Available online at http://scik.org
J. Math. Comput. Sci. 6 (2016), No. 4, 568-580
ISSN: 1927-5307

# THE MULTIPLICATIVE VERSION OF DEGREE DISTANCE AND THE MULTIPLICATIVE VERSION

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**Abstract.** In this paper, the multiplicative version of degree distance and the multiplicative version of Gutman index of Cartesian product of graphs are derived and the indices are evaluated for some well-known graphs such as hypercube, hypertorus and grid.

**Keywords:** Cartesian product; Multiplicative version of degree distance; Multiplicative version of Gutman index; Degree distance; Wiener index.

**2010 AMS Subject Classification:** 05C07, 05C76.

## 1. Introduction

Throughout this paper, we consider simple graphs which are finite, indirected graphs without loops and multiple edges. Suppose $G$ is a graph with a vertex set $V(G)$and an edge set $E(G)$. For a graph $G$, the degree of a vertex $v$ is the number of edges incident to $v$ and is denoted by $d_G(v)$. A topological index $Top(G)$ of a graph $G$ is a number with the property that for every

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Received November 26, 2015

graph $H$ is isomorphic to $G$ , $Top(H) = Top(G)$. Notations and definitions which are not given here can be found in [1] and [2]. The Wiener index of a graph $G$ denoted by $W(G)$ is defined as $W = W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v) = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)$. In [10], [11], the multiplicative version of the Wiener index was conceived by Gutman et al: $\pi = \pi(G) = \prod_{\{u,v\}\subseteq V(G)} d_G(u,v) = \frac{1}{2}\prod_{u,v\in V(G)} d_G(u,v)$.

The topological indices and graph invariants based on distances between vertices of a graph are widely used for characterizing molecular graphs, establishing relationships between structure and properties of molecules, predicting biological activity of chemical compounds and making their chemical applications [7]. The Wiener index is one of the most topological indices with high correlation with many physical and chemical indices of molecular compounds.

There are some topological indices based on degrees such as the first and second Zagreb indices of molecular graphs. The first and second kinds of Zagreb indices were first introduced in [6] (see also [5]). It is reported that these indices are useful in the study of anti-inflammatory activities of certain Chemical instances, and in other practical aspects. The first Zagreb index $M_1(G)$ and second Zagreb index $M_2(G)$ of graph $G$ are defined as $M_1(G) = \sum_{uv\in E(G)}[d_G(u) + d_G(v)] = \sum_{u\in V(G)} d_G^2(u)$ and $M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$.

The degree distance was introduced by Dobrynin and Kochetova [3] and Gutman [4] as a weighted version of the Wiener index. The degree distance of $G$, denoted by $DD(G)$, is defined as

$$DD(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)[d_G(u) + d_G(v)] = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)[d_G(u) + d_G(v)]$$

with the summation going over all pairs of vertices of $G$.

In [4], Gutman defined the Schultz index of the second kind which is now known as the Gutman index . The Gutman index of $G$ denoted by $Gut(G)$, is defined as

$$Gut(G) = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)[d_G(u)d_G(v)],$$

the summation runs over all the pairs of vertices of $G$. Recently , Todeschini et al [12, 13] have proposed the multiplicative variants of ordinary Zagreb indices, which are defined as follows:

$$\prod_1 = \prod_1(G) = \prod_{v\in V(G)} d_G^2(v) = \prod_{uv\in E(G)} [d_G(u) + d_G(v)]$$

and

$$\prod_2 = \prod_2(G) = \prod_{uv\in E(G)} [d_G(u)d_G(v)].$$

In this paper, we have found a new graph invariant named multiplicative version of degree distance and multiplicative version of the Gutman index, which can be seen as a weighted version of the Wiener index that is $DD^*(G) = \frac{1}{2}\prod_{u,v\in V(G)} d_G(u,v)[d_G(u)+d_G(u)]$ , $Gut^*(G) = \frac{1}{2}\prod_{u,v\in V(G)} d_G(u,v)[d_G(u)d_G(u)]$.

The Cartesian product of the graph $G_1$ and $G_2$,denoted by $G_1\Box G_2$ has the vertex set $V(G_1\Box G_2) = V(G_1)\times V(G_2)$ and $(u,x)(v,y)$ is an edge of $G_1\Box G_2$ is $u = v$ and $xy \in E(G_2)$ or $uv \in E(G_1)$ and $x = y$.

