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DIFFERENTIAL SUBORDINATION FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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Abstract. In this paper, we define some general classes of analytic functions by subordination. Moreover, by making use of the differential subordination of analytic functions, we investigate inclusion relationships among certain classes of analytic functions.

Keywords: Convex functions; close-to-convex functions; Starlike functions; Univalent functions.

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1. Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plan $\mathbb{C}$. Let A be the class of functions f which are analytic in the unit disk and these functions are given by

$$f(z) = 1 + \sum_{n=k}^{\infty} a_k z^k, \ \ n \in \mathbb{N}. \tag{1}$$

A function f analytic in Δ is said to be univalent in a domain D if

$$f(z_1) = f(z_2) \Longrightarrow z_1 = z_2 \ \ z_1, z_2 \in D.$$

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The class of all univalent functions f in Δ and have form (1) will be denoted by S.

A domain D is called convex if for every pair of points w_1 and w_2 in the interior of D, the line-segment joining w_1 to w_2 lies wholly in D. A function f which maps Δ onto a convex domain is called a convex function.The necessary and sufficient condition for $f \in S$ to be convex in Δ is that Re $\frac{(zf'(z))'}{f'(z)} > 0, \quad z \in \Delta$. The class off all functions convex and univalent in Δ is denoted by C.

A domain D is said to be starlike with respect to $w = 0$ if the linear segment joining $w = 0$ to any other point of D lies wholly in D. If a function f map Δ onto a starlike domain with respect to $w = 0$, then f is said to be starlike. The necessary and sufficient condition for $f \in S$ to be starlike is that

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0, \ \ z \in \Delta.$$

This class is denoted by S^*, and it was studied first by Alexander [23].

Let $f(z)$ and $g(z)$ be analytic in Δ. We say that $f(z)$ is subordinate to $g(z)$ if there exists a function $\phi(z)$ analytic (not necessarily univalent) in Δ satisfying $\phi(0) = 0$ and $|\phi(z)| < 1$ such that

$$f(z) = g(\phi(z)) \ \ (|z| < 1). \tag{2}$$

Subordination is denoted by $f(z) \prec g(z)$. For more details on univalent functions by subordination, we refer to [21, 22, 27, 34, 28, 35, 38, 39, 40, 41, 42].

Let B be the class of functions, analytic in Δ and of the form

$$w(z) = \sum_{n=1}^{\infty} b_n z^n, \ \ n \in N, \tag{3}$$

which satisfy the conditions $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \Delta$. Based on the class B Janowski [26] defined the class $P[A,B]$, as follows:

Let p be analytic function in Δ, given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{4}$$

Then $p(z)$ is said to be in the class $P[A,B]$; $-1 \leq B < A \leq 1$; if and only if, for $z \in \Delta$

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \ \ ; w \in B. \tag{5}$$

In [41], Ravichandran et.al defined the class $P_n[A,B]$ as follows:

For $-1 \le B < A \le 1$ and

$$p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + ..., \ \ n \in N,$$

we say that $p \in P_n[A,B]$ if

$$p(z) \prec \frac{1+Az}{1+Bz}, \ \ z \in \Delta.$$

The class with the property that $\frac{zf'(z)}{f(z)} \in P_n[A,B]$ is denoted by $ST_n[A,B]$. If $n = 1$, we drop the subscript. Also, Ravichandran et.al [41] obtained the following lemma:

Lemma 1.1 (see [41]) If $p \in P_n[A,B]$, then

$$\left| p(z) - \frac{1 - ABr^{2n}}{1 - B^2 r^{2n}} \right| \le \frac{(A-B)r^n}{1 - B^2 r^{2n}}, \ \ |z| = r < 1. \tag{6}$$

For the special case $p \in P_n(\alpha) = P_n[1-2\alpha, -1]$, we get

$$\left| p(z) - \frac{1 + (1-2\alpha)r^{2n}}{1 - r^{2n}} \right| \le \frac{2(1-\alpha)r^n}{1 - r^{2n}}, \ \ |z| = r < 1.$$

Let A^* denote the class of functions f analytic in the unit disk $\Delta = \{z : |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$, then $f \in S_n^*[A_j, B_j]$ if and only if

$$\frac{zf'(z)}{f(z)} \in P_n[A_j, B_j] \ \text{ and } \ f \in A^*.$$

The following lemma is useful in the sequel.

