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GENERALIZED STABILITY OF AN AQ-FUNCTIONAL EQUATION IN QUASI-(2;P)-BANACH SPACES

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Abstract. In this paper, we introduce and investigate the general solution of a new functional equation

$$\begin{aligned} f(\frac{x+y}{a}+\frac{z+w}{b})+f(\frac{x+y}{a}-\frac{z+w}{b}) &= \frac{1}{a^2}[(1+a)f(x+y)+(1-a)f(-x-y)] \\ &+ \frac{1}{b^2}[f(z+w)+f(-z-w)] \end{aligned}$$

where $a,b \geq 1$ and discuss its Generalized Hyers-Ulam-Rassias stability under the conditions such as even, odd, approximately even and approximately odd in quasi-(2;p)-Banach spaces.

Keywords: Generalized Hyers-Ulam-Rassias stability; AQ-functional equation; quasi-(2;p)-normed spaces; quadratic function; quasi-(2;p)-Banach spaces .

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1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] in 1940 concerning the stability of group homomorphisms. Let $(G_1,\cdot)$ be a group and let $(G_2,*)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if a mapping

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$h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x\cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $H : G_1 \longrightarrow G_2$ exists with $d(h(x), H(y)) < \varepsilon$ for all $x \in G_1$?

In 1941, Hyers [2] considered the case of approximately additive mappings $f : E \longrightarrow E^{'}$ where E and $E^{'}$ are Banach spaces. He proved the following theorem.

Theorem 1.1 [2] $E, E^{'}$ is Banach spaces and let $f : E \longrightarrow E^{'}$ be a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x \in E$ and $\varepsilon > 0$. Then the limit $l(x) = \lim_{n\longrightarrow\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $l : E \longrightarrow E^{'}$ is the unique additive mapping satisfying

$$\|f(x) - l(x)\| \le \varepsilon$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t(-\infty < t < +\infty)$ for each fixed $x \in E$, then l is linear.

From the above property, the additive functional equation $f(x+y) = f(x) + f(y)$ has Hyers-Ulam stability on $(E, E^{'})$.

The theorem of Hyers was generalized by Aoki [3] for additive mappings. In 1978, Rassias [4] considered an unbounded Cauchy difference for linear mappings. It states as follows:

Theorem 1.2 [4] Let $E, E^{'}$ be two Banach spaces and let $\theta \in [0, \infty)$ and $p \in [0, 1)$. If a function $f : E \longrightarrow E^{'}$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \theta[\|x\|^p + \|y\|^p]$$

for all $x \in E$. Then there exists a unique additive mapping $T : E \longrightarrow E^{'}$ such that

$$\|f(x) - T(x)\| \le \frac{2\theta}{2-2^p}\|x\|^p$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t(-\infty < t < +\infty)$ for each fixed $x \in E$, then l is linear.

The work of Rassias [4] has had a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds and this terminology is also applied to the case of other functional equations.

In this paper, we introduce and investigate the general solution of a new functional equation

$$\begin{aligned} f(\frac{x+y}{a}+\frac{z+w}{b})+f(\frac{x+y}{a}-\frac{z+w}{b}) &= \frac{1}{a^2}[(1+a)f(x+y)+(1-a)f(-x-y)] \\ &+\frac{1}{b^2}[f(z+w)+f(-z-w)] \end{aligned} \tag{1.1}$$

where $a,b\geq 1$ and discuss its Generalized Hyers-Ulam-Rassias stability in quasi-(2;p)-Banach spaces. It may be noted that $f(x)=ax^2+bx+c$ is a solution of the functional equation.

2. Preliminaries

Before giving the main results, we will present some preliminaries results.

Definition 2.1 [5] Let X be a linear space over $\mathbb{R}$ with $\dim X>1$. A quasi 2-norm is a real-valued function on $X\times X$ satisfying the following conditions:

(1) $\| x,y \|=0$ if and only if x and y are linearly dependent,

(2) $\| x,y \|=\| y,x \|$,

(3) $\| \alpha x,y \|=|\alpha|\ \| x,y \|$ for all $\alpha\in\mathbb{K}$,

(4) There is a constant $K\geq 1$ such that$\| x+y,z \|\leq K(\|x,z\|+\|y,z\|)$ for all $x,y,z\in X$. The pair $(X,\|\cdot,\cdot\|)$ is called a quasi 2-normed space if $\|\cdot,\cdot\|$ is a quasi 2-norm on X.

A quasi 2-norm $\|\cdot,\cdot\|$ is called quasi-(2;p)-norm $(0<p\leq 1)$ if

$$||x+y,z||^p\leq ||x,z||^p+||y,z||^p$$

for all $x,y,z\in X$. The pair $(X,\|\cdot,\cdot\|)$ is called a quasi-$(2;p)$-normed space if $\|\cdot,\cdot\|$ is a quasi-$(2;p)$-norm on X.

Definition 2.2 [10] A sequence $\{x_n\}$ in a quasi-$(2;p)$-normed space $(X,\|\cdot,\cdot\|)$ is called a Cauchy sequence if

$$\lim_{m,n\longrightarrow\infty}\|x_n-x_m,y\|=0$$

for all $y\in X$.

Definition 2.3 [10] A sequence $\{x_n\}$ in a quasi-$(2;p)$-normed space $(X,\|\cdot,\cdot\|)$ is called a convergent sequence if there is an $x \in X$ such that

$$\lim_{n \longrightarrow \infty} \|x_n - x, y\| = 0$$

for all $y \in X$. If $\{x_n\}$ converges to x, write $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and call x the limit of $\{x_n\}$.In this case,we also write $\lim_{n \longrightarrow \infty} x_n = x$.

Definition 2.4 [10] we say that a quasi-(2;p)-normed spaces $(X,\|\cdot,\cdot\|)$ is a quasi-(2;p)-Banach spaces if every Cauchy sequence in X is a convergent sequence.

We introduce a basic property of a quasi-(2;p)-normed space as follows. Let $(X,\|\cdot,\cdot\|)$ be linear quasi-(2;p)-normed space, $x \in X$ and $\| x, y \| = 0$ for each $y \in X$. suppose $x \neq 0$. Since $\dim X > 1$, choose $y \in X$ such that $\{x, y\}$ is linearly independent so we have $\|x, y\| \neq 0$, which is a contradiction. Therefore, we have the following lemma.