In this paper, we obtain the multiplicative version of degree distance and the multiplicative version of the Gutman index of the cartesian product of graphs. Also we apply some of our results to compute the exact multiplication version of degree distance and the exact multiplicative version of the Gutman index of $C_4$ nanotube, $C_4$ nanotorus and grid.

Throughout this paper the degree distance and the Gutman index are denoted as $DD^+(G)$ and $Gut^+(G)$. And the multiplicative version of degree distance and the multiplicative version of the Gutman index are denoted as $DD^*(G)$ and $Gut^*(G)$ respectively.

## 2. Multiplicative version of degree distance of cartesian product of graphs

We begin this section with standard inequality as follows:

**Lemma 2.1.** (Arithmetic Geometric inequality) *Let $x_1,x_2,...,x_n$ be non-negative numbers. Then* $\frac{x_1+x_2+...+x_n}{n} \geq \sqrt[n]{x_1x_2...x_n}$.

In this section, we compute the multiplicative version of the degree distance of the Cartesian product of $G_1\Box G_2$ of the graphs $G_1$ and $G_2$. Let $V(G_1) = \{u_0,u_1,...,u_{n_1-1}\}, V(G_2) = \{v_0,v_1,...,v_{n_2-1}\}$, and let $w_{ij}$ denote the vertex $(u_i,v_j)$ of $G_1\Box G_2$.

The following lemma follows from the definition of Cartesian product of two graphs.

**Lemma 2.2.** *Let $G_1$ and $G_2$ be two connected graphs. Let $w_{ij} = (u_i,v_j)$ and $w_{pq} = (u_p,v_q)$ be in $V(G_1\Box G_2)$. Then $d_{G_1\Box G_2}(w_{ij},w_{pq}) = d_{G_1}(u_i,u_p)+d_{G_2}(v_j,v_q)$ and $d_{G_1\Box G_2}(w_{ij}) = d_{G_1}(u_i)+d_{G_2}(u_j)$.*

**Theorem 2.3.** *If $G_1$ and $G_2$ are two connected graphs with $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$, where $n_1, n_2 \geq 2$, then* $DD^*(G) \leq 2^{3n_1n_2-1}\left[\frac{n_2DD^+(G_1)+4e(G_2)W(G_1)}{n_1n_2}\right]^{n_1n_2} \times \left[\frac{n_1DD^+(G_2)+4e(G_1)W(G_2)}{n_1n_2}\right]^{n_1n_2} \times \left(\frac{n_2(n_2-1)DD^+(G_1)+4(n_2-1)e(G_2)W(G_1)+4(n_1-1)e(G_1)W(G_2)+n_1(n_1-1)DD^+(G_2)}{n_1n_2}\right)^{n_1n_2}$,
*where $W(G)$, $DD(G)$ and $e(G)$ denote the Wiener index, degree distance and the number of edges of $G$ respectively.*

**Proof.** Let $G = G_1 \square G_2$.Then

$$\begin{aligned}
DD^*(G) &= \frac{1}{2}\prod_{w_{ij},w_{pq}\in V(G)} d_G(w_{ij},w_{pq})[d_G(w_{ij})+d_G(w_{pq})] \\
&= \frac{1}{2}\Big(\prod_{j=0}^{n_2-1}\prod_{i,p=0,i\neq p}^{n_1-1} d_G(w_{ij},w_{pj})[d_G(w_{ij})+d_G(w_{pj})] \\
&\times \prod_{i=0}^{n_1-1}\prod_{j,q=0,j\neq q}^{n_2-1} d_G(w_{ij},w_{iq})[d_G(w_{ij})+d_G(w_{iq})] \\
&\times \prod_{j,q=0,j\neq q}^{n_2-1}\prod_{i,p=0,i\neq p}^{n_1-1} d_G(w_{ij},w_{pq})[d_G(w_{ij})+d_G(w_{pq})]\Big) \\
&= \frac{1}{2}(A_1\times A_2\times A_3), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)
\end{aligned}$$

where $A_1$, $A_2$ and $A_3$ are the products of the above terms, in order. We calculate $A_1$, $A_2$ and $A_3$ of $(1)$ separately.

