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TIME-DEPENDENT AND STEADY STATE SOLUTION OF AN $M^{[x]}/(G)/1$ QUEUEING SYSTEM WITH SERVER'S LONG AND SHORT VACATIONS

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Abstract: We study the time-dependent and the steady state behaviour of an $M^{[x]}/(G)/1$ queue with server's long and short vacations. Customers arrive at the system in a Poisson stream in batches of variable size. The service time of a customer served by the server is assumed to have a general distribution. At each service completion epoch, the server may opt to take a long vacation with probability p_1 and a short vacation with probability p_2 or else with probability p_3 $(p_1 + p_2 + p_3 = 1)$ he may opt to continue to be available in the system. The long and short vacation periods of the server are assumed to have general distributions with different mean vacation times. We obtain time-dependent probability generating functions for the queue size. The corresponding steady state results including the stability condition, steady state probabilities of different states of the system, the mean queue size and mean waiting time in the queue as well as in the system have been derived in explicit and closed forms. Several particular cases of interest including some earlier known results have been derived. Many numerical examples confirming validity of the theoretical results have been peformed.

Keywords: poisson arrivals; general service times; long and short vacations; general vacation times.

2010 AMS Subject Classification: 60k25, 90B22.

1. Introduction

In recent years, queues with server vacations have emerged as one of the most important areas of queueing theory. Vaction queues with different vacation policies including exhaustive service, priorities or binomial schedules have been studied by many authors including Doshi [3],

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Shanthikumar [11], Cramer [2], Choi *et al.* [1], Takagi [12], Madan *et al*. [6], Maraghi *et al.* [5], Fadhil et al [4] and Madan [7, 8,9,10] to mention only a few.

In the present paper, we introduce a new vacation policy in a single server queueing sysetm, termed as a trinomial vacation policy. We assume that at every service completion epoch, the server may take a long vacation, or a short vacation or may continue staying in the system. We determine the time-dependent solution using Laplace Transform approach and derive corresponding steady state results from the time-dependent solution. The service times of the main server, the long vacation times and the short vacation times have all been assumed to have a general distrubution.

2. The Mathematical Model

The mathematical model is described by the following assumptions:

- Customers arrive at the system in a compound Poisson process in batches of variable size. Let $\lambda c_i dt$ (i = 1, 2, 3, . . .) be the first order probability that a batch of i customers arrives at the system during a short interval of time (t, $t+dt$), where $0 \le c_i \le 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches.
- The system has a single server who, after completion of service of a customer has the option to take a long vacation with probability p_1 or a short vacation with probability p_2 or else with probability p_3, may opt to continue providing service to the next customer, where $p_1 + p_2 + p_3 = 1$.
- The customers are served one by one on a first come, first-served basis.
- Let $B(s)$ and $b(s)$ respectively be the distribution function and the density function of the service time S of a customer and let $\mu(x)dx$ be the conditional probability of completion of a service, given that the elapsed time is x, so that

$$\mu(x) = \frac{b(x)}{1 - B(x)}, \tag{2.1}$$

and, therefore,

$$b(s) = \mu(s)\exp\left[-\int_0^v \mu_1(x)dx\right]. \tag{2.2}$$

- Let $A_1(v_1)$ and $a_1(v_1)$ respectively be the distribution function and the density function of the long vacation time V_1 of the server and let $\beta_1(x)dx$ be the conditional probability of completion of a long vacation of the server, given that the elapsed time of the long vacation is *x*, so that

$$\beta_1(x) = \frac{a_1(x)}{1 - A_1(x)}\ , \tag{2.3}$$

and, therefore,

$$a_1(v_1) = \beta_1(v_1)\exp\left[-\int_0^v \beta_1(x)dx\right]. \tag{2.4}$$

- Let $A_2(v_2)$ and $a_2(v_2)$ respectively be the distribution function and the density function of the short vacation time V_2 of the server and let $\beta_2(x)dx$ be the conditional probability of completion of a short vacation of the server, given that the elapsed time of a short vacation is *x*, so that

$$\beta_2(x) = \frac{a_2(x)}{1 - A_2(x)}, \tag{2.5}$$

and, therefore,

$$a_2(v_2) = \beta_2(v_2)\ \exp\left[-\int_0^v \beta_2(x)dx\right]. \tag{2.6}$$

- On completion of a vacation the server instantly joins the system and takes up a customer (at the head of the queue) for service if there are customers waiting in the queue. However, if on returning back the server finds the queue empty, he still joins the system and remains idle in the system until a new customer arrives.
- Various stochastic processes involved in the system are independent of each other.

3. Definitions and Equations Governing the System

We define:

$W_n(x,t) \equiv$ probability that at time t, there are n (≥ 0) customers in the queue excluding one customer in service with elapsed service time x,

$W_n(t) = \int\limits_{x=0}^{\infty} W_n(x,t) \equiv$ probability that at time t, there are n customers in the queue excluding one customer in service irrespective of the value of x,

$V_n^L(x,t) \equiv$ probability that at time t, the server is on long vacation with elapsed long vacation time x and there are n ($\geq$0) customers in the queue,

$V_n^L(t)$ as the probability that at time t, the server is on long vacation irrespective of the value of x and there are n ($\geq$0) customers in the queue,

$V_n^S(x,t) \equiv$ probability that at time t, the server is on short vacation with elapsed short vacation time x and there are n ($\geq$0) customers in the queue,

$V_n^S(t) \equiv$ probability that at time t, server is on short vacation irrespective of the value of x and there are n ($\geq$0) customers in the queue.

Let $P_n(t) = W_n(t) + V_n^L(t) + V_n^S(t)$ be the probability that at time t there are n ($\geq$0) customers in the queue irrespective of whether the server is provinding service or is on vacation. Let $Q(t) \equiv$ probability that at time t, there is no customer in the system and the server is idle.