Lemma 1.2 If $\psi(z) = \sum_{n=0}^{\infty} b_n z^n$ is regular in Δ, $\phi_1(z)$ and $h(z)$ are convex univalent in Δ such that $\psi(z) \prec \phi_1(z)$, then $\psi(z) * h(z) \prec \phi_1(z) * h(z)$, $z \in \Delta$, where

$$\phi_1(z) = \sum_{n=0}^{\infty} a_n z^n \ \text{ and } \ \psi(z) * \phi_1(z) = \sum_{n=0}^{\infty} b_n a_n z^n.$$

Recently in [3, 28], using subordination concept the author defined the following classes of analytic functions in the unit disk Δ;

$$P(m,n,\prec) = P_{k_1,k_2,k_3,...,k_n}^{m_1,m_2,m_3,...,m_n}[m,n,\prec; A_1,B_1,A_2,B_2,A_3,B_3,...,A_n,B_n],$$

$$P'(m,n,\prec) = P'^{m_1,m_2,m_3,...,m_n}_{k_1,k_2,k_3,...,k_n}[n,m,\prec; A_1,B_1,A_2,B_2,A_3,B_3,...,A_n,B_n].$$

Moreover, the author in [3] studied some of their basic properties.

In this paper, we define the following classes analytic functions of the single complex variable z in the unit disk $\Delta = \{z : |z| < 1\}$:

$$L(m,n,\prec) = L^{m_1,m_2,m_3,\ldots,m_n}_{k_1,k_2,k_3,\ldots,k_n}[n,m,\prec;A_1,B_1,A_2,B_2,A_3,B_3,\ldots,A_n,B_n],$$

and

$$W(m,n,\prec) = W^{m_1,m_2,m_3,\ldots,\alpha_n}_{k_1,k_2,k_3,\ldots,k_{n^*}}[m,n,\prec;A_1,B_1,A_2,B_2,A_3,B_3,\ldots,A_{n^*},B_n].$$

Also, we study some of their basic properties.

2. General Analytic Classes

Now, we give the following definitions.

Definition 2.1

Let $f \in A^*$, then $f \in L(m,n,\prec)$, if and only if, there exist functions $h_j, C_j \in S^*_n[A_j,B_j]$; $j = 1,2,3,\ldots,n$, such that

$$f'(z) = \prod_{j=1}^{n^*} \left\{\frac{h_j(z)}{z}\right\}^{\alpha_j \frac{(k_j+2)}{4}} \left\{\frac{C_j(z)}{z}\right\}^{-\alpha_j \frac{(k_j-2)}{4}}. \tag{7}$$

Definition 2.2 Let $f \in A^*$, then $f \in W(m,n,\prec)$ if and only if, there exist functions $h_j, C_j \in S^*_n[A_j,B_j]$; $j = 1,2,3,\ldots,n$, such that

$$f(z) = \prod_{j=1}^{N} \left\{h_j(z)\right\}^{\alpha_j \frac{(k_j+2)}{4}} \left\{C_j(z)\right\}^{-\alpha_j \frac{(k_j-2)}{4}}. \tag{8}$$

Remark 2.1 It follows from (7) and (8), that $f \in L(m,n,\prec)$, if and only if
$zf'(z) \in L(m,n,\prec)$.

Remark 2.2 The classes $L(m,n,\prec)$ and $W(m,n,\prec)$ generalizing some classes in [32].

Theorem 2.1 Let $f_i(z) \in L(m,n,\prec)$, $i = 1,2,3,\ldots n$ and and λ_i are positive constants. Then, $\sum\limits_{i=1}^{n} \lambda_i f_i(z) \in L(m,n,\prec)$ if and only if

$$\sum_{i=1}^{n} \lambda_i \frac{zf_i'(z)}{f_i(z)} \in P(m,n,\prec).$$

Proof. The proof is similar to the corresponding result in [32], so it will be omitted.