Lemma 2.5 Let $(X,\|\cdot,\cdot\|)$ be a linear quasi-$(2;p)$-normed space. If $x \in X$ and $\|x,y\| = 0$, for each $y \in X$, then $x = 0$.

3. odd case

In this section, we assume that E_1 is a real vector space, E_2 is a quasi-(2;p)-Banach space and $f(0) = 0$. For simplicity, given a mapping $f : E_1 \longrightarrow E_2$ and $Df : E_1 \times E_1 \times E_1 \times E_1 \longrightarrow E_2$ by

$$\begin{aligned} Df(x,y,z,w) = f(\frac{x+y}{a} + \frac{z+w}{b}) + f(\frac{x+y}{a} - \frac{z+w}{b}) - \frac{1}{a^2}[(1+a)f(x+y) + (1-a)f(-x-y)] \\ - \frac{1}{b^2}[f(z+w) + f(-z-w)] \end{aligned}$$

for all $x,y,z,w \in E_1$.

Lemma 3.1 [6] Let E_1 and E_2 denote real vectors spaces, if $f : E_1 \longrightarrow E_2$ is an even function satisfying (1.1) for all $x,y,z,w \in E_1$, then f is quadratic.

Lemma 3.2 [6] Let E_1 and E_2 denote real vectors spaces, if $f : E_1 \longrightarrow E_2$ is an odd function satisfying (1.1) for all $x,y,z,w \in E_1$, then f is additive.

Theorem 3.3 Let $\phi : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \longrightarrow [0,\infty)$ be a function such that

$$\widetilde{\phi}(x,y,z,w,v) = \sum_{i=0}^{\infty} a^{ip}\phi(\|\frac{x}{a^i},v\|,\|\frac{y}{a^i},v\|,\|\frac{z}{a^i},v\|,\|\frac{w}{a^i},v\|)^p < \infty \tag{3.1}$$

for all $x,y,z,w,v \in E_1$. If $f : E_1 \longrightarrow E_2$ is an odd mapping satisfies

$$\|Df(x,y,z,w),v\| \le \phi(\|x,v\|,\|y,v\|,\|z,v\|,\|w,v\|) \tag{3.2}$$

for all $x,y,z,w,v \in E_1$. Then there exists a unique additive mapping $A : E_1 \longrightarrow E_2$ satisfying the equation (1.1) such that

$$\|f(x)-A(x),v\| \le \frac{a}{2}\widetilde{\phi}(x,0,0,0,v)^{\frac{1}{p}} \tag{3.3}$$

Proof. Using oddness and $f(0)=0$ in (3.2) we have

$$\|f(\frac{x+y}{a}+\frac{z+w}{b})+f(\frac{x+y}{a}-\frac{z+w}{b})-\frac{2}{a}f(x+y),v\|^p \le \phi(\|x,v\|,\|y,v\|,\|z,v\|,\|w,v\|)^p \tag{3.4}$$

for all $x,y,z,w,v \in E_1$. Replace (y,z,w) by $(0,0,0)$ in (3.4) we have

$$\|2f(\frac{x}{a})-\frac{2}{a}f(x),v\|^p \le \phi(\|x,v\|,0,0,0)^p \tag{3.5}$$

Again replacing x by ax in (3.5) and multiply both sides by $(\frac{a}{2})^p$ yields

$$\|af(x)-f(ax),v\|^p \le (\frac{a}{2})^p\phi(\|ax,v\|,0,0,0)^p \tag{3.6}$$

for all $x,v \in E_1$. Again replacing x by $\frac{x}{a^{i+1}}$ in (3.6) we have

$$\|af(\frac{x}{a^{i+1}})-f(\frac{x}{a^i}),v\|^p \le (\frac{a}{2})^p\phi(\|\frac{x}{a^i},v\|,0,0,0)^p \tag{3.7}$$

so,

$$\begin{aligned}\|a^m f(\frac{x}{a^m})-a^n f(\frac{x}{a^n}),v\|^p &\le \sum_{i=m}^{n-1}\|a^i f(\frac{x}{a^i})-a^{i+1}f(\frac{x}{a^{i+1}}),v\|^p \\ &= \sum_{i=m}^{n-1} a^{ip}\|af(\frac{x}{a^{i+1}})-f(\frac{x}{a^i}),v\|^p \\ &\le \sum_{i=m}^{n-1}\frac{a^{(i+1)p}}{2^p}\cdot\phi(\|\frac{x}{a^i},v\|,0,0,0)^p\end{aligned} \tag{3.8}$$

for all $x, v \in E_1$ and for any $n > m \geq 0$. Since the right-hand side of inequality (3.8) tend to 0 as $m \longrightarrow \infty$. We conclude that $\{a^n f(\frac{x}{a^n})\}$ is a Cauchy sequence in E_2 and so it converges. Because of the completeness of E_2, we can define a mapping $A : E_1 \longrightarrow E_2$ by

$$A(x) = \lim_{n\longrightarrow\infty} a^n f(\frac{x}{a^n})$$

for all $x \in E_1$. By (3.1) and (3.2), we obtain that

$$\begin{aligned} \|DA(x,y,z,w),v\|^p &= \lim_{n\longrightarrow\infty} a^{np}\|Df(\frac{x}{a^n},\frac{y}{a^n},\frac{z}{a^n},\frac{w}{a^n}),v\|^p \\ &\leq \lim_{n\longrightarrow\infty} a^{np}\phi(\|\frac{x}{a^n},v\|,\|\frac{y}{a^n},v\|,\|\frac{z}{a^n},v\|,\|\frac{w}{a^n},v\|)^p = 0 \end{aligned}$$

for all $x,y,z,w,v \in E_1$. Hence the mapping $A : E_1 \longrightarrow E_2$ satisfies (1.1). Note that f is an odd mapping ,we obtain

$$A(x) + A(-x) = \lim_{n\longrightarrow\infty} a^n f(\frac{x}{a^n}) + a^n f(-\frac{x}{a^n}) = 0$$

for all $x \in E_1$. So $A(x) = -A(-x)$. Using lemma 3.2, A is additive. Taking $m = 0, n \longrightarrow \infty$ in (3.8), we get

$$\begin{aligned} \|f(x) - A(x),v\|^p &\leq \sum_{i=0}^{\infty} \frac{a^{(i+1)p}}{2^p}\phi(\|\frac{x}{a^i},v\|,0,0,0)^p \\ &= (\frac{a}{2})^p\widetilde{\phi}(x,0,0,0,v) \end{aligned}$$

so,

$$\|f(x) - A(x),v\| \leq \frac{a}{2}\widetilde{\phi}(x,0,0,0,v)^{\frac{1}{p}}$$

We get the inequality (3.3). To prove the uniqueness of the additive mapping A, let us assume that there exists a additive mapping $A' : E_1 \longrightarrow E_2$ satisfies (1.1) and (3.3). Using $f(0) = 0$ and oddness in (1.1), we get

$$f(\frac{x+y}{a} + \frac{z+w}{b}) + f(\frac{x+y}{a} - \frac{z+w}{b}) = \frac{2}{a}f(x+y) \tag{3.9}$$

