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THE ADOMIAN DECOMPOSITION METHOD FOR EIGENVALUE PROBLEMS

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Abstract. In this paper, The Adomian decomposition method (ADM) is a powerful method which considers the approximate solution of a non-linear equation as an infinite series which usually converges to the exact solution. This method is proposed to solve some eigenvalue problems. It is shown that the series solutions converges to the exact solution for each problem and we obtain the eigenvalues of these problems.

Keywords: Adomian decomposition method; eigenvalue problem; convergence.

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1. Introduction

The Adomian decomposition method (ADM) was firstly introduced by George Adomian in 1981 and developed in [1]. This method has been applied to solve differential and integral equations of linear and non-linear problems in mathematics, physics, biology and chemistry and upto now a large number of research papers have been published to show the feasibility of the decomposition method.

The main advantage of this method is that it can be applied directly to all types of differential and integral equations, linear or non-linear, homogeneous or inhomogeneous, with constant

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or variable coefficients. Another important advantage is that the method is capable of greatly reducing the size of computation work while still maintaining high accuracy of the numerical solution [2]. The ADM decomposes a solution into an infinite series which converges rapidly to the exact solution. The convergence of the ADM has been investigated by a number of authors [3, 4].

The non-linear problems are solved easily and elegantly without linearising the problem by using ADM. It also avoids linearisation, perturbation and discretization unlike other classical techniques [5].

2. The Adomian decomposition method

Consider the differential equation

$$Ly + Ry + Ny = g(x), \tag{1}$$

where N is a non-linear operator, L is the highest order derivative which is assumed to be invertible and R is a linear differential operator of order less than L. Making Ly subject of the formula, we get

$$Ly = g(x) - Ry - Ny. \tag{2}$$

By solving (2) for Ly, since L is invertible, we can write

$$L^{-1}Ly = L^{-1}g(x) - L^{-1}Ry - L^{-1}Ny. \tag{3}$$

For initial value problems we conveniently define L^{-1} for $L = \dfrac{d^n}{dx^n}$ as the $n-$fold definite integration from 0 to x. If L is a second-order operator, L^{-1} is a two fold integral and so by solving (3) for y, we get

$$y = A + Bx + L^{-1}g(x) - L^{-1}Ry - L^{-1}Ny, \tag{4}$$

where A and B are constants of integration and can be found from the initial or boundary conditions.

The Adomian method consists of approximating the solution of (1) as an infinite series

(5) $$y(x) = \sum_{n=0}^{\infty} y_n(x)$$

and decomposing the non-linear operator N as

(6) $$N(y) = \sum_{n=0}^{\infty} A_n,$$

where A_n are Adomian polynomials [6, 7] of $y_0, y_1, y_2, \ldots, y_n$ given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots.$$

Substituting (5) and (6) into (4) yields

$$\sum_{n=0}^{\infty} y_n = A + Bx + L^{-1}g(x) - L^{-1}R\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right).$$

The recursive relationship is found to be

$$\begin{aligned} y_0 &= g(x) \\ y_{n+1} &= -L^{-1}Ry_n - L^{-1}A_n. \end{aligned}$$

Using the above recursive relationship, we can construct the solution y as

(7) $$y = \lim_{n\to\infty} \Phi_n(y),$$

where

(8) $$\Phi_n(y) = \sum_{i=0}^{n} y_i.$$

3. Applications to eigenvalue problems

Problem I

Consider the differential equation

$$y'' + \lambda y = 0, \quad 0 < x < \infty$$

with the conditions $y(0)=0$, y and y' are finite as $x\to\infty$. The equation can be written as

$$\begin{aligned} Ly &= -\lambda y \\ y(0) &= 0, \end{aligned}$$

where $L=\dfrac{d^2}{dx^2}$ is the differential operator. Operating on both sides with the inverse operator of L (namely $L^{-1}[\cdot]=\int_0^t\int_0^x[\cdot]\,dsdx$) to get

$$y(x)=A+Bx-L^{-1}\lambda y,$$

where A and B are constant of integration. Applying ADM technique yields

$$\sum_{n=0}^{\infty}y_n=A+Bx-L^{-1}\lambda\sum_{n=0}^{\infty}y_n.$$

Thus we obtain

$$\begin{aligned} y_0 &= A+Bx, \\ y_{n+1} &= -\lambda L^{-1}y_n, \quad n=0,1,2,\ldots. \end{aligned}$$