Then following usual probability reasoning based on the underlying assumptions of the model, the system has the following set of integro differential-difference time-dependent forward equations:

$$\frac{\partial}{\partial x} W_n(x,t) + \frac{\partial}{\partial t} W_n(x,t) + (\lambda + \mu(x)) W_n(x,t) = \lambda \sum_{i=1}^{n-1} c_i W_{n-i}(x,t), \; n \geq 1, \tag{3.1}$$

$$\frac{\partial}{\partial x} W_0(x,t) + \frac{\partial}{\partial t} W_0(x,t) + (\lambda + \mu(x)) W_0(x,t) = 0, \tag{3.2}$$

$$\frac{d}{dt} Q(t) + \lambda Q(t) = \int\limits_0^{\infty} V_0^L(x,t)\beta_1(x)dx + \int_0^{\infty} V_0^S(x,t)\beta_2(x)dx + p_3 \int\limits_0^{\infty} W_0(x,t)\mu_1(x)dx, \tag{3.3}$$

$$\frac{\partial}{dx}V_n^L(x,t)+\frac{\partial}{\partial t}V_n^L(x,t)+(\lambda+\beta_1(x))V_n^L(x,t)=\lambda\sum_{i=1}^{n-1}c_iV_{n-1}^L(t)\ ,\ n\geq 1, \tag{3.4}$$

$$\frac{\partial}{dx}V_0^L(x,t)+\frac{\partial}{\partial t}V_0^L(x,t)+(\lambda+\beta_1(x))V_0^L(x,t)=0\ , \tag{3.5}$$

$$\frac{\partial}{dx}V_n^S(x,t)+\frac{\partial}{\partial t}V_n^S(x,t)+(\lambda+\beta_2(x))V_n^S(x,t)=\lambda\sum_{i=1}^{n-1}c_iV_{n-1}^{(2)}(t),\quad n\geq 1, \tag{3.6}$$

$$\frac{d}{dx}V_0^S(x,t)+\frac{\partial}{\partial t}V_0^S(x,t)+(\lambda+\beta_2(x))V_0^S(t)=0\,. \tag{3.7}$$

Equations (3.1) through (3.7) are to be solved subject to the following boundary conditions:

$$\begin{aligned} W_n(0,t) = {} & p_3\int_0^\infty W_{n+1}(x,t)\mu(x)dx+\int_0^\infty V_{n+1}^L(x,t)\beta_1(x)dx \\ & +\int_0^\infty V_{n+1}^S(x,t)\beta_2(x)dx+\lambda\, c_{n+1}Q(t),\quad n\geq 0\,, \end{aligned} \tag{3.8}$$

$$V_n^L(0,t)=\ p_1\int_0^\infty W_n(x,t)\mu_1(x)dx\ ,\ n\geq 0\,, \tag{3.9}$$

$$V_n^S(0,t)=\ p_2\int_0^\infty W_n(x,t)\mu_1(x)dx\ ,\ n\geq 0\,. \tag{3.10}$$

We assume that initially there is no customer in the system and the server is idle so that the initial conditions are

$$Q(0)=1,\ W_n(0)=0\ ,\ V_n^L(0)=0\ \text{and}\ V_n^S(0)=0,\text{for}\ n\geq 0\ . \tag{3.11}$$

4. The Time-Dependent Solution

We define the following probability generating functions (pgf's):

$$W(x,z,t)=\sum_{n=0}^{\infty}z^nW_n(x,t)\,,\quad W(z,t)=\sum_{n=0}^{\infty}z^nW_n(t)=\sum_{n=0}^{\infty}z^n\int_0^{\infty}W_n(x,t)\,,$$

$$V^L(x,z,t)=\sum_{n=0}^{\infty}z^nV^L{}_n(x,t)\,,\ V^L(z,t)=\sum_{n=0}^{\infty}z^nV_n^L(t)=\sum_{0}^{\infty}z^n\int_0^{\infty}V^L_n(x,t),$$

$$V^S(x,z,t)=\sum_{n=0}^{\infty}z^nV^S{}_n(x,t)\,,\ V^S(z,t)=\sum_{n=0}^{\infty}z^nV_n^S(t)=\sum_{0}^{\infty}z^nV_n^S(x,t)\,,$$

$$P_Q(z,t)=\sum_{n=0}^{\infty}z^nP_n(t)\,,\ C(z)=\sum_{i=1}^{\infty}c_iz^i\ |\,z\,|\,\le 1\,. \tag{4.1}$$

Further, we define Laplace transform (LT) of a function f(t) as

$$f^*(s)=LT(f(t)=\int_0^{\infty}e^{-st}f(t)dt,\quad Rel(s)>0, \tag{4.2}$$

with the well known property of LT as

$$\text{LT}(\frac{d}{dt}f(t))=\text{s }f^*(s)-f(0)\,. \tag{4.3}$$

Next, we define the LT of the generating functions defined in (4.1) as follows:

$$W^*(x,z,s)=\sum_{n=0}^{\infty}z^nW_n^*(x,s)\,,\quad W^*(z,s)=\sum_{n=0}^{\infty}z^nW_n^*(s)=\sum_{n=0}^{\infty}z^n\int_0^{\infty}W_n^*(x,s)dx\,,$$

$$V^{L^*}(x,z,s)=\sum_{n=0}^{\infty}z^nV^{L^*}{}_n(x,s)\,,\ V^{L*}(z,s)=\sum_{n=0}^{\infty}z^nV_n^{L*}(s)=\sum_{n=0}^{\infty}z^n\int_0^{\infty}V^{L^*}{}_n(x,s)dx\ ,$$

$$V^{S^*}(x,z,s)=\sum_{n=0}^{\infty}z^nV^{S^*}{}_n(x,s)\,,\ V^{S*}(z,s)=\sum_{n=0}^{\infty}z^nV_n^{S*}(s)=\sum_{n=0}^{\infty}z^n\int_0^{\infty}V^{S^*}{}_n(x,s)dx\ ,$$

$$P_q^*(z,s)=\sum_{n=0}^{\infty}z^nP_n^*(s)\,,\ |\,z\,|\,\le 1\,. \tag{4.4}$$

We take Laplace transform of equations (3.1) through (3.10) and use (3.11), (4.2), (4.3 and (4.4)). Thus we have

$$\frac{\partial}{\partial x}W_n{}^*(x,s)+(s+\lambda+\mu(x))W_n{}^*(x,s)=\lambda\sum_{i=1}^{n-1}c_iW_{n-i}{}^*(x,s)$$

