Available online at http://scik.org
J. Math. Comput. Sci. 6 (2016), No. 5, 730-740
ISSN: 1927-5307

# SOME NEW TWO STEP ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS USING STEFFENSEN'S METHOD

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**Abstract:** In this paper, we introduce the comparative study of some new two step iterative methods for solving nonlinear equations by using Steffensen's method. Some examples are also discussed. These new methods can be viewed as significant modification and improvement of the Steffensen's method. Numerical comparisons are made with other exiting methods to show the performance of the present methods.

**Key words:** Steffensen's method; iterative; convergence; nonlinear equation; two steps.

## 1. Introduction

Numerical analysis is the area of mathematics and computer sciences that creates, analyzes and implements algorithms for solving numerically the problems of continuous mathematics. Such problems originate generally from real - world applications of algebra, geometry and calculus and they involve variables which vary continuously: these problems occur throughout the natural sciences, social sciences, engineering, medicine and business. New two step iterative methods for finding the approximate solutions of the nonlinear equation $f(x) = 0$ are

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Received February 2, 2016

being developed using several different techniques including Taylor series, quadrature formulas, homotopy and decomposition techniques, see [ 1, 2, 4, 7, 9-16 ] and the references therein. There are several different methods in the literature for computation of the root of the nonlinear equation. The most famous of these methods is the classical Newton's method (NM) [15].

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Therefore the Newton's method was modified by Steffensen's method [5, 6, 8]

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)}$$

Newton's method and Steffensen's method are of second order converges, [17, 18]
We use Predictor - corrector methods, we shall now discuss the application of the explicit and implicit multistep methods, for the solution of the initial value problems. We use explicit ( predictor ) method for predicting a value and then use the implicit ( corrector ) method iteratively until the convergence is obtained [8].

## Definition and Notation:

Let $\alpha$ $\varepsilon$ R and $x_n$ $\varepsilon$ R , n = 0, 1, 2, 3, . .. Then the sequence $x_n$ is said to be convergence to $\alpha$ if $\lim_{n\to\infty}$ | $x_n$ - $\alpha$ | = 0. If there exists a constant c $\rangle$ 0 , an integer $n_0 \geq 0$ and P $\geq$ 0 such that for all n $\rangle$ $n_0$.

$$|x_{n+1} - \alpha| \leq c|x_n - \alpha|^P$$

then $x_n$ is said to be convergence to $\alpha$ with convergence order at least P. If P = 2, the convergence is to be quadratic or if P = 3 then it is cubic.

Notation: The notation $e_n = x_n - \alpha$, is the error in the $n^{th}$ iteration

The equation $e_{n+1} = ce_n^p + O(e_n^{p+1})$ is called the error equation. By substituting $e_n = x_n - \alpha$ for all n in any iterative method and simplifying. We obtain the error equation for that method. The value of P obtained is called the order of this method. [3]

Inspired and motivated by the ongoing research activities in this area, we suggest and analyze a new iterative method for solving nonlinear equations. To derive these iterative methods, we show that the nonlinear function can be approximated by a new series which can be obtained by using the trapezoidal rule for approximating the integral in conjunction with the fundamental theorem. This new expansion is used to suggest these new iterative methods for solving nonlinear equations. We also consider the convergence analysis of these methods [15].

## 2. Iterative methods:

Numerical analysis is the area of mathematics and computer sciences, which arise in various fields of pure and applied sciences can be formulated in terms of nonlinear equations of the type.

$$f(x) = 0 \tag{1}$$

we assume that $\alpha$ is a simple root of (1) and $\gamma$ is an initial guess sufficiently close to $\alpha$ Now using the trapezoidal rule and fundamental theorem of calculus, one can show that the function $f(x)$ can be approximated by the series [15]

$$f(x) = f(\gamma) + \frac{x-\gamma}{2}[f'(x) + f'(\gamma)] \tag{2}$$

where $f'(x)$ is the differential of $f$.

From (1) and (2), we have

$$x = \gamma - 2\frac{f(\gamma)}{f'(\gamma)} - (x-\gamma)\frac{f'(x)}{f'(\gamma)} \tag{3}$$

Using (3), one can suggest the following iterative method for solving the nonlinear equations (1).