First we compute

$$\begin{aligned}
A_1 &= \prod_{j=0}^{n_2-1}\prod_{i,p=0,i\neq p}^{n_1-1} d_G(w_{ij},w_{pj})\big[d_G(w_{ij})+d_G(w_{pj})\big] \\
&= \prod_{j=0}^{n_2-1}\prod_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i,u_p)\big[d_{G_1}(u_i)+d_{G_2}(v_j)+d_{G_1}(u_p)+d_{G_2}(v_j)\big]_{by Lemma (2.2)} \\
&= \prod_{j=0}^{n_2-1}\prod_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i,u_p)[d_{G_1}(u_i)+d_{G_1}(u_p)+2d_{G_2}(v_j)] \\
&\leq \Big(\frac{\sum_{j=0}^{n_2-1}\sum_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i,u_p)[d_{G_1}(u_i)+d_{G_1}(u_p)+2d_{G_2}(v_j)]}{n_1n_2}\Big)^{n_1n_2}_{by Lemma (2.1)} \\
&= \Big[\frac{\sum_{j=0}^{n_2-1} 2DD^+(G_1)+4W(G_1)d_{G_2}(v_j)}{n_1n_2}\Big]^{n_1n_2}
\end{aligned}$$

by the definition of Wiener index and the degree distance of a graph

$$A_1 \leq \Big[\frac{2n_2DD^+(G_1)+8e(G_2)W(G_1)}{n_1n_2}\Big]^{n_1n_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Next we compute

$$\begin{aligned}
A_2 &= \prod_{i=0}^{n_1-1}\prod_{j,q=0,j\neq q}^{n_2-1} d_G(w_{ij},w_{iq})[d_G(w_{ij})+d_G(w_{iq})] \\
&= \prod_{i=0}^{n_1-1}\prod_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j,v_q)\big[d_{G_1}(u_i)+d_{G_2}(v_j)+d_{G_1}(u_i)+d_{G_2}(v_j)\big]_{byLemma(2.2)} \\
&= \prod_{i=0}^{n_1-1}\prod_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j,v_q)[2d_{G_1}(u_i)+d_{G_2}(v_j)+d_{G_2}(v_q)] \\
&\leq \Big[\frac{\sum_{i=0}^{n_1-1}\sum_{j,q=0,i\neq q}^{n_2-1} d_{G_2}(v_j,v_q)[2d_{G_1}(u_i)+d_{G_2}(v_j)+d_{G_2}(v_q)]}{n_1n_2}\Big]^{n_1n_2}_{byLemma(2.1)} \\
&= \Big[\frac{\sum_{i=0}^{n_1-1} 2DD^+(G_2)+4W(G_2)d_{G_1}(u_i)}{n_1n_2}\Big]^{n_1n_2}
\end{aligned}$$

by the definition of Wiener index and the degree distance of a graph

$$A_2 \leq \Big[\frac{2n_1DD^+(G_2)+8e(G_1)W(G_2)}{n_1n_2}\Big]^{n_1n_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Next we compute

$$\begin{aligned}
A_3 &= \prod_{j,q=0,j\neq q}^{n_2-1}\prod_{i,p=0,i\neq p}^{n_1-1} d_G(w_{ij},w_{pq})\big[d_G(w_{ij})+d_G(w_{pq})\big] \\
&= \prod_{j=0,q=0,j\neq q}^{n_2-1}\prod_{i,p=0,i\neq p}^{n_i-1} \big[d_{G_1}(u_i,u_p)+d_{G_2}(v_j,v_q)\big] \times \Big[d_{G_1}(u_i)+d_{G_2}(v_j)+d_{G_i}(u_p)+d_{G_{2(v_q)}}\Big] \\
&\leq \Big[\frac{\sum_{j,q=0,j\neq q}^{n_2-1}\sum_{i,p=0,i\neq p}^{n_1-1}[d_{G_1}(u_i,u_p)+d_{G_2}(v_j,v_q)]+\big[d_{G_1}(u_i)+d_{G_2}(v_j)+d_{G_1}(u_p)+d_{G_2}(v_q)\big]}{n_1n_2}\Big]^{n_1n_2} \\
&= \Big[\frac{S}{n_1n_2}\Big]^{n_1n_2}.
\end{aligned}$$

Here

$$\begin{aligned}
S = &\sum_{j,q=0,j\neq q}\Big[2DD^+(G_1)+2W(G_1)[d_{G_2}(v_j)+d_{G_2}(v_q)]+d_{G_2}(v_j,v_q)4(n_1-1)e(G_1) \\
&+n_1(n_1-1)d_{G_2}(v_j,v_q)[d_{G_2}(v_j)+d_{G_2}(v_q)]\Big],
\end{aligned}$$

where

$$\sum_{i,p=0,i\neq p}^{n_1-1} 1 = 2\binom{n_1}{2} \quad and \quad \sum_{i,p=0,i\neq p}^{n_1-1} [d_{G_1}(u_i)+d_{G_1}(u_p)] = 4(n_1-1)e(G_1)$$

and by the definition of Wiener index and the additive version of degree distance of a graph.