Replacing (z,w) by $(0,0)$ in (3.9), we obtain

$$f(\frac{x+y}{a}) = \frac{1}{a}f(x+y) \tag{3.10}$$

Replacing x by y in (3.10), we obtain

$$f(\frac{2y}{a}) = \frac{1}{a}f(2y) \tag{3.11}$$

Replacing $2y$ by ax in (3.11), we obtain

$$f(ax) = af(x) \tag{3.12}$$

Then it follows that $A^{'}(ax) = aA^{'}(x), A^{'}(a^m x) = a^m A^{'}(x)$. We have

$$\begin{aligned}
\|A(x) - A^{'}(x), v\|^p &= \|\frac{A(a^m x)}{a^m} - \frac{A^{'}(a^m x)}{a^m}, v\|^p \\
&\leq \frac{1}{a^{mp}}\|A(a^m x) - f(a^m x), v\|^p + \frac{1}{a^{mp}}\|A^{'}(a^m x) - f(a^m x), v\|^p \\
&\leq \frac{2}{a^{mp}} \cdot \frac{a^p}{2^p}\widetilde{\phi}(a^m x, 0, 0, 0, v) \longrightarrow 0 \quad as \quad m \longrightarrow \infty
\end{aligned}$$

for all $x, v \in E_1$. Therefore A is unique.

Corollary 3.4 Let E_1 be a quasi-2-normed linear space and E_2 be a quasi-(2;p)-Banach space. Let θ, r be real numbers such that $\theta \geq 0, r > 1$. Suppose that a odd mapping $f : E_1 \longrightarrow E_2$ satisfies

$$\|Df(x,y,z,w), v\| \leq \theta(\|x,v\|^r + \|y,v\|^r + \|z,v\|^r + \|w,v\|^r)$$

for all $x, y, z, w, v \in E_1$. Then there exists a unique additive mapping $A : E_1 \longrightarrow E_2$ satisfying the equation (1.1) such that

$$\|f(x) - A(x), v\| \leq \frac{\theta}{2}\|x,v\|^r \frac{a}{\sqrt[p]{1 - a^{(1-r)p}}}$$

4. even case

Theorem 4.1 Let $\phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty)$ be a function such that

$$\widetilde{\phi}(x,y,z,w,v) = \sum_{i=0}^{\infty} \frac{\phi(\|a^{i+1}x, v\|, \|a^{i+1}y, v\|, \|a^{i+1}z, v\|, \|a^{i+1}w, v\|)^p}{a^{2ip}} < \infty \tag{4.1}$$

for all $x, y, z, w, v \in E_1$. If $f : E_1 \longrightarrow E_2$ is an even mapping satisfies

$$\|Df(x,y,z,w), v\| \leq \phi(\|x,v\|, \|y,v\|, \|z,v\|, \|w,v\|) \tag{4.2}$$

for all $x,y,z,w,v \in E_1$. Then there exists a unique quadratic mapping $A : E_1 \longrightarrow E_2$ satisfying the equation (1.1) such that

$$\|f(x)-A(x),v\| \leq \frac{1}{2}\widetilde{\phi}(x,0,0,0,v)^{\frac{1}{p}} \tag{4.3}$$

Proof. Using evenness and $f(0)=0$ in (4.2) we have

$$\begin{aligned}&\|f(\frac{x+y}{a}+\frac{z+w}{b})+f(\frac{x+y}{a}-\frac{z+w}{b})-\frac{2}{a^2}f(x+y)+\frac{2}{b^2}f(z+w),v\|^p\\&\leq \phi(\|x,v\|,\|y,v\|,\|z,v\|,\|w,v\|)^p\end{aligned} \tag{4.4}$$

for all $x,y,z,w,v \in E_1$. Replace (y,z,w) by $(0,0,0)$ in (4.4) we have

$$\|2f(\frac{x}{a})-\frac{2}{a^2}f(x),v\|^p \leq \phi(\|x,v\|,0,0,0)^p \tag{4.5}$$

Again replacing x by ax in (4.5) and dividing both sides by 2^p yields

$$\|f(x)-\frac{1}{a^2}f(ax),v\|^p \leq \frac{1}{2^p}\phi(\|ax,v\|,0,0,0)^p \tag{4.6}$$

for all $x,v \in E_1$. Again replacing x by a^ix in (4.6) we have

$$\|f(a^ix)-\frac{1}{a^2}f(a^{i+1}x),v\|^p \leq \frac{1}{2^p}\phi(\|a^{i+1}x,v\|,0,0,0)^p \tag{4.7}$$

so,

$$\begin{aligned}\|\frac{f(a^mx)}{a^{2m}}-\frac{f(a^nx)}{a^{2n}},v\|^p &\leq \sum_{i=m}^{n-1}\|\frac{f(a^ix)}{a^{2i}}-\frac{f(a^{i+1}x)}{a^{2(i+1)}},v\|^p\\&=\sum_{i=m}^{n-1}\frac{1}{a^{2ip}}\|f(a^ix)-\frac{1}{a^2}f(a^{i+1}x),v\|^p\\&\leq \sum_{i=m}^{n-1}\frac{1}{a^{2ip}}\cdot\frac{1}{2^p}\phi(\|a^{i+1}x,v\|,0,0,0)^p\end{aligned} \tag{4.8}$$

for all $x,v \in E_1$ and for any $n > m \geq 0$. Since the right-hand side of inequality (4.8) tend to 0 as $m \longrightarrow \infty$. We conclude that $\{\frac{f(a^nx)}{a^{2n}}\}$ is a Cauchy sequence in E_2 and so it converges. Because of the completeness of E_2, we can define a mapping $A : E_1 \longrightarrow E_2$ by