$$\frac{\partial}{\partial x}W_n^{\ *}(x,s)+(s+\lambda+\mu(x))W_n^{\ *}(x,s)=\lambda\sum_{i=1}^{n-1}c_iW_{n-i}^{\ *}(x,s)\text{ , n≥1,}\tag{4.5}$$

$$\frac{\partial}{\partial x}W_0^{\ *}(x,s)+(s+\lambda+\mu(x))W_0^{\ *}(x,s)=0,\tag{4.6}$$

$$(s+\lambda)Q^*(s)=1+\int_0^\infty V_0^L(x,s)+\int_0^\infty V^{*S}{}_0(x,s)\beta_2(x)dx+p_3\int_0^\infty W_0^{\ *}(x,s)\mu_1(x)dx,\tag{4.7}$$

$$\frac{\partial}{dx}V_n^{*L}(x,s)+(s+\lambda+\beta_1(x))V_n^{*L}(x,s)=\lambda\sum_{i=1}^{n-1}c_iV_{n-i}^{*L}(s)\text{ , }n\geq 1,\tag{4.8}$$

$$\frac{\partial}{dx}V_0^{*L}(x,s)+(s+\lambda+\beta_1(x))V_0^{*L}(x,s)=0,\tag{4.9}$$

$$\frac{\partial}{dx}V_n^{*S}(x,s)+\frac{\partial}{\partial t}V_n^{*S}(x,s)+(s+\lambda+\beta_2(x))V_n^{*S}(x,s)=\lambda\sum_{i=1}^{n-1}c_iV_{n-i}^{*S}(s)\text{ , }n\geq 1,\tag{4.10}$$

$$\frac{\Im}{dx}V_0^{*S}(x,s)+\frac{\partial}{\partial t}V_0^{*S}(s+\lambda+\beta_2(x))V_0^{S}(s)=0,\tag{4.11}$$

$$\begin{aligned}W_n^{\ *}(0,s)=p_3\int_0^\infty W_{n+1}^*(x,s)\mu(x)dx+\int_0^\infty V_{n+1}^{*L}(x,s)\beta_1(x)dx\\+\int_0^\infty V_{n+1}^{*S}(x,s)\beta_2(x)dx+\lambda\, c_{n+1}Q^*(s),\quad n\geq 0,\end{aligned}\tag{4.12}$$

$$V_n^{*L}(0,s)=p_1\int_0^\infty W_n^*(x,s)\mu(x)dx\text{ , }n\geq 0,\tag{4.13}$$

$$V_n^{*S}(0,s)=p_2\int_0^\infty W_n^*(x,s)\mu(x)dx\text{ , }n\geq 0.\tag{4.14}$$

We perform the operation $\sum_{n=1}^{\infty}(4.5)z^n+(4.6)$, use (4.4) and simplify. Thus we obtain

$$\frac{\partial}{\partial x}W^*(x,z,s)+(s+\lambda-\lambda C(z)+\mu(x))W^*(x,z,s)=0.\tag{4.15}$$

Similarly perform $\sum_{n=1}^{\infty}(4.8)z^n + (4.9)$ and $\sum_{n=1}^{\infty}(4.10)z^n + (4.11)$, use (4.4) and simplify. Then we obtain

$$\frac{\partial}{\partial x}V^{*L}(x,z,s) + (s+\lambda-\lambda C(z)+\beta_1(x))V^{*L}(x,z,s) = 0\,, \tag{4.16}$$

$$\frac{\partial}{\partial x}V^{*S}(x,z,s) + (s+\lambda-\lambda C(z)+\beta_2(x))V^{*S}(x,z,s) = 0\,. \tag{4.17}$$

We again perform similar opeartions on (4.12) and use (4.4) and simplify. This yields

$$\begin{aligned} zW^*(0,z,s) = {} & p_3\int_0^\infty W^*(x,z,s)\mu(x)dx + \int_0^\infty V^{*L}(x,z,s)\,\beta_1(x)dx + \int_0^\infty V^{*S}(x,z,s)\beta_2(x)dx + \\ & -\left(\int_0^{\aleph} V_0^L(x,s) + \int_0^\infty V^{*S}_0(x,s)\beta_2(x)dx + p_3\int_0^\infty W_0{}^*(x,s)\mu_1(x)dx\right) + \lambda C(z)Q^*(s)\,, \end{aligned} \tag{4.18}$$

On using (4.7), equation (4.18) yields

$$\begin{aligned} zW^*(0,z,s) = {} & p_3\int_0^\infty W^*(x,z,s)\mu(x)dx + \int_0^\infty V^{*L}(x,z,s)\,\beta_1(x)dx + \int_0^\infty V^{*S}(x,z,s)\beta_2(x)dx + \\ & + \ 1-\big(s+\lambda-\lambda C(z)\big)Q^*(s)\,. \end{aligned} \tag{4.19}$$

And yet again, a similar operation on equations (4.13) and (4.14), using (4.4), we obtain

$$V^{*L}(0,z,s) = p_1\int_0^\infty W^*(x,z,s)\mu(x)dx\,, \tag{4.20}$$

$$V^{*S}(0,z,s) = p_2\int_0^\infty W^*(x,z,s)\mu(x)dx\,. \tag{4.21}$$

Now, we integrate equations (4.15), (4.16) and (4.17) beween *0* and *x*, we obtain

$$W^*(x,z,s) = W^*(0,z,s)\ \exp\left[-\big(s+\lambda-\lambda C(z)\big)x - \int_0^x \mu(t)dt\right], \tag{4.22}$$

$$V^{*L}(x,z,s) = V^{*L}(0,z,s)\exp\left[\ -\left(\ s+\lambda-\lambda C(z)+\mu_2+\frac{\mu_2}{z}\right)x - \int_0^x \beta_1(t)dt\ \right], \tag{4.23}$$

$$V^{*S}(x,z,s) = V^{*}S(0,z,s)\ \exp\left[-\left(s+\lambda-\lambda C(z)\right)x - \int_0^x \beta_2(t)dt\right], \tag{4.24}$$

where $W^*(0,z,s)$, $V^{*L}(0,z,s)$ and $V^{*S}(0,z,s)$ are given by (4.19), (4.20) and (4.21) respectively.