$A_3 \leq$

$$\left[\frac{2n_2(n_2-1)DD^+(G_1)+8(n_2-1)e(G_2)W(G_1)+8(n_1-1)e(G_1)W(G_2)+2n_1(n_1-1)DD^+(G_2)}{n_1n_2}\right]^{n_1n_2} \quad (4)$$

Using $(2),(3)$ and $(4)$ in $(1)$, we get

$$DD^*(G) \leq \frac{1}{2}\left[\left(\frac{2n_2DD^+(G_1)+8e(G_2)W(G_1)}{n_1n_2}\right)^{n_1n_2} \times \left(\frac{2n_1DD^+(G_2)+8e(G_1)W(G_2)}{n_1n_2}\right)^{n_1n_2}\right.$$

$$\left.\times \left(\frac{2n_2(n_2-1)DD^+(G_1)+8(n_2-1)e(G_2)W(G_1)+8(n_1-1)e(G_1)W(G_2)+2n_1(n_1-1)DD^+(G_2)}{n_1n_2}\right)^{n_1n_2}\right]$$

$$DD^*(G) \leq 2^{3n_1n_2-1}\left[\frac{n_2DD^+(G_1)+4e(G_2)W(G_1)}{n_1n_2}\right]^{n_1n_2} \times \left[\frac{n_1DD^+(G_2)+4e(G_1)W(G_2)}{n_1n_2}\right]^{n_1n_2}$$

$$\times \left[\frac{n_2(n_2-1)DD^+(G_1)+4(n_2-1)e(G_2)W(G_1)+4(n_1-1)e(G_1)W(G_2)+n_1(n_1-1)DD^+(G_2)}{n_1n_2}\right]^{n_1n_2}.$$

**Lemma 2.4.** [10] *Let $P_n$ and $C_n$ denote the path and the cycle $n$ on vertices respectively.*

$$(1).For, n \geq 2, W(P_n) = \frac{n(n^2-1)}{6}$$

$$(2).For, n \geq 3, W(C_n) = \begin{cases} \frac{n^3}{8}, & if\ n\ is\ even, \\ \frac{n(n^2-1)}{8}, & if\ n\ is\ odd. \end{cases}$$

**Lemma 2.5**. *Let $P_n$ and $C_n$ denote the path and the cycle on $n$ vertices, respectively.*

$$(1)For, n \geq 2, DD(P_n) = \frac{n(n-1)(2n-1)}{3}$$

$$(2)For, n \geq 3, DD(C_n) = \begin{cases} \frac{n^3}{2}, & if\ n\ is\ even, \\ \frac{n(n^2-1)}{2}, & if\ n\ is\ odd. \end{cases}$$

*Using Theorem* 2.3 *, Lemmas* 2.4 *and* 2.5 *, we obtain the exact multiplication version of degree distance of the following graphs.*

**Corollary 2.6.** The graphs $R = P_m \Box C_n, S = C_m \Box C_n, n \geq 3$ and $m \geq 2$ and $T = P_m \Box P_n, m, n \geq 2$ are known as $C_4$ nanotube,$C_4$ nanotorus and grid respectively. (1) $DD^*(P_m \Box P_n)$

$$\begin{aligned} \leq \quad & \frac{2^{3mn-1}}{3^{3mn}}\Big(\frac{(m-1)(4mn+n-2m-1)}{n}\Big)^{mn} \times \Big(\frac{(n-1)(4mn+m-2n-1)}{m}\Big)^{mn} \\ \times \quad & \Big(\frac{(2(n-1)(m-1)((2mm-1)(m+n)-(m-n)^2))}{m}\Big)^{mn} \end{aligned}$$