$$A(x) = \lim_{n\longrightarrow\infty}\frac{f(a^nx)}{a^{2n}}$$

for all $x \in E_1$. By (4.1) and (4.2), we obtain that

$$\begin{aligned}\|Df(x,y,z,w),v\|^p &= \lim_{n\longrightarrow\infty}\frac{1}{a^{2np}}\|Df(a^nx,a^ny,a^nz,a^nw),v\|^p\\ &\leq \lim_{n\longrightarrow\infty}\frac{1}{a^{2np}}\phi(\|a^nx,v\|,\|a^ny,v\|,\|a^nz,v\|,\|a^nw,v\|)^p = 0\end{aligned}$$

for all $x,y,z,w,v \in E_1$. Hence the mapping $A : E_1 \longrightarrow E_2$ satisfies (1.1). Note that f is an even mapping ,we obtain

$$A(x) - A(-x) = \lim_{n\longrightarrow\infty}\frac{f(a^nx)}{a^{2n}} - \frac{f(-a^nx)}{a^{2n}} = 0$$

for all $x \in E_1$. So $A(x) = A(-x)$. Using lemma 3.1 A is quadratic. Taking $m = 0, n \longrightarrow \infty$ in (4.8), we get

$$\begin{aligned}\|f(x) - A(x),v\|^p &\leq \sum_{i=o}^{\infty}\frac{1}{a^{2ip}}\cdot\frac{1}{2^p}\phi(\|a^{i+1}x,v\|,0,0,0)^p\\ &= (\frac{1}{2})^p\widetilde{\phi}(x,0,0,0,v)\end{aligned}$$

so,

$$\|f(x) - A(x),v\| \leq \frac{1}{2}\widetilde{\phi}(x,0,0,0)^{\frac{1}{p}}$$

We get the inequality (4.4). To prove the uniqueness of the quadratic mapping A, let us assume that there exists a quadratic mapping $A' : E_1 \longrightarrow E_2$ satisfies (1.1) and (4.3). Using $f(0) = 0$ and evenness in (1.1), we get

$$f(\frac{x+y}{a} + \frac{z+w}{b}) + f(\frac{x+y}{a} - \frac{z+w}{b}) = \frac{2}{a^2}f(x+y) + \frac{2}{b^2}f(z+w) \tag{4.9}$$

Replacing (z,w) by $(0,0)$ in (4.9), we obtain

$$f(\frac{x+y}{a}) = \frac{1}{a^2}f(x+y) \tag{4.10}$$

Replacing x by 0 in (4.10), we obtain

$$f(\frac{y}{a}) = \frac{1}{a^2}f(y) \tag{4.11}$$

Replacing y by ax in (4.11), we obtain

$$f(ax) = a^2f(x) \tag{4.12}$$

Then it follows that $A^{'}(ax) = a^2A^{'}(x), A^{'}(a^m x) = a^{2m}A^{'}(x)$. We have

$$\begin{aligned}
\|A(x)-A^{'}(x),v\|^p &= \|\frac{A(a^m x)}{a^{2m}} - \frac{A^{'}(a^m x)}{a^{2m}}, v\|^p \\
&\le \frac{1}{a^{2mp}}\|A(a^m x)-f(a^m x),v\|^p + \frac{1}{a^{2mp}}\|A^{'}(a^m x)-f(a^m x),v\|^p \\
&\le \frac{2}{a^{2mp}}\cdot\frac{1}{2^p}\widetilde{\phi}(a^m x,0,0,0,v) \longrightarrow 0
\end{aligned}$$

as $m \longrightarrow \infty$ for all $x, v \in E_1$. Therefore A is unique.

Corollary 4.2 Let E_1 be a quasi-2-normed linear space and E_2 be a quasi-(2;p)-Banach space. Let θ, r be real numbers such that $\theta \ge 0, r < 2$. Suppose that a even mapping $f : E_1 \longrightarrow E_2$ satisfies

$$\|Df(x,y,z,w),v\| \le \theta(\|x,v\|^r + \|y,v\|^r + \|z,v\|^r + \|w,v\|^r)$$

for all $x,y,z,w,v \in E_1$. Then there exists a unique quadratic mapping $A : E_1 \longrightarrow E_2$ satisfying the equation (1.1) such that

$$\|f(x)-A(x),v\| \le \frac{\theta}{2}\|x,v\|^r \frac{a^r}{\sqrt[p]{1-a^{(r-2)p}}}$$

Corollary 4.3 Let E_1 be a quasi-2-normed linear space and E_2 be a quasi-(2;p)-Banach space. Let θ be real numbers such that $\theta \ge 0$. Suppose that a even mapping $f : E_1 \longrightarrow E_2$ satisfies

$$\|Df(x,y,z,w),v\| \le \theta$$

for all $x,y,z,w,v \in E_1$. Then there exists a unique quadratic mapping $A : E_1 \longrightarrow E_2$ satisfying the equation (1.1) such that

$$\|f(x)-A(x),v\| \le \frac{\theta}{2}\frac{1}{\sqrt[p]{1-a^{-2p}}}$$

5. Approximately even case

Lemma 5.1 Let $\phi : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \longrightarrow [0,\infty)$ be a given mapping. Suppose that a mapping $f : E_1 \longrightarrow E_2$ satisfies

$$\|Df(x,y,z,w),v\| \le \phi(\|x,v\|,\|y,v\|,\|z,v\|,\|w,v\|) \tag{5.1}$$

for all $x,y,z,w,v \in E_1$. We have

$$\begin{aligned}&\|f(x)-\frac{1+a^n}{2a^{2n}}f(a^nx)+\frac{a^n-1}{2a^{2n}}f(-a^nx),v\|^p\\&\leq\sum_{k=1}^{n}\{[(\frac{1+a^{k-1}}{4\cdot a^{2k-2}})^p+(\frac{a^{k-1}-1}{4\cdot a^{2k-2}})^p]\phi(\|\frac{a^k}{2}x,v\|,\|\frac{a^k}{2}x,v\|,\|\frac{a^k}{2}x,v\|,\|\frac{a^k}{2}x,v\|)^p\}\end{aligned}\tag{5.2}$$

for all $x,v \in E_1$ and $n \in \mathbb{N}$.