We again integrate (4.22), (4.23) and (4.24) w.r.t. x by parts and use equations (2.1) to (2.6). Thus we obtain

$$W^*(z,s) = W^*(0,z,s)\left[\frac{1-B^*[s+\lambda-\lambda C(z)]}{s+\lambda-\lambda C(z)}\right], \tag{4.25}$$

where $B^*[s+\lambda-\lambda z] = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dB(x)$ is the Laplace-Stieltjes transform of the service time S,

$$V^{*L}(z,s) = V^{*L}(0,z,s)\ \left[\frac{1\text{- A}_1^*[s+\lambda-\lambda C(z)]}{s+\lambda-\lambda C(z)}\right], \tag{4.26}$$

where $A_1^*[s+\lambda-\lambda C(z)] = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dA_1(x)$ is the Laplace-Stieltjes transform of the long vacation time V_1,

$$V^{*S}(z,s) = V^{*S}(0,z,s)\left[\frac{1-A_2^*[s+\lambda-\lambda C(z)]}{s+\lambda-\lambda C(z)}\right], \tag{4.27}$$

where $A_2^*[s+\lambda-\lambda C(z)] = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dA_2(x)$ is the Laplace-Stieltjes transform of the short vacation time V_2.

Next, we shall determine the integrals $\int_0^\infty W^*(x,z,s)\mu(x)dx$, $\int_0^\infty V^{*L}(x,z,s)\beta_1(x)dx$ and $\int_0^\infty V^{*S}(x,z,s)\beta_2(x)dx$ appearing in the right hand sides of equations (4.19), (4.20) and (4.21).

For this purpose, we multiply (4.22), (4.23) and (4.23) by $\mu(x)$, $\beta_1(x)$ and $\beta_2(x)$ respectively , integrate each w.r.t. x and use (2.2), (2.4) and (2.6) respectively. Thus we have

$$\int_0^\infty W^*(x,z,s)\mu(x)dx = W^*(0,z,s)\ B^*[s+\lambda-\lambda C(z)\,]\ , \tag{4.28}$$

$$\int_0^\infty V^{*L}(x,z,s)\beta_1(x)dx = V^{*L}(0,z,s)\ A_1^*[s+\lambda-\lambda C(z)]] , \tag{4.29}$$

$$\int_0^\infty V^{*S}(x,z,s)\beta_2(x)dx = V^{*S}(0,z,s)\ A_2^*[s+\lambda-\lambda C(z)] . \tag{4.30}$$

Now, we use equations (4.28), (4.29) and (4.30) in (4.19), (4.20) and (4.21) and on simplifying we obtain

$$W^*(0,z,s) = \frac{\left[1-\left(s+\lambda-\lambda C(z)\right)Q^*(s)\right]}{z-B^*[s+\lambda-\lambda C(z)]\left[\begin{array}{l} p_1A_1^*[s+\lambda-\lambda C(z)] \\ \qquad + p_2A_2^*[s+\lambda-\lambda C(z)]+p_3\end{array}\right]} , \tag{4.31}$$

$$V^{*L}(0,z,s) = \frac{p_1B^*[s+\lambda-\lambda C(z)]\left[1-\left(s+\lambda-\lambda C(z)\right)Q^*(s)\right]}{z-B^*[s+\lambda-\lambda C(z)]\left[\begin{array}{l} p_1A_1^*[s+\lambda-\lambda C(z)] \\ \qquad + p_2A_2^*[s+\lambda-\lambda C(z)]+p_3\end{array}\right]} , \tag{4.32}$$

$$V^{*S}(0,z,s) = \frac{p_2B^*[s+\lambda-\lambda C(z)]\left[1-\left(s+\lambda-\lambda C(z)\right)Q^*(s)\right]}{z-B^*[s+\lambda-\lambda C(z)]\left[\begin{array}{l} p_1A_1^*[s+\lambda-\lambda C(z)] \\ \qquad + p_2A_2^*[s+\lambda-\lambda C(z)]+p_3\end{array}\right]} . \tag{4.33}$$

Then using equations (4.31), (4.32) and (4.33) into equations (4.25), (4.26) and (4.27) respectively, we obtain

$$W^*(z,s) = \frac{\left[1-\left(s+\lambda-\lambda C(z)\right)Q^*(s)\right]\left[\dfrac{1-B^*[s+\lambda-\lambda C(z)]}{s+\lambda-\lambda C(z)}\right]}{z-B^*[s+\lambda-\lambda C(z)]\left[\begin{array}{l} p_1A_1^*[s+\lambda-\lambda C(z)] \\ \qquad + p_2A_2^*[s+\lambda-\lambda C(z)]+p_3\end{array}\right]} , \tag{4.34}$$

$$V^{*L}(z,s) = \frac{p_1B^*[s+\lambda-\lambda C(z)]\left[1-\left(s+\lambda-\lambda C(z)\right)Q^*(s)\right]\left[\dfrac{1-A_1^*[s+\lambda-\lambda C(z)]}{s+\lambda-\lambda C(z)}\right]}{z-B^*[s+\lambda-\lambda C(z)]\left[\begin{array}{l} p_1A_1^*[s+\lambda-\lambda C(z) \\ \qquad + p_2A_2^*[s+\lambda-\lambda C(z)]+p_3\end{array}\right]} , \tag{4.35}$$

$$V^{*S}(z,s)=\frac{p_2B^*[s+\lambda-\lambda C(z)][1-(s+\lambda-\lambda C(z))]Q^*(s)\left[\dfrac{1-A_2^*[s+\lambda-\lambda C(z)]}{s+\lambda-\lambda C(z)}\right]}{z-B^*[s+\lambda-\lambda C(z)]\left[\begin{array}{l}p_1A_1^*[s+\lambda-\lambda C(z)]\\ \quad +p_2A_2^*[s+\lambda-\lambda C(z)]+p_3\end{array}\right]}\ . \tag{4.36}$$