2)$DD^*(P_m \Box C_n)$

$$\leq \begin{cases} 2^{3mn-1}\left(\frac{1}{3}(m-1)(4m+1)\right)^{mn} \times \left(\frac{n^2}{2m}(2m-1)\right)^{mn}\Big(\frac{\frac{m}{3}(n-1)(m-1)(4m+1)+\frac{n^2}{2}(m-1)(2m-1)}{m}\Big)^{mn}, \\ \text{if } n \text{ is even.} \\ 2^{3mn-1}(\frac{1}{3}(m-1)(4m+1))^{mn}(\frac{1}{2m}(n^2-1)(2m-1))^{mn} \\ \times\Big(\frac{\frac{m}{3}(n-1)(m-1)(4m+1)+\frac{1}{2}(m-1)(n^2-1)(2m-1)}{m}\Big)^{mn}, \\ \text{if } n \text{ is odd.} \end{cases}$$

$$3).DD(C_m \Box C_n) \leq \begin{cases} 2^{3mn-1}m^{2mn} \times n^{2mn} \times \left(m^2(n-1)+n^2(m-1)\right)^{mn}, \\ \text{if } m \text{ is even and } n\text{is even.} \\ 2^{3mn-1}(m^{2mn}) \times (n^2-1)^{mn} \times \left((n-1)(m^2+mn+m-n-1)\right)^{mn}, \\ \text{if } m \text{ is even and } n\text{is odd.} \\ 2^{3mn-1}(m^2-1)^{mn} \times n^{2mn} \times \left((m-1)(n^2+mn+n-m-1)\right)^{mn}, \\ \text{if } m \text{ is odd and } n\text{is even.} \\ 2^{3mn-1}(m^2-1)^{mn} \times (n^2-1)^{mn} \times \left((n-1)(m-1)(m+n+2)\right)^{mn}, \\ \text{if } m \text{ is odd and } n \text{ is odd.} \end{cases}$$

## 3. Multiplicative version of Gutman index of Cartesian product of graphs.

In this section, we compute the Multiplicative version of Gutman index of the Cartesian product, $G_1 \Box G_2$ of the graphs $G_1$ and $G_2$.

**Theorem 3.1.** *If $G_1$ and $G_2$ are two connected graphs with $\mid V(G_1) \mid = n_1$ and $\mid V(G_2) \mid = n_2$, where $n_1, n_2 \geq 2$, then*

$$\begin{aligned} Gut^*(G_1 \Box G_2) &\leq 2^{3n_1n_2-1}\Big[\frac{n_2Gut^+(G_1)+2e(G_2)DD^+(W_1)M_1(G_2)}{n_1n_2}\Big]^{n_1n_2} \\ &\times \Big[\frac{n_1Gut^+(G_2)+2e(G_1)DD^+(G_2)+W(G_1)M_1(G_2)}{n_1n_2}\Big]^{n_1n_2} \\ &\times \Big[\frac{S_1}{n_1n_2}\Big]^{n_1n_2}, \end{aligned}$$

*Here*

$$S_1 = n_2(n_2-1)Gut^+(G_1)+2(n_2-1)e(G_2)DD^+(G_1)+W(G_1)(4e(G_2)^2-M_1(G_2))$$
$$+W(G_2)(4e(G_1)^2-M_1(G_1))+2(n_1-1)e(G_1)DD^+(G_2)^+n_1(n_1-1)Gut^+(G_2),$$

*where $Gut(G)^+, W(G), M_1(G)$ and $DD^+(G)$ denote the Gutman index, the Wiener index, the first Zagreb index and the degree distance of G.*

**Proof.** Let $G = G_1 \Box G_2$. Then

$$\begin{aligned} Gut^*(G) &= \frac{1}{2}\prod_{w_{ij},w_{pq}\in V(G)} d_G(w_{ij},w_{pq})[d_G(w_{ij})d_G(w_{pq})] \\ &= \frac{1}{2}\{\prod_{j=0}^{n_2-1}\prod_{i,p=0,i\neq p}^{n_1-1} d_G(w_{ij},w_{pj})[d_G(w_{ij})d_G(w_{pj})] \\ &\times \prod_{i=0}^{n_1-1}\prod_{j,q=0,j\neq p}^{n_2-1} d_G(w_{ij},w_{iq})[d_G(w_{ij})d_G(w_{iq})] \\ &\times \prod_{j,q=0,j\neq q}^{n_2-1}\prod_{i,p=0,i\neq p}^{n_1-1} d_G(w_{ij},w_{pq})[d_G(w_{ij})d_G(w_{pq})]\} \\ &= \frac{1}{2}(A_1\times A_2\times A_3), \qquad\qquad (5) \end{aligned}$$

where $A_1, A_2$ and $A_3$ are the products of the above terms, in order. We calculate $A_1, A_2$ and $A_3$ separately.