## Theorem 1:

For a given initial choice $x_0$, compute approximate solution $x_{n+1}$ by the iterative scheme [13].

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)} - (x_{n+1} - x_n)\frac{f'(x_{n+1})}{f'(x_n)} \tag{4}$$

$$n = 0, 1, 2, 3, ............$$

Theorem 1 is an implicit iterative method to implement theorem 1, we use the predictor C corrector technique. Using the Steffensen's method as a predictor and equation (4) as a corrector, we suggest and analyze the following iterative method for solving the nonlinear equation (1) and this is the main motivation of this note.

## Theorem 2:

For a given initial choice $x_0$, compute approximate solution $x_{n+1}$ by the iterative schemes.

$$y_n = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)}$$

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)} - (y_n - x_n)\frac{f'(y_n)}{f'(x_n)}$$

$$n = 0, 1, 2, 3, .............$$

From theorem 2, we can deduce the following iterative method for solving the nonlinear equations $f(x) = 0$ which appears to be new one.

## Theorem 3:

From equation (1) and equation (2) we can have

$$x_{n+1} = x_n - \frac{2f(x_n)}{[f'(x_{n+1}) + f'(x_n)]}$$

This is fixed point formulation enable us to suggest the following iterative method for solution the nonlinear equation.

## Theorem 4:

For a given initial choice $x_0$ , find the approximate solution $x_{n+1}$, by the iterative schemes,

$$y_n = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)}$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{[f'(y_n) + f'(x_n)]}$$

$$n = 0, 1, 2, 3, .........$$

## 3. Convergence analysis

let us now discuss the convergence analysis of the above theorem 2.

## Theorem 2.1:

let $\alpha \in I$ be a simple zero of sufficiently differential function $f : I \subseteq R \to R$ for an open interval I, if $x_0$ is sufficiently close to $\alpha$ then the two step iterative method defined by theorem 2 second order convergence.

Proof: Let $\alpha$ be a simple zero of $f$. Than by expanding $f(x_n)$ and $f'(x_n)$ about $\alpha$ we have

$$f(x_n) = e_n c_1 + e_n^2 c_2 + e_n^3 c_3 + O(e_n^4) \tag{5}$$

$$f'(x_n) = 1 + \frac{2c_2}{c_1} e_n + \frac{3c_3}{c_1} e_n^2 + \frac{4c_4}{c_1} e_n^3 + O(e_n^4) \tag{6}$$

Where $c_k = \frac{1}{k!} f^{(k)}(\alpha)$

$$k = 1, 2, 3, ...........$$

and $e_n = x_n - \alpha$

from (5), we have

$$[f(x_n)]^2 = c_1^2 e_n^2 + 2c_1c_2e_n^3 + c_2^2 e_n^4 + O(e_n^5) \tag{7}$$

$$f(x_n + f(x_n)) = c_1^2 e_n + (3c_1c_2 + c_1^2 c_2 + 2c_2^2)e_n^2 + O(e_n^3) \tag{8}$$

From (7) and (8), we have

$$\frac{[f(x_n)]^2}{f(x_n) + f(x_n))} = e_n - \left(\frac{c_2}{c_1} + c_2 + 2\frac{c_2^2}{c_1^2}\right) e_n^2 + O(e_n^3) \tag{9}$$

From (9), we have

$$y_n = \alpha + \left(\frac{c_2}{c_1} + c_2 + 2\frac{c_2^2}{c_1^2}\right) e_n^2 + O(e_n^3) \tag{10}$$

From (10), we have

$$f'(y_n) = f'(\alpha)\left[1 + \left(\frac{2c_2^2}{c_1^2} + \frac{2c_2^2}{c_1} + \frac{4c_2^3}{c_1^3}\right) e_n^2 + O(e_n^4)\right] \tag{11}$$

From (6) and (11), we have

$$\frac{f'(y_n)}{f'(x_n)} = 1 - \frac{2c_2}{c_1}e_n + \left(\frac{6c_2^2}{c_1^2} + \frac{2c_2^2}{c_1} + \frac{4c_2^3}{c_1^3} - \frac{3c_3}{c_1}\right) e_n^2 + O(e_n^3) \tag{12}$$