Proof. We use mathematical induction on n to prove lemma. Putting $x=y=z, w=-x$ in (5.1) yields

$$\|2f(\frac{2}{a}x)-\frac{1+a}{a^2}f(2x)+\frac{a-1}{a^2}f(-2x),v\|^p\leq\phi(\|x,v\|,\|x,v\|,\|x,v\|,\|x,v\|)^p\tag{5.3}$$

for all $x,v \in E_1$. Replacing x by $\frac{ax}{2}$ in (5.3) and dividing by 2^p gives

$$\|f(x)-\frac{1+a}{2a^2}f(ax)+\frac{a-1}{2a^2}f(-ax),v\|^p\leq\frac{1}{2^p}\phi(\|\frac{ax}{2},v\|,\|\frac{ax}{2},v\|,\|\frac{ax}{2},v\|,\|\frac{ax}{2},v\|)^p\tag{5.4}$$

for all $x,v \in E_1$. Note that (5.4) proves the validity of inequality (5.2) for the case $n=1$.Assume that inequality (5.2) holds for $n \in \mathbb{N}$. Replacing x by a^nx in (5.4) yields

$$\begin{aligned}&\|f(a^nx)-\frac{1+a}{2a^2}f(a^{n+1}x)+\frac{a-1}{2a^2}f(-a^{n+1}x),v\|^p\\&\leq\frac{1}{2^p}\phi(\|\frac{a^{n+1}x}{2},v\|,\|\frac{a^{n+1}x}{2},v\|,\|\frac{a^{n+1}x}{2},v\|,\|\frac{a^{n+1}x}{2},v\|)^p\end{aligned}\tag{5.5}$$

We have the following relation:

$$\begin{aligned}\|\quad & f(x)-\frac{1+a^{n+1}}{2a^{2(n+1)}}f(a^{n+1}x)+\frac{a^{n+1}-1}{2a^{2(n+1)}}f(-a^{n+1}x),v\|^p\\
\leq\quad & \|f(x)-\frac{1+a^n}{2a^{2n}}f(a^nx)+\frac{a^n-1}{2a^{2n}}f(-a^nx),v\|^p\\
+\quad & (\frac{1+a^n}{2a^{2n}})^p\|f(a^nx)-\frac{1+a}{2a^2}f(a^{n+1}x)+\frac{a-1}{2a^2}f(-a^{n+1}x),v\|^p\\
+\quad & (\frac{a^n-1}{2a^{2n}})^p\|-f(-a^nx)+\frac{1+a}{2a^2}f(-a^{n+1}x)-\frac{a-1}{2a^2}f(a^{n+1}x),v\|^p\\
\leq\quad & \sum_{k=1}^{n}\{[(\frac{1+a^{k-1}}{4\cdot a^{2k-2}})^p+(\frac{a^{k-1}-1}{4\cdot a^{2k-2}})^p]\phi(\|\frac{a^k}{2}x,v\|,\|\frac{a^k}{2}x,v\|,\|\frac{a^k}{2}x,v\|,\|\frac{a^k}{2}x,v\|)^p\}\\
+\quad & (\frac{1+a^n}{2a^{2n}})^p\cdot\frac{1}{2^p}\phi(\|\frac{a^{n+1}x}{2},v\|,\|\frac{a^{n+1}x}{2},v\|,\|\frac{a^{n+1}x}{2},v\|,\|\frac{a^{n+1}x}{2},v\|)^p\\
+\quad & (\frac{a^n-1}{2a^{2n}})^p\cdot\frac{1}{2^p}\phi(\|\frac{a^{n+1}x}{2},v\|,\|\frac{a^{n+1}x}{2},v\|,\|\frac{a^{n+1}x}{2},v\|,\|\frac{a^{n+1}x}{2},v\|)^p\\
\leq\quad & \sum_{k=1}^{n+1}\{[(\frac{1+a^{k-1}}{4\cdot a^{2k-2}})^p+(\frac{a^{k-1}-1}{4\cdot a^{2k-2}})^p]\phi(\|\frac{a^k}{2}x,v\|,\|\frac{a^k}{2}x,v\|,\|\frac{a^k}{2}x,v\|,\|\frac{a^k}{2}x,v\|)^p\}\end{aligned}$$

for all $x, v \in E_1$. This proves the validity of inequality (5.2) for the case $n+1$.

Theorem 5.2 Let $\phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty)$ be a function such that

$$\widetilde{\phi}(x,y,z,w,v) = \sum_{i=0}^{\infty} \frac{\phi(\|a^i x, v\|, \|a^i y, v\|, \|a^i z, v\|, \|a^i w, v\|)^p}{a^{ip}} < \infty \tag{5.6}$$

for all $x, y, z, w, v \in E_1$. Let $\psi : \mathbb{R} \longrightarrow [0, \infty)$ satisfies

$$\lim_{n \longrightarrow \infty} \frac{\psi(\|a^n x, v\|)}{a^n} = 0 \tag{5.7}$$

for all $x \in E_1$. If $f : E_1 \longrightarrow E_2$ is a mapping satisfies

$$\|f(x) - f(-x), v\| \le \psi(\|x, v\|) \tag{5.8}$$

for all $x, v \in E_1$ and

$$\|Df(x,y,z,w), v\| \le \phi(\|x,v\|, \|y,v\|, \|z,v\|, \|w,v\|) \tag{5.9}$$

for all $x, y, z, w, v \in E_1$. Then there exists a unique quadratic mapping $A : E_1 \longrightarrow E_2$ satisfying the equation (1.1) such that

$$\begin{aligned} &\|f(x) - A(x), v\| \\ &\le \{\sum_{k=1}^{\infty} [(\frac{1+a^{k-1}}{4 \cdot a^{2k-2}})^p + (\frac{a^{k-1}-1}{4 \cdot a^{2k-2}})^p] \cdot \phi(\|\frac{a^k}{2}x, v\|, \|\frac{a^k}{2}x, v\|, \|\frac{a^k}{2}x, v\|, \|\frac{a^k}{2}x, v\|)^p\}^{\frac{1}{p}} \end{aligned} \tag{5.10}$$