First we compute

$$\begin{aligned} A_1 &= \prod_{j=0}^{n_2-1} \prod_{i,p=0,i\neq p}^{n_1-1} d_G(w_{ij},w_{pj})[d_G(w_{ij})d_G(w_{pj})] \\ &= \prod_{j=0}^{n_2-1} \prod_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i,u_p)(d_{G_1}(u_i)+d_{G_2}(v_j))(d_{G_1}(u_p)+d_{G_2}(v_j)) by\ Lemma (2.2) \\ &\leq \Big[\frac{\sum_{j=0}^{n_2-1}\sum_{i,p=0,i\neq p}^{n_1-1} d_{G_1}(u_i,u_p)\big(d_{G_1}(u_i)+d_{G_2}(v_j)\big)\big(d_{G_1}(u_p)+d_{G_2}(v_j)\big)}{n_1n_2}\Big]^{n_1n_2} by\ Lemma (2.1) \\ &= \Big[\frac{\sum_{j=0}^{n_2-1} 2Gut^+(G_1)+2d_{G_2}(v_j)DD^+(G_1)+2d_{G_2}^2(v_j)W(G_1)}{n_1n_2}\Big]^{n_1n_2} \end{aligned}$$

by the definitions of the Gutman index, the degree distance and the Wiener index of a graph.

$$A_1 \leq \Big[\frac{2n_2Gut^+(G_1)+4e(G_2)DD^+(G_1)+2W(G_1)M_1(G_2)}{n_1n_2}\Big]^{n_1n_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Next we compute $A_2 = \prod_{i=0}^{n_1-1}\prod_{j,q=0,j\neq q}^{n_2-1} d_G(w_{ij},w_{iq})d_G(w_{ij})d_G(w_{iq})$

$$= \prod_{i=0}^{n_1-1}\prod_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j,v_q)[d_{G_1}(u_i)+d_{G_2}(v_j)][d_{G_1}(u_i)+d_{G_2}(v_q)]_{byLemma(2.2)}$$

$$\leq \Big[\frac{\sum_{i=0}^{n_1-1}\sum_{j,q=0,i\neq q}^{n_2-1} d_{G_2}(v_j,v_q)[2d_{G_1}(u_i)+d_{G_2}(v_j)][d_{G_1}(u_i)]+d_{G_2}(v_q)}{n_1n_2}\Big]^{n_1n_2}_{byLemma(2.1)}$$

$$= \Big[\frac{\sum_{i=0}^{n_1-1} 2Gut^+(G_2)+2d_{G_1}(u_i)DD^+(G_2)+2d_{G_1}^2(v_j)W(G_2)}{n_1n_2}\Big]^{n_1n_2}$$

by the definition of the Gutman index,degree distance and the Wiener index of a graph

$$A_2 \leq \Big[\frac{2n_1Gut^+(G_2)+4e(G_1)DD^+(G_2)+2W(G_2)M_1(G_1)}{n_1n_2}\Big]^{n_1n_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Finally we compute

$$A_3 = \prod_{j,q=0,j\neq q.}^{n_2-1}\prod_{i,p=0,i\neq p}^{n_1-1} d_G(w_{ij},w_{pq})[d_G(w_{ij})d_G(w_{pq})]$$

$$= \prod_{j=0,q=0,j\neq q.}^{n_2-1}\prod_{i,p=0,i\neq p}^{n_i-1}[d_{G_1}(u_i,u_p)+d_{G_2}(v_j,v_q)][d_{G_1}(u_i)+d_{G_2}(v_j)][d_{G_1}(u_p)+d_{G_{2(v_q)}}]$$

$$\leq \Big[\frac{\sum_{j,q=0,j\neq q}^{n_2-1}\sum_{i,p=0,i\neq p}^{n_1-1}[d_{G_1}(u_i,u_p)+d_{G_2}(v_j,v_q)][d_{G_1}(u_i)+d_{G_2}(v_j)][d_{G_1}(u_p)+d_{G_2}(v_q)]}{n_1n_2}\Big]^{n_1n_2}$$