Now, we give the following result:

Theorem 2.2 Let $f_i(z) \in L(m,n,\prec),\ i = 1,2,3,\ldots n$ and λ_i are positive constants. Then,

$$\sum_{i=1}^{n} \lambda_i f_i(z) \left| \frac{(zf_i'(z))'}{f_i'(z)} \right| \leq \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{s} \left| \frac{m_j k_j}{2} \right| \left| \frac{1+A_j r^n}{1+B_j r^n} \right|.$$

Proof. Let $f_i(z) \in L(m,n,\prec),\ i = 1,2,3,\ldots n$ and λ_i. Then, as in [26, 32], we deduce

$$\log f_i'(z) = \sum_{j=1}^{s} m_j \left\{ (\frac{k_j+2}{4}) \log(\frac{h_j(z)}{z}) - (\frac{k_j-2}{4}) \log \frac{C_j(z)}{z} \right\}.$$

Differentiating both sides with respect to z it follows that

$$\frac{f_i''(z)}{f_i'(z)} = \sum_{j=1}^{s} m_j \left\{ (\frac{k_j+2}{4})(\frac{zh_j'(z)-h_j(z)}{z^2})(\frac{z}{h_j(z)} - (\frac{k_j-2}{4})(\frac{zC_j'(z)-C_j(z)}{z^2})(\frac{z}{C_j(z)}) \right\}.$$

Therefore

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i \left(1 + \frac{zf_i''(z)}{f_i'(z)} \right) &= \sum_{i=1}^{n} \lambda_i \left(\frac{(zf_i'(z))'}{f_i'(z)} \right) \\ &= \sum_{i=1}^{n} \lambda_i \left(\sum_{j=1}^{s} m_j \left\{ (\frac{k_j+2}{4})(\frac{zh_j'(z)}{h_j(z)}) - (\frac{k_j-2}{4})(\frac{zC_j'(z)}{C_j(z)}) \right\} \right). \end{aligned} \tag{9}$$

Since,

$$\left| \frac{zh_j'(z)}{h_j(z)} \right| \leq \frac{1+A_j r^n}{1+B_j r^n}, \tag{10}$$

Then, we get that

$$\sum_{i=1}^{n} \lambda_i \left| \frac{(zf'(z))'}{f'(z)} \right| \leq \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{s} \left| \frac{\alpha_j k_j}{2} \right| \left| \frac{1+A_j r^n}{1+B_j r^n} \right|,$$

The proof is therefore completely established.

Theorem 2.3 Let $f_i(z) \in W(m,n,\prec),\ i = 1,2,3,\ldots n$ and λ_i are positive constants. Then,

$$\sum_{i=1}^{n} \lambda_i f_i(z) \left| \frac{(zf_i'(z))'}{f_i'(z)} \right| \leq \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{s} \left| \frac{m_j k_j}{2} \right| \left| \frac{1+A_j r^n}{1+B_j r^n} \right|.$$

Proof. The proof is very much akin to the proof of Theorem 2.2, so it will be omitted.

3. Main results

Theorem 3.1 Let $\log(|f'(z)|) \in L(m,n,\prec)$. Then

$$\sum_{j=1}^{s}\left\{\frac{[1-B_jr^n]^{k_j+2}}{[1+B_jr^n]^{k_j-2}}\right\}^{\frac{m_j}{4}\frac{(A_j-B_j)}{B_j+\gamma(B_j)}} - \frac{r^n}{2}\Big[\sum_{s^*=1}^{s} m_{s^*}\ell_{s^*}A_{s^*}\gamma(B_{s^*})\Big]\Bigg\} \le \log(|f'(z)|)$$

$$\le \sum_{j=1}^{s}\left\{\frac{[1+B_jr^n]^{k_j+2}}{[1-B_jr^n]^{k_j-2}}\right\}^{\frac{m_j}{4}\frac{(A_j-B_j)}{B_j+\gamma(B_j)}} + \left\{\frac{r^n}{2}\Big[\sum_{s^*=1}^{s} m_{s^*}\ell_{s^*}A_{s^*}\gamma(B_{s^*})\Big]\right\}, \tag{11}$$

where $\gamma(B_s)$ is defined as above. The function $\log(|f'(z)|)$ given by

$$\log(|f'(z)|) = \sum_{j=1}^{s}\left\{\frac{[1+B_j\delta_jz^n]^{k_j+2}}{[1-B_j\eta_jz^n]^{k_j-2}}\right\}^{\frac{\alpha_j}{4}\frac{(A_j-B_j)}{B_j+\gamma(B_j)}} + \left\{\frac{z^n}{2}\Big[\sum_{s^*=1}^{*} sm_{s^*}k_{s^*}A_{s^*}\gamma(B_{s^*})\Big]\right\},$$

shows that the above result is sharp.

Proof. The proof follows from Lemmas 1.1, 1.2 and Definition 2.1.