Proof. It follows from (5.2) and (5.8) that we have

$$\begin{aligned} &\|f(x) - \frac{f(a^n x)}{a^{2n}}, v\|^p \\ &\le \|f(x) - \frac{1+a^n}{2a^{2n}} f(a^n x) + \frac{a^n - 1}{2a^{2n}} f(-a^n x), v\|^p + (\frac{a^n-1}{2a^{2n}})^p \| - f(a^n x) + f(-a^n x), v\|^p \\ &\le \sum_{k=1}^{n} [(\frac{1+a^{k-1}}{4 \cdot a^{2k-2}})^p + (\frac{a^{k-1}-1}{4 \cdot a^{2k-2}})^p] \cdot \phi(\|\frac{a^k}{2}x, v\|, \|\frac{a^k}{2}x, v\|, \|\frac{a^k}{2}x, v\|, \|\frac{a^k}{2}x, v\|)^p \\ &+ (\frac{a^n-1}{2a^{2n}})^p \psi(\|a^n x, v\|)^p \end{aligned} \tag{5.11}$$

for all $x, v \in E_1$ and $n \in \mathbb{N}$. By virtue of (4.11), for $n, m \in \mathbb{N}$ with $n > m$, we obtain

$$\begin{aligned}
&\|\frac{f(a^m x)}{a^{2m}} - \frac{f(a^n x)}{a^{2n}}, v\|^p \\
&= \frac{1}{a^{2mp}}\|f(a^m x) - \frac{f(a^{n-m}\cdot a^m x)}{a^{2(n-m)}}, v\|^p \\
&\leq \frac{1}{a^{2mp}}\sum_{k=1}^{n-m}[(\frac{1+a^{k-1}}{4\cdot a^{2k-2}})^p + (\frac{a^{k-1}-1}{4\cdot a^{2k-2}})^p]\cdot\phi(\|\frac{a^{k+m}}{2}x, v\|, \|\frac{a^{k+m}}{2}x, v\|, \|\frac{a^{k+m}}{2}x, v\|, \|\frac{a^{k+m}}{2}x, v\|)^p \\
&+ [\frac{a^{n-m}-1}{2a^{2(n-m)}}]^p\psi(\|a^{n-m}x, v\|)^p
\end{aligned} \tag{5.12}$$

for all $x, v \in E_1$ and $n \in \mathbb{N}$. From (5.6) and (5.7), the right-hand side of inequality (5.12) tends to 0 as $m \longrightarrow \infty$, the sequence $\{\frac{f(a^n x)}{a^{2n}}\}$ is a Cauchy sequence. Completeness of E_2 allows us to assume that there exists a mapping A so that

$$A(x) = \lim_{n\longrightarrow\infty}\frac{f(a^n x)}{a^{2n}}$$

for all $x \in E_1$. By (5.9), we obtain that

$$\begin{aligned}
\|DA(x,y,z,w), v\|^p &= \lim_{n\longrightarrow\infty}\frac{1}{a^{2np}}\|Df(a^n x, a^n y, a^n z, a^n w), v\|^p \\
&\leq \lim_{n\longrightarrow\infty}\frac{1}{a^{np}}\frac{\phi(\|a^n x, v\|, \|a^n y, v\|, \|a^n z, v\|, \|a^n w, v\|)^p}{a^{np}} \longrightarrow 0
\end{aligned}$$

as $n \longrightarrow \infty$ for all $x, y, z, w, v \in E_1$ and so the mapping A satisfies (1.1). We have the following results

$$\begin{aligned}
\|A(x) - A(-x), v\|^p &= \lim_{n\longrightarrow\infty}\|\frac{f(a^n x)}{a^{2n}} - \frac{f(-a^n x)}{a^{2n}}, v\|^p \\
&\leq \frac{1}{a^{2np}}\psi(\|a^n x, v\|)^p \longrightarrow 0
\end{aligned}$$

as $n \longrightarrow \infty$ for all $x, v \in E_1$. So, $A(x) = A(-x)$ and A is quadratic. Taking $m = 0, n \longrightarrow \infty$ in (5.12), we get (5.10).

Next, we prove the uniqueness of A. A satisfies (1.1) and putting $y = z = w = 0$, we have

$$2A(\frac{x}{a}) - \frac{1+a}{a^2}A(x) + \frac{a-1}{a^2}A(-x) = 0 \tag{5.13}$$

for all $x \in E_1$. Using evenness of A and replacing x by ax in (5.13), we have

$$A(ax) = a^2 A(x)$$

for all $x \in E_1$. So,we assume that $A^{'} : E_1 \longrightarrow E_2$ be another quadratic mapping satisfying (1.1) and (5.10),we calculate

$$\begin{aligned}
&\|A(x) - A^{'}(x), v\|^p \\
&= \|\frac{A(a^n x)}{a^{2n}} - \frac{A^{'}(a^n x)}{a^{2n}}, v\|^p \\
&\leq \frac{1}{a^{2np}}\|A(a^n x) - f(a^n x), v\|^p + \frac{1}{a^{2np}}\|f(a^n x) - A^{'}(a^n x), v\|^p \\
&\leq \frac{2}{a^{2np}} \cdot \sum_{k=1}^{\infty}[(\frac{1+a^{k-1}}{4 \cdot a^{2k-2}})^p + (\frac{a^{k-1}-1}{4 \cdot a^{2k-2}})^p] \cdot \phi(\|\frac{a^{k+n}}{2}x, v\|, \|\frac{a^{k+n}}{2}x, v\|, \|\frac{a^{k+n}}{2}x, v\|, \|\frac{a^{k+n}}{2}x, v\|)^p \\
&\longrightarrow 0
\end{aligned}$$

as $n \longrightarrow \infty$ for all $x \in E_1$.