$$= \left[\frac{S_2}{n_1,n_2}\right]^{n_1n_2},$$

$$\textit{where} \qquad S_2 = \sum_{j,q=0,j\neq q.}^{n_2-1}\sum_{i,q=0,i\neq p}\Big[d_{G_1}(u_i,u_p)d_{G_1}(u_i)d_{G_1}(u_p)+d_{G_1}(v_i,u_p)d_{G_1}(v_i)d_{G_2}(v_q)$$
$$+d_{G_1}(u_i,u_p)d_{G_2}(v_j)d_{G_1}(u_p)+d_{G_1}(u_i,u_p)d_{G_2}(v_j)d_{G_2}(v_q)$$
$$+d_{G_2}(v_j,v_q)d_{G_1}(u_i)d_{G_1}(u_p)+d_{G_2}(v_j,v_q)d_{G_1}(u_i)d_{G_2}(v_q)$$
$$+d_{G_2}(v_j,v_q)d_{G_2}(v_j)d_{G_1}(u_p)+d_{G_2}(v_j,v_q)d_{G_2}(v_j)d_{G_2}(v_q)\Big].$$
$$\leq \Big[\frac{S_3}{n_1,n_2}\Big]^{n_1n_2}$$

$$\textit{Here} \qquad S_3 = \sum_{j,q=0,j\neq q}^{n_2-1}\Big[2GutG_1+[d_{G_2}(v_q)+d_{G_2}(v_j)]DD(G_1)+2W(G_1)d_{G_2}(v_j)d_{G_2}(v_q)$$
$$+d_{G_2}(v_j,v_q)(4e(G_1)^2-M_1(G_1))+2(n_1-1)d_{G_2}(v_j,v_q)(d_{G_2}(v_j))$$
$$+d_{G_2}(v_q))e(G_1)+n_1(n_1-1)d_{G_2}(v_j,v_q)d_{G_2}(v_j)d_{G_2}(v_q)\Big].$$

by the definitions of the Gutman index, degree distance, the Wiener index and first Zagreb index of a graph.

$$A_3 \quad \leq \quad \Big[\frac{S_4}{n_1n_2}\Big]^{n_1n_2}, \dots\dots\dots\dots\dots\dots\dots\dots (8)$$

$$\textit{where}, \qquad S_4 = 2n_2(n_2-1)Gut(G_1)+4(n_2-1)e(G_2)DDD(G_1)+2W(G_1)(4e(G_2)^2-M_1(G_2))$$
$$+2W(G_2)(4e(G_1)^2-M_1(G_1))+4(n_1-1)e(G_1)DD(G_2)+2n_1(n_1-1)Gut(G_2).$$

$$\textit{Since} \qquad \sum_{j,q=0,j\neq q}^{n_2-1} d_{G_2}(v_j)d_{G_2}(v_q) = 4e(G_2)^2-M_1(G_2)$$

Using $(6),(7)$ and $(8)$ in $(5)$, we get

$$\begin{aligned}
Gut^*(G_1\Box G_2) &\leq 2^{3n_1n_2-1}\Big[\frac{n_2Gut^+(G_1)+2e(G_2)DD^+(W_1)M_1(G_2)}{n_1n_2}\Big]^{n_1n_2}\\
&\times \Big[\frac{n_1Gut^+(G_2)+2e(G_1)DD^+(G_2)+W(G_1)M_1(G_2)}{n_1n_2}\Big]^{n_1n_2}\\
&\times \Big[\frac{S_1}{n_1n_2}\Big]^{n_1n_2},\\
S_1 &= n_2(n_2-1)Gut^+(G_1)+2(n_2-1)e(G_2)DD^+(G_1)+W(G_1)(4e(G_2)^2-M_1(G_2))\\
&+ W(G_2)(4e(G_1)^2-M_1(G_1))+2(n_1-1)e(G_1)DD^+(G_2)^+n_1(n_1-1)Gut^+(G_2).
\end{aligned}$$

This completes the proof.