Further, on adding (4.34), (4.35) and (4.36), we obtain

$$P_q^*(z,s)=W^*(z,s)+V^{*(1)}(z,s)+V^{*(2)}(s)\,. \tag{4.37}$$

Now, if we put $z=1$ (and consequenltly $C(z)=1$) in (4.37), and use the normalizing condition $Q^*(s)+P_q^*(1,s)=1/s$ and deal with a lot of algebra, we find the following expression for $Q^*(s)$, which can be verified to be of the 0/0 form at *z=1*. Therefore, we take limit as $z\to 1$ and apply L'Hopital's rule on the following expression

$$Q^*(s)=\lim_{z\to 1}\frac{\begin{array}{l}\left(\dfrac{1}{s}\right)\left[1-B^*\left[s+\lambda-\lambda C(z)\right]\left[p_1A_1^*\left[s+\lambda-\lambda C(z)\right]+p_2A_2^*\left[s+\lambda-\lambda C(z)\right]+p_3\right]\right]\\ -\left(\dfrac{1-B^*\left[s+\lambda-\lambda C(z)\right]}{s+\lambda-\lambda C(z)}\right)-p_1B^*\left[s+\lambda-\lambda C(z)\right]\left(\dfrac{1-A_1^*\left[s+\lambda-\lambda C(z)\right]}{s+\lambda-\lambda C(z)}\right)\\ -p_2B^*\left[s+\lambda-\lambda C(z)\right]\left(\dfrac{1-A_{21}^*\left[s+\lambda-\lambda C(z)\right]}{s+\lambda-\lambda C(z)}\right)\end{array}}{\begin{array}{l}\left[1-B^*\left[s+\lambda-\lambda C(z)\right]\left[p_1A_1^*\left[s+\lambda-\lambda C(z)\right]+p_2A_2^*\left[s+\lambda-\lambda C(z)\right]+p_3\right]\right]\\ -\left(1-B^*\left[s+\lambda-\lambda C(z)\right]\right)-p_1B^*\left[s+\lambda-\lambda C(z)\right]\left(\dfrac{1-A_1^*\left[s+\lambda-\lambda C(z)\right]}{s+\lambda-\lambda C(z)}\right)\\ -p_2B^*\left[s+\lambda-\lambda C(z)\right]\left(\dfrac{1-A_{21}^*\left[s+\lambda-\lambda C(z)\right]}{s+\lambda-\lambda C(z)}\right)\end{array}}\ . \tag{4.38}$$

On further simplyfying the above we obtain

$$Q^*(s)=\frac{1}{s}\left(1-\lambda E(I)\left[\left(B^*\right)'(s)+p_1\left(A_1^*\right)'(s)+p_2\left(A_2^*\right)'(s)\right]\right), \tag{4.39}$$

Where $E(I)$ is the mean size of the arriving batch.

Thus, all desired probability generating functions have been determined completely.

5. The Steady State Results

Assuming that the steady state exists, we define the corresponding steady state probabilities and the probability generating functions by droping the argument *t* and, for that matter, the argument *s*, wherever it appears in the time-dependent analysis upto now. Then, the corresponding steady state results can be obtained by applying the well-known Tauberian property,

$$\underset{s\to 0}{Lim}\ sf^{*}(s) = \underset{t\to\infty}{Lim}\,\mathrm{f(t)}. \tag{5.1}$$

Next, multiplying both sides of equations (4.34), (4.35) and (4.36) by *s*, taking limit as $s \to 0$, applying the property (5.1) and simplifying, we obtain

$$W(z) = \frac{\left[B^{*}[\lambda-\lambda C(z)]-1\right]Q}{z-B^{*}[\lambda-\lambda C(z)]\left[\begin{array}{l} p_1 A_1^{*}[\lambda-\lambda C(z)] \\ \qquad + p_2 A_2^{*}[\lambda-\lambda C(z)] + p_3 \end{array}\right]}\ , \tag{5.2}$$

$$V^{L}(z) = \frac{p_1 B^{*}[\lambda-\lambda C(z)]\left[A_1^{*}[\lambda-\lambda C(z)]-1\right]Q}{z-B^{*}[\lambda-\lambda C(z)]\left[\begin{array}{l} p_1 A_1^{*}[\lambda-\lambda C(z)] \\ \qquad + p_2 A_2^{*}[\lambda-\lambda C(z)] + p_3 \end{array}\right]}\ , \tag{5.3}$$

$$V^{S}(z) = \frac{p_2 B^{*}[\lambda-\lambda C(z)]Q\left[A_2^{*}[\lambda-\lambda C(z)]-1\right]Q}{z-B^{*}[\lambda-\lambda C(z)]\left[\begin{array}{l} p_1 A_1^{*}[\lambda-\lambda C(z)] \\ \qquad + p_2 A_2^{*}[\lambda-\lambda C(z)] + p_3 \end{array}\right]}\ . \tag{5.4}$$

Further, we have on adding (5.2), (5.3) and (5.4),

$$P_q(z) = W(z) + V^{(1)}(z) + V^{(2)}(z)\,. \tag{5.5}$$

In order to determine the above generating functions completely, we shall now determine the only unknown constant *Q* by using the normalizing condition

$$P_q(1) + Q = 1\,. \tag{5.6}$$

We see that for $z=1$, each of $W(z)$, $V^L(z)$, $V^S(z)$ and in (5.2), (5.3) and (5.4) is indeterminate of the 0/0 form. Therefore, applying L'Hopital's rule, we obtain , on simplifying

$$W(1) = \lim_{z\to 1} W(z) = \frac{\lambda E(I)E(S)Q}{1-\lambda E(I)\left[E(S)+p_1E(V_1)+p_2E(V_2)\right]}, \tag{5.7}$$

$$V^L(1) = \lim_{z\to 1} V^L(z) = \frac{p_1\lambda E(I)E(V_1)Q}{1-\lambda E(I)\left[E(S)+p_1E(V_1)+p_2E(V_2)\right]}, \tag{5.8}$$

$$V^S(1) = \lim_{z\to 1} V^S(z) = \frac{p_2\lambda E(I)E(V_2)Q}{1-\lambda E(I)\left[E(S)+p_1E(V_1)+p_2E(V_2)\right]}, \tag{5.9}$$

Where $E(S)$ is the mean service time, $E(V_1)$ is the mean duration of the long vacation and $E(V_2)$ is the mean duration of the short vacation.