6. Approximately odd case

Lemma 6.1 Let $\phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty)$ be a given mapping. Suppose that a mapping $f : E_1 \longrightarrow E_2$ satisfies

$$\|Df(x, y, z, w), v\| \leq \phi(\|x, v\|, \|y, v\|, \|z, v\|, \|w, v\|) \tag{6.1}$$

for all $x, y, z, w, v \in E_1$. We have

$$\begin{aligned}
&\|f(x) - \frac{a^n + a^{2n}}{2} f(\frac{x}{a^n}) - \frac{a^{2n} - a^n}{2} f(-\frac{x}{a^n}), v\|^p \\
&\leq \sum_{k=1}^{n}[(\frac{a^{2k}+a^k}{4})^p + (\frac{a^{2k}-a^k}{4})^p]\phi(\|\frac{x}{2a^{k-1}}, v\|, \|\frac{x}{2a^{k-1}}, v\|, \|\frac{x}{2a^{k-1}}, v\|, \|\frac{x}{2a^{k-1}}, v\|)^p
\end{aligned} \tag{6.2}$$

for all $x, v \in E_1$ and $n \in \mathbb{N}$.

proof. Replacing x by $\frac{x}{a}$ in (5.4), we have

$$\|f(\frac{x}{a}) - \frac{1+a}{2a^2} f(x) + \frac{a-1}{2a^2} f(-x), v\|^p \leq \frac{1}{2^p}\phi(\|\frac{x}{2}, v\|, \|\frac{x}{2}, v\|, \|\frac{x}{2}, v\|, \|\frac{x}{2}, v\|)^p \tag{6.3}$$

Replacing x by $-x$ in (6.3), we have

$$\|f(-\frac{x}{a}) - \frac{1+a}{2a^2} f(-x) + \frac{a-1}{2a^2} f(x), v\|^p \leq \frac{1}{2^p}\phi(\|\frac{x}{2}, v\|, \|\frac{x}{2}, v\|, \|\frac{x}{2}, v\|, \|\frac{x}{2}, v\|)^p \tag{6.4}$$

From (6.3) and (6.4), we get

$$\|f(x)-\frac{a+a^2}{2}f(\frac{x}{a})-\frac{a^2-a}{2}f(-\frac{x}{a}),v\|^p \leq (\frac{a^2+a}{4})^p\phi(\|\frac{x}{2},v\|,\|\frac{x}{2},v\|,\|\frac{x}{2},v\|,\|\frac{x}{2},v\|)^p + (\frac{a^2-a}{4})^p\phi(\|\frac{x}{2},v\|,\|\frac{x}{2},v\|,\|\frac{x}{2},v\|,\|\frac{x}{2},v\|)^p \tag{6.5}$$

for all $x,v\in E_1$. Note that (6.5) proves the validity of inequality (6.2) for the case $n=1$.Assume that inequality (6.2) holds for $n\in\mathbb{N}$. Replacing x by $\frac{x}{a^n}$ in (6.5), we get

$$\|f(\frac{x}{a^n})-\frac{a+a^2}{2}f(\frac{x}{a^{n+1}})-\frac{a^2-a}{2}f(-\frac{x}{a^{n+1}}),v\|^p \leq (\frac{a^2+a}{4})^p\phi(\|\frac{x}{2a^n},v\|,\|\frac{x}{2a^n},v\|,\|\frac{x}{2a^n},v\|,\|\frac{x}{2a^n},v\|)^p + (\frac{a^2-a}{4})^p\phi(\|\frac{x}{2a^n},v\|,\|\frac{x}{2a^n},v\|,\|\frac{x}{2a^n},v\|,\|\frac{x}{2a^n},v\|)^p$$

so,

$$\begin{aligned}
&\|f(x)-\frac{a^{n+1}+a^{2(n+1)}}{2}f(\frac{x}{a^{n+1}})-\frac{a^{2(n+1)}-a^{n+1}}{2}f(-\frac{x}{a^{n+1}}),v\|^p\\
&\leq \|f(x)-\frac{a^n+a^{2n}}{2}f(\frac{x}{a^n})-\frac{a^{2n}-a^n}{2}f(-\frac{x}{a^n}),v\|^p\\
&+(\frac{a^{2n}+a^n}{2})^p\|f(\frac{x}{a^n})-\frac{a+a^2}{2}f(\frac{x}{a^{n+1}})-\frac{a^2-a}{2}f(-\frac{x}{a^{n+1}}),v\|^p\\
&+(\frac{a^{2n}-a^n}{2})^p\|f(-\frac{x}{a^n})-\frac{a+a^2}{2}f(-\frac{x}{a^{n+1}})-\frac{a^2-a}{2}f(\frac{x}{a^{n+1}}),v\|^p\\
&\leq \sum_{k=1}^{n}[(\frac{a^{2k}+a^k}{4})^p+(\frac{a^{2k}-a^k}{4})^p]\phi(\|\frac{x}{2a^{k-1}},v\|,\|\frac{x}{2a^{k-1}},v\|,\|\frac{x}{2a^{k-1}},v\|,\|\frac{x}{2a^{k-1}},v\|)^p\\
&+(\frac{a^{2n+2}+a^{n+1}}{4})^p\phi(\|\frac{x}{2a^n},v\|,\|\frac{x}{2a^n},v\|,\|\frac{x}{2a^n},v\|,\|\frac{x}{2a^n},v\|)^p\\
&+(\frac{a^{2n+2}-a^{n+1}}{4})^p\phi(\|\frac{x}{2a^n},v\|,\|\frac{x}{2a^n},v\|,\|\frac{x}{2a^n},v\|,\|\frac{x}{2a^n},v\|)^p
\end{aligned}$$

for all $x,v\in E_1$. This proves the validity of inequality (6.2) for the case $n+1$.

Theorem 6.2 Let $\phi:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\longrightarrow[0,\infty)$ be a function such that

$$\widetilde{\phi}(x,y,z,w,v)=\sum_{i=0}^{\infty}a^{2ip}\phi(\|\frac{x}{a^i},v\|,\|\frac{y}{a^i},v\|,\|\frac{z}{a^i},v\|,\|\frac{w}{a^i},v\|)^p<\infty \tag{6.6}$$

for all $x,y,z,w,v\in E_1$. Let $\psi:\mathbb{R}\longrightarrow[0,\infty)$ satisfies

$$\lim_{n\longrightarrow\infty}a^n\psi(\|\frac{x}{a^n},v\|)=0 \tag{6.7}$$

for all $x \in E_1$. If $f : E_1 \longrightarrow E_2$ is a mapping satisfies

$$\|f(x)+f(-x),v\| \leq \psi(\|x,v\|) \tag{6.8}$$

for all $x,v \in E_1$.and

$$\|Df(x,y,z,w),v\| \leq \phi(\|x,v\|,\|y,v\|,\|z,v\|,\|w,v\|) \tag{6.9}$$

for all $x,y,z,w,v \in E_1$. Then there exists a unique additive mapping $A : E_1 \longrightarrow E_2$ satisfying the equation (1.1) such that