**Lemma 3.2.** *Let $P_n$ and $C_n$ denote the path and the cycle on nvertices, respectively.*
$(1)$*For,*$n\geq 2, Gut(P_n)=\frac{(n-1)(2n^2-4n+3)}{3}$

$$(2)For, n\geq 3, Gut(C_n)=\begin{cases}\frac{n^3}{2}, & if\ n\ is\ even\\ \frac{n(n^2-1)}{2}, & if\ n\ is\ odd\end{cases}$$

$(3)$*For,*$n\geq 2, M_1(P_n)=4n-6 and M_1(P_1)=0$
$(4)$*For,*$n\geq 3, M_1(C_n)=4n$ *Using Theorem* 3.1 *, Lemmas* 2.4 2.5 *and* 3.2*, we obtain the exact multiplication version of Gutman index of the following graphs.*

**Corollary 3.3.** *The graphs $R=P_m\Box C_n, S=C_m\Box C_n, n\geq 3$ and $m\geq 2$ and $T=P_m\Box P_n, m,n\geq 2$ are known as $C_4$ nanotube,$C_4$ nanotorus and grid respectively. 1) $Gut(P_m\Box P_n)$*

$$\begin{aligned}
&\leq\ 2^{3mn-1}\times\Big(\frac{(m-1)}{mn}((2mn-m)(m-1)+n)\Big)^{mn}\times\Big(\frac{(n-1)}{mn}((2mn-n)(n-1)+m)\Big)^{mn}\\
&\times\ \Big(\frac{(n-1)(m-1)}{3mn}(6mn+5)(m+n)-4(m^2+n^2+3mn)\\
&+\ \frac{1}{6mn}(4-2mn)(m^2+n^2)+2m^4(m-2)+2n^4(n-2)+3(m^3+n^3)-5(m+n)+4mn\Big)^{mn}
\end{aligned}$$

*2)* $Gut^*(P_m\square C_n)$

$$\begin{aligned} &\leq 2^{3mn-1}\times\Big(\frac{(m-1)}{3mn}((2m)(2m+1)(m-2)+6mn(m-1)-3)\Big)^{mn}\times\Big(\frac{n^2}{4m}(8m-7)\Big)^{mn} \\ &\times \Big[\frac{(m-1)}{3mn}\Big(3n(n-1)+2m^3(2m-1)-2m(m-1)+4m^3n(n-1) \\ &+ 2mn(m-n)-4mn(mn-1)\Big)+\Big(\frac{n^2}{4m}(6m^2-8m+2n^2+7)\Big)\Big]^{mn}, if n is even \end{aligned}$$

$$\begin{aligned} Gut^*(P_m\square C_n) &\leq 2^{3mn-1}\times\Big(\frac{(m-1)}{3mn}((2m)(2m+1)(m-2)+6mn(m-1)-3)\Big)^{mn}\Big(\frac{n^2-1}{4m}(8m-7)\Big)^{mn} \\ &\times \Big[\frac{(m-1)}{3mn}\big(3n(n-1)+2m^3(2m-1)-2m(m-1)+4m^3n(n-1) \\ &+ 2mn(m-n)-4mn(mn-1)\big)+\Big(\frac{n^2-1}{4m}(6m^2-8m+2n^2+7)\Big)\Big]^{mn}, if n is odd \end{aligned}$$

*3)* $Gut^*(C_m\square C_n)\leq 2^{4mn-1}m^{2mn}n^{2mn}\Big[3\Big(mn(m+n)-(m^2+n^2)\Big)+\frac{m^3(m^2-n)+n^3(n^2-m)}{mn}\Big]^{mn}$*, if m is even and n is even*

$Gut^*(C_m\square C_n)\leq 2^{4mn-1}m^{2mn}(n^2-1)^{2mn}\Big[3(n-1)\Big(m^2+(n+1)(m-1)\Big)+\Big(\frac{m^3(m^2-n)+(n^3-mn)(n^2-1)}{mn}\Big)\Big]^{mn}$,
*if m is even and n is odd*

$Gut^*(C_m\square C_n)\leq 2^{4mn-1}(m^2-1)^{mn}n^{2mn}\Big[3(m-1)\big(n^2+(n-1)(m+1)\big)+\Big(\frac{(m^3-mn)(m^2-1)+n^3(n^2-m)}{mn}\Big)\Big]^{mn}$,
*if m is odd and n is even*

$Gut^*(C_m\square C_n)\leq 2^{4mn-1}(m^2-1)^{mn}(n^2-1)^{mn}\Big[3\Big((m^2-1)(n-1)+(n^2-1)(m-1)\Big)$
$+\Big(\frac{(m^3-mn)(m^2-1)+(n^3-mn(n^2-1)}{mn})\Big)\Big]^{mn}$*, if m is odd and n is odd.*

## Conflict of Interests

The authors declare that there is no conflict of interests.

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