Using the condition $y(0)=0$, we have $A=0$ and therefore $y_0=Bx$. Therefore we have

$$\begin{aligned} y_1(x) &= -\lambda L^{-1}(y_0)=-\lambda B\frac{x^3}{3!}. \\ y_2(x) &= -\lambda L^{-1}(y_1)=\lambda^2 B\frac{x^5}{5!}. \\ y_3(x) &= -\lambda L^{-1}(y_2)=-\lambda^3 B\frac{x^7}{7!}, \end{aligned}$$

and so on. Considering these components, the solution can be approximated as
$y(x)=\Phi_n(y)=\sum_{i=0}^{\infty}y_i(x)$, with the following expansions

$$\begin{aligned} \Phi_1 &= Bx-\lambda\frac{x^3}{3!}. \\ \Phi_2 &= Bx-\lambda B\frac{x^3}{3!}+\lambda^2 B\frac{x^5}{5!}. \\ \Phi_3 &= Bx-\lambda B\frac{x^3}{3!}+\lambda^2 B\frac{x^5}{5!}-\lambda^3 B\frac{x^7}{7!}. \\ \vdots &= \vdots \end{aligned}$$

contains the exact power series expansion of the closed form solution

$$y(x) = \frac{B}{\sqrt{\lambda}} \sin(\sqrt{\lambda} x).$$

Problem II

Consider the following differential equation

$$y'' + \lambda y = 0, \quad 0 < x < \pi$$

with the Neumann boundary conditions $y'(0) = 0$ and $y'(\pi) = 0$. Applying the Adomian decomposition method, the equation can be written as

$$Ly = -\lambda y,$$

where $L = \dfrac{d^2}{dx^2}$ is the differential operator. Operating on both sides with the inverse operator of L (namely $L^{-1}[\cdot] = \int_0^t \int_0^x [\cdot]\, ds dx$) to get

$$y(x) = A + Bx - L^{-1}\lambda y$$

where A and B are constants of integration. Applying ADM technique yields

$$\sum_{n=0}^{\infty} y_n = A + Bx - L^{-1}\lambda \sum_{n=0}^{\infty} y_n.$$

Thus we obtain

$$\begin{aligned} y_0 &= A + Bx \\ y_{n+1} &= -\lambda L^{-1} y_n, \quad n = 0, 1, 2, \ldots. \end{aligned}$$

Using the condition $y'(0) = 0$, we have $B = 0$ and therefore $y_0 = A$. Therefore we have

$$\begin{aligned} y_1(x) &= -\lambda L^{-1}(y_0) = -\lambda A \frac{x^2}{2!}. \\ y_2(x) &= -\lambda L^{-1}(y_1) = \lambda^2 A \frac{x^4}{4!}. \\ y_3(x) &= -\lambda L^{-1}(y_2) = -\lambda^3 A \frac{x^6}{6!}, \end{aligned}$$

and so on. Considering these components, the solution can be approximated as
$y(x) = \Phi_n(y) = \sum_{i=0}^{\infty} y_i(x)$, with the following expansions

$$\begin{aligned}
\Phi_1 &= A - \lambda A \frac{x^2}{2!}. \\
\Phi_2 &= A - \lambda A \frac{x^2}{2!} + \lambda^2 A \frac{x^4}{4!}. \\
\Phi_3 &= A - \lambda A \frac{x^2}{2!} + \lambda^2 A \frac{x^4}{4!} - \lambda^3 A \frac{x^6}{6!}. \\
\vdots &= \vdots
\end{aligned}$$

contains the exact power series expansion of the closed form solution

$$y(x) = A\cos(\sqrt{\lambda}x).$$

Using the other condition $y'(\pi) = 0$, the eigenvalues are computed exactly

$$\lambda = n^2, \quad n = 0, 1, 2, \ldots.$$

4. Conclusion

In this paper, we showed the accuracy, applicability and simplicity of the Adomian decomposition method applied to some eigenvalue problems. This method is very powerful and an efficient technique for solving different kinds of problems arising in various fields of science and engineering and present a rapid convergence for the solution.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] G. Adomian, A review of the decomposition method in applied mathematics, J. Math. Anal. Appl. 135 (1988), 501-544.

[2] S. Somali, G. Gokmen, Adomian decomposition method for non-Linear Sturm-Liouville problems, Surveys Math. Appll. 2 (2007), 11-20.

[3] Y. Cherruault, Convergence of Adomian method, Kybernetes, 18 (1989), 31-38.

[4] I. Mustafa, On numerical solution of one-dimensional non-linear Burger's equation and convergence of the decomposition method, Appl. Math. Comput. 170 (2005), 76-85.

[5] H.R. Marasi, M. Nikbakht, The adomian decomposition method for boundary eigenvalue problems, Australian J. Basic Appl. Sci. 5 (2011), 2106-2111.

[6] J. Biazar, S.M. Shafiof, A simple algorithm for calculating adomian polynomials, Int. J. Contemp. Math. Sc. 2 (2007), 975-982.

[7] J. Biazar, M. Pourabd, A maple program for computing adomian polynomials, Int. Math. Math. Forum 39 (2006), 1919-1924.