$$\begin{aligned}
&\|f(x)-A(x),v\| \\
&\leq \{\sum_{k=1}^{\infty}[(\frac{a^{2k}+a^k}{4})^p+(\frac{a^{2k}-a^k}{4})^p]\phi(\|\frac{x}{2a^{k-1}},v\|,\|\frac{x}{2a^{k-1}},v\|,\|\frac{x}{2a^{k-1}},v\|,\|\frac{x}{2a^{k-1}},v\|)^p\}^{\frac{1}{p}}
\end{aligned} \tag{6.10}$$

proof. It follows from Lemma 6.1 and (6.8) that we have

$$\begin{aligned}
&\|f(x)-a^n f(\frac{x}{a^n}),v\|^p \\
&\leq \|f(x)-\frac{a^{2n}+a^n}{2}f(\frac{x}{a^n})-\frac{a^{2n}-a^n}{2}f(-\frac{x}{a^n}),v\|^p+(\frac{a^{2n}-a^n}{2})^p\|f(\frac{x}{a^n})+f(-\frac{x}{a^n}),v\|^p \\
&\leq \sum_{k=1}^{n}[(\frac{a^{2k}+a^k}{4})^p+(\frac{a^{2k}-a^k}{4})^p]\phi(\|\frac{x}{2a^{k-1}},v\|,\|\frac{x}{2a^{k-1}},v\|,\|\frac{x}{2a^{k-1}},v\|,\|\frac{x}{2a^{k-1}},v\|)^p \\
&+(\frac{a^{2n}-a^n}{2})^p\psi(\|\frac{x}{a^n},v\|)^p
\end{aligned} \tag{6.11}$$

for all $x,v \in E_1$ and $n \in \mathbb{N}$. By virtue of (6.11), for $n,m \in \mathbb{N}$ with $n > m$, we obtain

$$\begin{aligned}
&\|a^m f(\frac{x}{a^m})-a^n f(\frac{x}{a^n}),v\|^p \\
&= a^{mp}\|f(\frac{x}{a^m})-a^{n-m}f(\frac{x}{a^{n-m}\cdot a^m}),v\|^p \\
&\leq a^{mp}\sum_{k=1}^{n-m}[(\frac{a^{2k}+a^k}{4})^p+(\frac{a^{2k}-a^k}{4})^p]\phi(\|\frac{x}{2a^{k+m-1}},v\|,\|\frac{x}{2a^{k+m-1}},v\|,\|\frac{x}{2a^{k+m-1}},v\|,\|\frac{x}{2a^{k+m-1}},v\|)^p \\
&+(\frac{a^{2n-m}-a^n}{2})^p\psi(\|\frac{x}{a^{n-m}},v\|)^p
\end{aligned} \tag{6.12}$$

for all $x,v \in E_1$ and $n \in \mathbb{N}$. From (6.6) and (6.7), the right-hand side of inequality (6.12) tends to 0 as $m \longrightarrow \infty$, the sequence $\{a^n f(\frac{x}{a^n})\}$ is a Cauchy sequence. Completeness of E_2 allows us to assume that there exists a mapping A so that

$$A(x)=\lim_{n\longrightarrow\infty}a^{n}f(\frac{x}{a^{n}})$$

for all $x\in E_1$. By (6.9), we obtain that

$$\begin{aligned}\|DA(x,y,z,w),v\|^{p} &= \lim_{n\longrightarrow\infty}a^{np}\|Df(\frac{x}{a^{n}},\frac{y}{a^{n}},\frac{z}{a^{n}},\frac{w}{a^{n}}),v\|^{p}\\ &\leq \lim_{n\longrightarrow\infty}a^{np}\phi(\|\frac{x}{a^{n}},v\|,\|\frac{y}{a^{n}},v\|,\|\frac{z}{a^{n}},v\|,\|\frac{w}{a^{n}},v\|)^{p}\longrightarrow 0\end{aligned}$$

as $n\longrightarrow\infty$ for all $x,y,z,w,v\in E_1$ and so the mapping A satisfies (1.1). We have the following results

$$\begin{aligned}\|A(x)+A(-x),v\|^{p} &= \lim_{n\longrightarrow\infty}\|a^{n}f(\frac{x}{a^{n}})+a^{n}f(-\frac{x}{a^{n}}),v\|^{p}\\ &\leq \lim_{n\longrightarrow\infty}a^{np}\psi(\|\frac{x}{a^{n}},v\|)^{p}\longrightarrow 0\end{aligned}$$

as $n\longrightarrow\infty$ for all $x,v\in E_1$. So, $A(x)=-A(-x)$ and A is additive. Taking $m=0,n\longrightarrow\infty$ in (6.12), we get (6.10).

Next, we prove the uniqueness of A. A satisfies (1.1) and putting $y=z=w=0$, we have

$$2A(\frac{x}{a})-\frac{1+a}{a^{2}}A(x)+\frac{a-1}{a^{2}}A(-x)=0 \tag{6.13}$$

for all $x\in E_1$. Using oddness of A and replacing x by ax in (6.13), we have

$$A(ax)=aA(x)$$

for all $x\in E_1$. So,we assume that $A^{'}:E_1\longrightarrow E_2$ be another quadratic mapping satisfying (1.1) and (6.5),we calculate

$$\begin{aligned}&\|A(x)-A^{'}(x),v\|^{p}\\ &=\|\frac{A(a^{n}x)}{a^{n}}-\frac{A^{'}(a^{n}x)}{a^{n}},v\|^{p}\\ &\leq\frac{1}{a^{np}}\|A(a^{n}x)-f(a^{n}x),v\|^{p}+\frac{1}{a^{np}}\|f(a^{n}x)-A^{'}(a^{n}x),v\|^{p}\\ &\leq\frac{2}{a^{np}}\sum_{k=1}^{\infty}[(\frac{a^{2k}+a^{k}}{4})^{p}+(\frac{a^{2k}-a^{k}}{4})^{p}]\phi(\|\frac{a^{n}x}{2a^{k-1}},v\|,\|\frac{a^{n}x}{2a^{k-1}},v\|,\|\frac{a^{n}x}{2a^{k-1}},v\|,\|\frac{a^{n}x}{2a^{k-1}},v\|)^{p}\longrightarrow 0\end{aligned}$$

as $n\longrightarrow\infty$ for all $x\in E_1$. This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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