Using (5.7), (5.8) and (5.9) in (5.6), we obtain on simplifying

$$Q = 1-\lambda E(I)\left[E(S)+p_1E(V_1)+p_2E(V_2)\right]. \tag{5.10}$$

Note that (5.10) gives the steady state probability that the system is empty and the server is idle but available in the system. Further note that using the value of Q from (5.10) into equations (5.2), (5.3) and (5.4), we have completely and explicilty detrmined all the steady probability functions namely, $W(z)$, $V^L(z)$, $V^S(z)$ and for that matter $P_q(z)$. Further, if we substitute the value of Q obtained in (5.10) into (5.7), (5.8) and (5.9), we obtain

$$W(1) = \lambda E(I)E(S) \tag{5.11}$$

$$V^L(1) = p_1\lambda E(I)E(S) \tag{5.12}$$

$$V^S(1) = p_2\lambda E(I)E(S) \tag{5.13}$$

We may note that the value of Q found in (5.10) can be alternatively found from equation (4.39) as follows: Taking limit as $s \to 0$ and using $\left(B^*\right)'(0) = E(S)$, $\left(A_1^*\right)'(0) = E(V_1)$ and $\left(A_2^*\right)'(0) = E(V_2)$ for $s = 0$, we obtain

$$Q = \lim_{s \to 0} sQ^*(s) = 1 - \lambda E(I)\left[E(S) + p_1 E(V_1) + p_2 E(V_2)\right]. \tag{5.14}$$

Incidently, it also verifies the result obtained in (4.39).

Note that the stability condition under which the steady state exists emerges from (5.10 A). Thus we have

$$\lambda E(I)\left[E(S) + p_1 E(V_1) + p_2 E(V_2)\right] < 1. \tag{5.15}$$

Further, ρ , the utilisation factor of the system (the proportion of time the server is busy) is given by (5.11). Thus we note that (5.11) gives the steady state probability that the server is busy providing service to customers, (5.12) gives the steady state probability that the server is on a long vacation and (5.13) gives the steady state probability that the server is on a short vacation at any random point of time.

Let L_Q denote the mean queue size at a random epoch. Then using (5.5), we obtain

$$L_Q = \frac{d}{dz} P_Q(z)\Big|_{z=1}$$

$$= \frac{\lambda^2 \left(E(I)\right)^2 \left[E(S^2) + p_1 E(V_1^2) + p_2 E(V_2^2) + E(S)\left(p_1 E(V_1) + p_2 E(V_2)\right)\right] + \lambda E\left(I(I-1)\right)\left[E(S) + p_1 E(V_1) + p_2 E(V_2)\right]}{2\left[1 - \lambda E(I)\left(E(S) + p_1 E(V_1) + p_2 E(V_2)\right)\right]}. \tag{5.16}$$

Next, the steady state average number of customers in the system, the average waiting time in the queue and the average waiting time in thesystem are respectively given by

$$L = L_q + \rho, \qquad W_q = \frac{L_q}{\lambda}, \qquad W = \frac{L}{\lambda} \tag{5.17}$$

6. Particular Cases

Case 1 : Single arrivals, exponetial service times, exponential duration of long vacation and exponential duration of short vacation

In this case we have

$E(I)=1$, $E(I(I-1))=0$, $E(S)=\dfrac{1}{\mu}$, $E(S^2)=\dfrac{2}{\mu^2}$ $E(V_1)=\dfrac{1}{\beta_1}$, $E(V_1{}^2)=\dfrac{2}{\beta_1{}^2}$, $E(V_2)=\dfrac{1}{\beta_2}$,

$E(V_2{}^2)=\dfrac{2}{\beta_2{}^2}$, $B^*[\lambda-\lambda C(z)]=\left(\dfrac{\mu}{\mu+\lambda-\lambda C(z)}\right)$, $A_1^*[\lambda-\lambda C(z)]=\left(\dfrac{\beta_1}{\beta_1+\lambda-\lambda C(z)}\right)$ and

$A_2^*[\lambda-\lambda C(z)]=\left(\dfrac{\beta_2}{\beta_2+\lambda-\lambda C(z)}\right)$. Consequently,

$B^*[\lambda-\lambda C(z)]-1=\left(\dfrac{\mu}{\mu+\lambda-\lambda C(z)}-1\right)$

With these substitutions in the main results we obtain the following results for this case.

$$W(z)=\frac{-\left(\dfrac{\lambda-\lambda C(z)}{\mu+\lambda-\lambda C(z)}\right)Q}{z-B^*[\lambda-\lambda C(z)]\begin{bmatrix} p_1A_1^*[\lambda-\lambda C(z)] \\ +p_2A_2^*[\lambda-\lambda C(z)]+p_3\end{bmatrix}}, \tag{6.1}$$

$$V^L(z)=\frac{-p_1\left(\dfrac{\mu}{\mu+\lambda-\lambda C(z)}\right)\left(\dfrac{\lambda-\lambda C(z)}{\beta_1+\lambda-\lambda C(z)}\right)Q}{z-B^*[\lambda-\lambda C(z)]\begin{bmatrix} p_1A_1^*[\lambda-\lambda C(z)] \\ +p_2A_2^*[\lambda-\lambda C(z)]+p_3\end{bmatrix}}, \tag{6.2}$$

$$V^S(z)=\frac{-p_2\left(\dfrac{\mu}{\mu+\lambda-\lambda C(z)}\right)\left(\dfrac{\lambda-\lambda C(z)}{\beta_2+\lambda-\lambda C(z)}\right)Q}{z-B^*[\lambda-\lambda C(z)]\begin{bmatrix} p_1A_1^*[\lambda-\lambda C(z)] \\ +p_2A_2^*[\lambda-\lambda C(z)]+p_3\end{bmatrix}}, \tag{6.3}$$

Where

$$Q = 1 - \lambda\left[\left(\frac{1}{\mu}\right) + p_1\left(\frac{1}{\beta_1}\right) + p_2\left(\frac{1}{\beta_2}\right)\right], \tag{6.4}$$

