n_i})\| &\le \|y_{n_i} - x_{n_i}\| + \|Q_C(x_{n_i} - \lambda_0 Ax_{n_i}) - y_{n_i}\|\\ &\le \|(x_{n_i} - \lambda_0 Ax_{n_i}) - (x_{n_i} - \lambda_{n_i} Ax_{n_i})\| + \|y_{n_i} - x_{n_i}\|\\ &\le \|y_{n_i} - x_{n_i}\| + \|\lambda_{n_i} - \lambda_0\|K,\end{aligned}$$

where K is an appropriate constant such that $K \ge \sup_{n\ge 1}\{\|Ax_n\|\}$. Using (2.4), one has

$$\lim_{i\to\infty}\|Q_C(x_{n_i} - \lambda_0 Ax_{n_i}) - x_{n_i}\| = 0.$$

On the other hand, we know that $Q_C(I-\lambda_0 A)$ is nonexpansive. Indeed, for $x,y\in C$, from Lemma 1.3, we see that

$$\begin{aligned}\|Q_C(I-\lambda_0 A)x-Q_C(I-\lambda_0 A)y\|^2 &\le \|(I-\lambda_0 A)x-(I-\lambda_0 A)y\|^2\\ &\le \|x-y\|^2-2\lambda_0\langle Ax-Ay,J(x-y)\rangle+2K^2\lambda_0^2\|Ax-Ay\|^2\\ &\le \|x-y\|^2+2\lambda_0(\lambda_0 K^2-\alpha)\|Ax-Ay\|^2\\ &\le \|x-y\|^2.\end{aligned}$$

It follows from Lemma 1.6 that $w\in F(Q_C(I-\lambda_0 A))$. This in turn implies $w\in VI(C,A)=F(Q_C(I-\lambda_0 A))$. From (2.6), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},J(x_n-\bar{x})\rangle &= \lim_{i\to\infty}\langle f(\bar{x})-\bar{x},J(x_{n_i}-\bar{x})\rangle\\ &= \lim_{i\to\infty}\langle f(\bar{x})-\bar{x},J(w-\bar{x})\rangle\le 0.\end{aligned}\tag{2.7}$$

Finally, we show that $x_n\to\bar{x}$ as $n\to\infty$. Observe that

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 &= \langle\alpha_n f(x_n)+\beta_n x_n+\gamma_n y_n-\bar{x},J(x_{n+1}-\bar{x})\rangle\\ &= \alpha_n\langle f(x_n)-\bar{x},J(x_{n+1}-\bar{x})\rangle+\beta_n\langle x_n-\bar{x},J(x_{n+1}-\bar{x})\rangle+\gamma_n\langle y_n-\bar{x},J(x_{n+1}-\bar{x})\rangle\\ &\le \alpha_n\|f(x_n)-f(\bar{x})\|\|x_{n+1}-\bar{x}\|+\alpha_n\langle f(\bar{x})-\bar{x},J(x_{n+1}-\bar{x})\rangle+\beta_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\|\\ &\quad+\gamma_n\|y_n-\bar{x}\|\|x_{n+1}-\bar{x}\|\\ &\le \big(1-\alpha_n(1-\kappa)\big)\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\|+\alpha_n\langle f(\bar{x})-\bar{x},J(x_{n+1}-\bar{x})\rangle.\end{aligned}$$

This implies that

$$\|x_{n+1}-\bar{x}\|^2\le\big(1-\alpha_n(1-\kappa)\big)\|x_n-\bar{x}\|^2+2\alpha_n\langle f(\bar{x})-\bar{x},J(x_{n+1}-\bar{x})\rangle$$

From assumptions $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$ and applying Lemma 1.4, we obtain that

$$\lim_{n\to\infty}\|x_n-\bar{x}\|=0.$$

This completes the proof.

In the framework of Hilbert space, 2-uniformly smooth and uniformly convex Banach spaces are reduced to Hilbert space and sunny nonexpansive retraction $Q_C$from E onto C is reduced to metric projection $Proj_C$. Then Theorem 2.1 is reduced to the following.

Corollary 2.2. *Let E be a Hilbert space and let C be a nonempty closed convex subset of E. Let $A : C \to E$ be an α-inverse-strongly monotone mapping with $VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$, $y_n = Proj_C(x_n - \lambda_n Ax_n)$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n$, $n \geq 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0, \alpha/K^2)$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| = 0$. Then $\{x_n\}$ converges strongly to $\bar{x}$, where $\bar{x} = Proj_{VI(C,A)} f\bar{x}$.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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