Theorem 3.2 Let $\log(|f_i'(z)|) \in L(m,n,\prec)$. Then

$$\sum_{i=1}^{n}\lambda_i\left[\sum_{j=1}^{s}\left\{\frac{[1-B_jr^n]^{k_j+2}}{[1+B_jr^n]^{k_j-2}}\right\}^{\frac{m_j}{4}\frac{(A_j-B_j)}{B_j+\gamma(B_j)}} - \frac{r^n}{2}\Big[\sum_{s^*=1}^{s} m_{s^*}\ell_{s^*}A_{s^*}\gamma(B_{s^*})\Big]\right] \le \sum_{i=1}^{n}\lambda_i\Big[\log(|f_i'(z)|)\Big]$$

$$\le \sum_{i=1}^{n}\lambda_i\left[\sum_{j=1}^{s}\left\{\frac{[1+B_jr^n]^{k_j+2}}{[1-B_jr^n]^{k_j-2}}\right\}^{\frac{m_j}{4}\frac{(A_j-B_j)}{B_j+\gamma(B_j)}} + \left\{\frac{r^n}{2}\Big[\sum_{s^*=1}^{s} m_{s^*}\ell_{s^*}A_{s^*}\gamma(B_{s^*})\Big]\right],\right. \tag{12}$$

where $\gamma(B_s)$ is defined as above. The function $\log(|f_i'(z)|)$ given by

$$\log(|f_i'(z)|) = \sum_{j=1}^{s}\left\{\frac{[1+B_j\delta_jz^n]^{k_j+2}}{[1-B_j\eta_jz^n]^{k_j-2}}\right\}^{\frac{\alpha_j}{4}\frac{(A_j-B_j)}{B_j+\gamma(B_j)}} + \left\{\frac{z^n}{2}\Big[\sum_{s^*=1}^{*} sm_{s^*}k_{s^*}A_{s^*}\gamma(B_{s^*})\Big]\right\},$$

shows that the above result is sharp.

Proof. The proof follows from Theorem 3.1.

Theorem 3.3 Let $\log(|f_i'(z)|) \in W(m,n,\prec)$. Then

$$\sum_{i=1}^{n}\lambda_i\left[\sum_{j=1}^{s}\left\{\frac{[1-B_jr^n]^{k_j+2}}{[1+B_jr^n]^{k_j-2}}\right\}^{\frac{m_j}{4}\frac{(A_j-B_j)}{B_j+\gamma(B_j)}} - \frac{r^n}{2}\Big[\sum_{s^*=1}^{s} m_{s^*}\ell_{s^*}A_{s^*}\gamma(B_{s^*})\Big]\right] \le \sum_{i=1}^{n}\lambda_i\Big[\log(|f_i'(z)|)\Big]$$

$$\le \sum_{i=1}^{n}\lambda_i\left[\sum_{j=1}^{s}\left\{\frac{[1+B_jr^n]^{k_j+2}}{[1-B_jr^n]^{k_j-2}}\right\}^{\frac{m_j}{4}\frac{(A_j-B_j)}{B_j+\gamma(B_j)}} + \left\{\frac{r^n}{2}\Big[\sum_{s^*=1}^{s} m_{s^*}\ell_{s^*}A_{s^*}\gamma(B_{s^*})\Big]\right],\right. \tag{13}$$

where $\gamma(B_s)$ is defined as above. The function $\log(|f_i'(z)|)$ given by

$$\log(|f_i'(z)|) = \sum_{j=1}^{s} \left\{ \frac{[1+B_j\delta_j z^n]^{k_j+2}}{[1-B_j\eta_j z^n]^{k_j-2}} \right\}^{\frac{\alpha_j}{4}\frac{(A_j-B_j)}{B_j+\gamma(B_j)}} + \left\{ \frac{z^n}{2}[\sum_{s^*=1}^{*} sm_{s^*}k_{s^*}A_{s^*}\gamma(B_{s^*})] \right\},$$

shows that the above result is sharp.

Proof. The proof can be obtained by using Lemma 1.2 and Definition 2.2.

Remark 3.1 The new results in this paper extend and improve a lot of known results (see [23, 27, 32, 33]).

Remark 3.2 It is still an open problem to study subordination concept in spaces of analytic functions which defined by integral norms. For various definitions of such analytic classes, we refer to [1, 7, 9, 11, 12, 13, 14, 15, 16, 17, 18, 24, 25, 36, 37] and others.

Remark 3.3 It is still an open problem to extend the concept of subordination to Clifford analysis. For more details on some classes of quaternion function spaces, we refer to [1, 2, 4, 5, 6, 8, 10, 19, 20, 29, 30, 31].

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