Stability Condition:

$$\lambda\left[\left(\frac{1}{\mu}\right) + p_1\left(\frac{1}{\beta_1}\right) + p_2\left(\frac{1}{\beta_2}\right)\right] < 1, \tag{6.5}$$

$$\rho = W(1) = \frac{\lambda}{\mu}, \qquad V^L(1) = p_1\left(\frac{\lambda}{\beta_1}\right), \qquad V^S(1) = p_2\left(\frac{\lambda}{\beta_2}\right), \tag{6.6}$$

$$L_Q = \frac{\lambda^2\left[\frac{2}{\mu^2} + p_1{}^2\left(\frac{2}{\beta_1{}^2}\right) + p_2{}^2\left(\frac{2}{\beta_2{}^2}\right) + \left(\frac{1}{\mu}\right)\left(p_1\left(\frac{1}{\beta_1}\right) + p_2\left(\frac{1}{\beta_2}\right)\right)\right]}{2\left[1 - \lambda\left\{\frac{1}{\mu} + p_1\left(\frac{1}{\beta_1}\right) + p_2\left(\frac{1}{\beta_2}\right)\right\}\right]}. \tag{6.7}$$

$$L = \rho + L_q = \rho + \frac{\lambda^2\left[\frac{2}{\mu^2} + p_1{}^2\left(\frac{2}{\beta_1{}^2}\right) + p_2{}^2\left(\frac{2}{\beta_2{}^2}\right) + \left(\frac{1}{\mu}\right)\left(p_1\left(\frac{1}{\beta_1}\right) + p_2\left(\frac{1}{\beta_2}\right)\right)\right]}{2\left[1 - \lambda\left\{\frac{1}{\mu} + p_1\left(\frac{1}{\beta_1}\right) + p_2\left(\frac{1}{\beta_2}\right)\right\}\right]}, \tag{6.08}$$

$$W_q = \frac{L_q}{\lambda} = \frac{\lambda\left[\frac{2}{\mu^2} + p_1{}^2\left(\frac{2}{\beta_1{}^2}\right) + p_2{}^2\left(\frac{2}{\beta_2{}^2}\right) + \left(\frac{1}{\mu}\right)\left(p_1\left(\frac{1}{\beta_1}\right) + p_2\left(\frac{1}{\beta_2}\right)\right)\right]}{2\left[1 - \lambda\left\{\frac{1}{\mu} + p_1\left(\frac{1}{\beta_1}\right) + p_2\left(\frac{1}{\beta_2}\right)\right\}\right]}, \tag{6.09}$$

The steady state average waiting time in the system,

$$W = \frac{L}{\lambda} = \frac{\lambda^2}{\mu} + \frac{\lambda\left[\frac{2}{\mu^2} + p_1{}^2\left(\frac{2}{\beta_1{}^2}\right) + p_2{}^2\left(\frac{2}{\beta_2{}^2}\right) + \left(\frac{1}{\mu}\right)\left(p_1\left(\frac{1}{\beta_1}\right) + p_2\left(\frac{1}{\beta_2}\right)\right)\right]}{2\left[1 - \lambda\left\{\frac{1}{\mu} + p_1\left(\frac{1}{\beta_1}\right) + p_2\left(\frac{1}{\beta_2}\right)\right\}\right]}, \tag{6.10}$$

Case 2 : No short vacations

The results of this acse can be derived from the main results by putting $p_2 = 0$.

Case 3: No long vacations

In this case, we let $p_1 = 0$ in the above results.

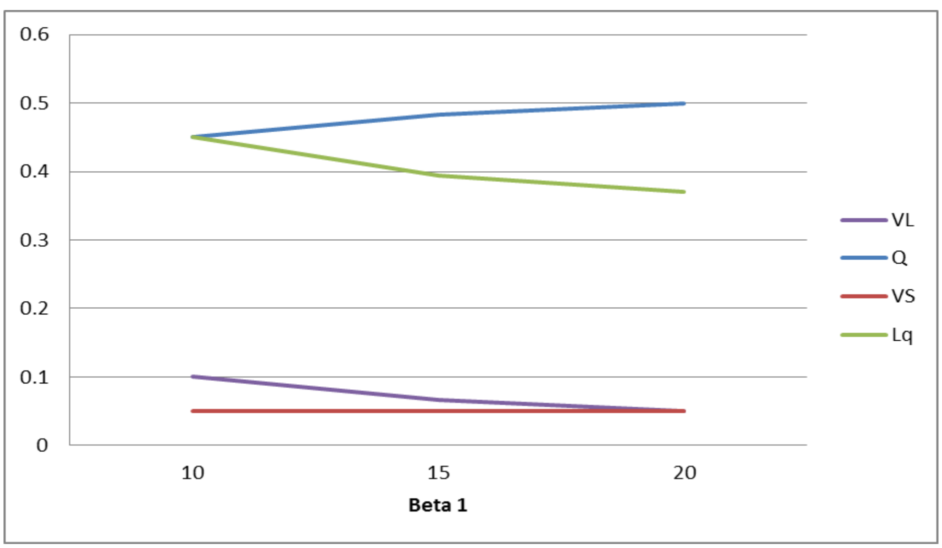
Case 4: No server vacations

In this case we let $p_1 = 0$, $p_2 = 0$ and $p_3 = 1$ in the main results.

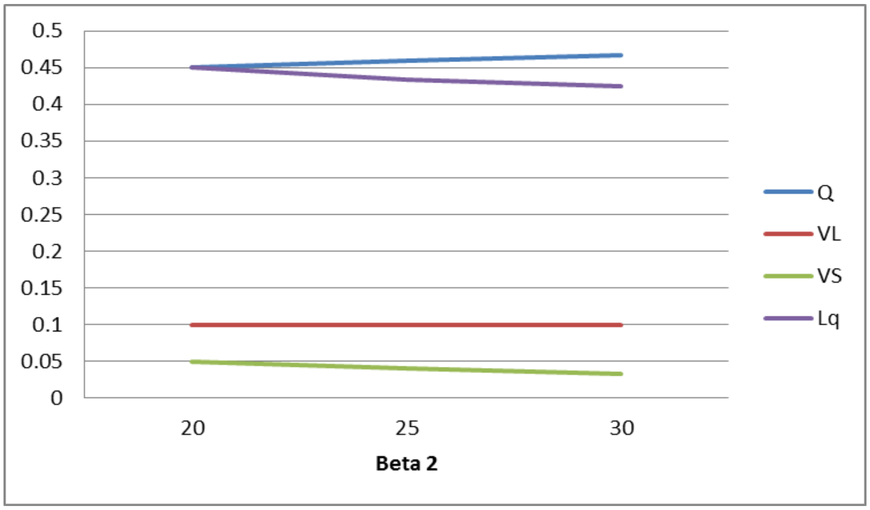
7. **Numerical Examples and Graphical Presentations**

We base our numerical example on the results of case 1 and arbitrarily fix $\lambda = 2$, $\mu = 5$, and varying values of $\beta_1 = 10, 15, 20$, $\beta_2 = 20, 25, 30$, $p_1 = 0, 0.5, 1$ and $p_2 = 0, 0.5, 1$. These values are chosen so as the stability condition (6.5) is not violated.

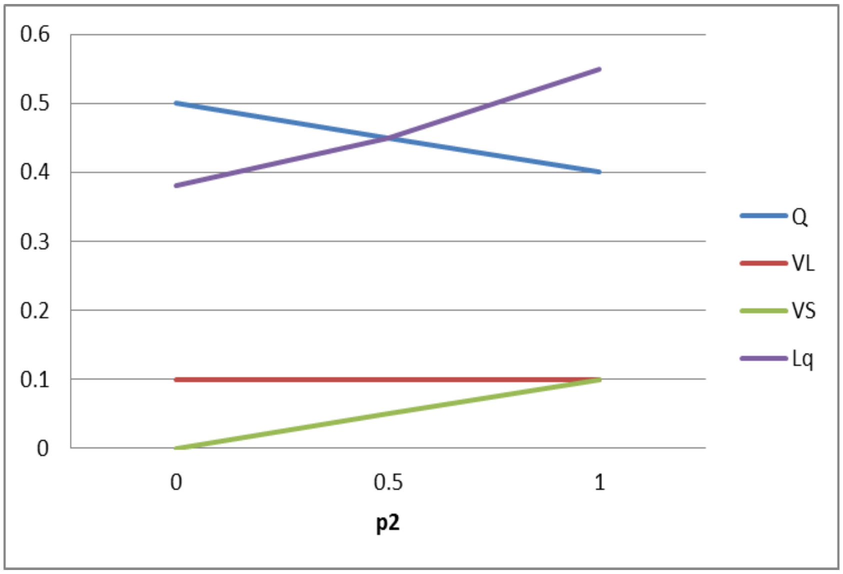
β_1	β_2	p_1	p_2	Q	$V^L(1)$	$V^S(1)$	L_Q
10	20	0.5	0.5	0.45	0.1	0.05	0.45
15	20	0.5	0.5	0.4833	0.0667	0.05	0.3937
20	20	0.5	0.5	0.5	0.05	0.05	0.37



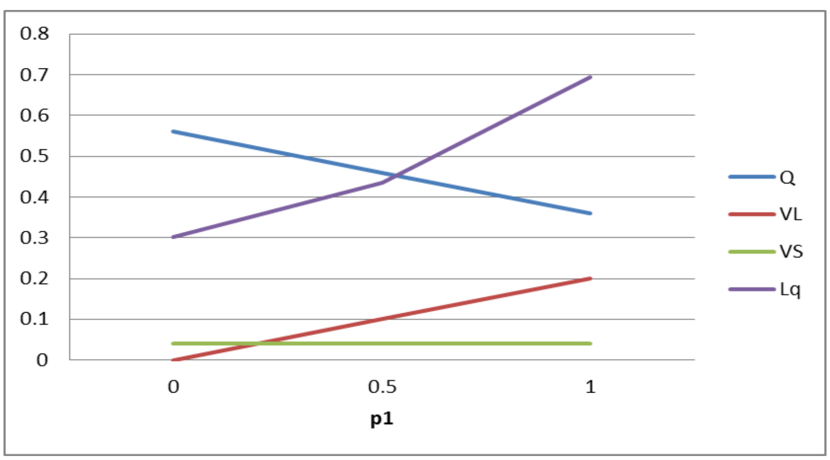
β_1	β_2	p_1	p_2	Q	$V^L(1)$	$V^S(1)$	L_Q
10	20	0.5	0.5	0.45	0.1	0.05	0.45
10	25	0.5	0.5	0.46	0.1	0.04	0.4339
10	30	0.5	0.5	0.4667	0.1	0.0332	0.4238



β_1	β_2	p_1	p_2	Q	$V^L(1)$	$V^S(1)$	L_Q
10	20	0.5	0	0.5	0.1	0	0.38
10	20	0.5	0.5	0.45	0.1	0.05	0.45
10	20	0.5	1	0.4	0.1	0.1	0.55



β_1	β_2	p_1	p_2	Q	$V^L(1)$	$V^S(1)$	L_Q
10	25	0	0.5	0.56	0	0.04	0.30285
10	25	0.5	0.5	0.46	0.1	0.04	0.4339
10	25	1	0.5	0.36	0.2	0.04	0.6933



Trends and Conclusions

From the above graphs we note the following trends and conclusions:

(1) For fixed values of λ, μ, β_2, p_1 and p_2, as β_1 increases, Q increases, $V^L(1)$ decreases, $V^S(1)$ remains constant and L_Q, W_Q, L and W all decrease.

(2) For fixed values of λ, μ, β_1, p_1 and p_2, as β_2 increases, Q increases, $V^S(1)$ decreases, $V^L(1)$ remains constant and L_Q, W_Q, L and W all decrease.

(3) For fixed values of λ, μ, β_1, β_2 and p_2, as p_1 increases, Q decreases, $V^L(1)$ increases, $V^S(1)$ remains constant and L_Q, W_Q, L and W all increase.

(4) For fixed values of λ, μ, β_1, β_2 and p_2, as p_2 increases, Q decreases, $V^S(1)$ increases, $V^L(1)$ remains constant and L_Q, W_Q, L and W all increase.

All results are as expected.

Conflict of Interests

The authors declare that there is no conflict of interests.

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