From (10), we have

$$(y_n - x_n) = -e_n + \left(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2}\right) e_n^2 + O(e_n^3) \tag{13}$$

From (12) and (13), we have

$$(y_n - x_n)\frac{f'(y_n)}{f'(x_n)} = -e_n + \left(\frac{3c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2}\right) e_n^2 + O(e_n^3) \tag{14}$$

From (5) and (6) we have

$$\frac{f(x_n)}{f'(x_n)} = e_n + \left(\frac{c_2}{c_1} - \frac{2c_2}{c_1}\right) e_n^2 + \left(\frac{-2c_3}{c_1} - \frac{2c_2^2}{c_1^2}\right) e_n^3 + O(e_n^4) \tag{15}$$

From (14) and (15), we have

$$x_{n+1} = \alpha + \left(\frac{-c_2}{c_1} - c_2 - \frac{2c_2^2}{c_1^2}\right) e_n^2 + \left(\frac{4c_3}{c_1} + \frac{4c_2^2}{c_1^2}\right) e_n^3 + O(e_n^4)$$

which implies that

$$e_{n+1} = \left(\frac{-c_2}{c_1} - c_2 - \frac{2c_2^2}{c_1^2}\right) e_n^2 + \left(\frac{4c_3}{c_1} + \frac{4c_2^2}{c_1^2}\right) e_n^3 + O(e_n^4)$$

This shows that Theorem 2 is second order convergence

## 4. Numerical Examples

We present some example to illustrate the root of the newly developed two step iterative method, see Table 1. We compare the Newton method (NM) and Steffensen's method (SM).

We suggest the following new two step iterative methods, which will denote by NEW TWO STEP, Theorem 2 (NTS 1) and Theorem 4 (NTS 2). All computations are performed using MATLAB. The following examples are used for numerical testing.

$f_1(x) = \sin(x) - 1 - x^3$

$f_2(x) = \cos(x) - xe^x$

$f_3(x) = 3x - \sqrt{1+\sin(x)}$

$f_4(x) = \cos(x) - \sqrt{x} + 1$

$f_5(x) = x + \sin(x) - x^3$

$f_6(x) = e^x - 1.5 - \tan^{-1}(x)$

$f_7(x) = x\log_{10}(x) - 1.2$

$f_8(x) = \sin(x) - 1 + x$

$f_9(x) = x\tan(x) + 1$

$f_{10}(x) = x^2 - 9$

As for the convergence criteria, it was required that the distance of two consecutive approximations $\delta$ and also displayed is the number of iterations to approximate the zero (IT), the approximate zero $x_n$ and the value $f(x_n)$.

Table 1: ( Numerical Examples and Comprison)

| *method* | *IT* | $x_n$ | $f(x_n)$ | $\delta$ |
|---|---|---|---|---|
| $f_1$, $x_0$=-1 | | | | |
| NM | 4 | -1.24905214850119 | -1.99840144432528e-014 | 0.000227912085001725 |
| SM | 6 | -1.24905214850119 | | 7.16056545615724e-009 |
| NTS-1 | 6 | -1.24905214850119 | | 8.06346234227817e-009 |
| NTS-2 | 4 | -1.24905214850119 | | 3.55673268614965e-011 |
| $f_2$, $x_0$=1 | | | | |
| NM | 6 | 0.517757363682458 | 0.802866819140127 | 1.50990331349021e-013 |
| SM | 6 | 0.517757363682458 | | 2.87492252226684e-012 |
| NTS-1 | 7 | 0.517757363682458 | | 6.22145102102678e-009 |
| NTS-2 | 5 | 0.517757363682458 | | 1.50990331349021e-013 |
| $f_3$, $x_0$=0.5 | | | | |
| NM | 4 | 0.391846907002648 | -4.44089209850063e-016 | 1.01410879693731e-010 |
| SM | 4 | 0.391846907002648 | | 5.32907051820075e-015 |
| NTS-1 | 4 | 0.391846907002648 | | 6.93445301180873e-013 |
| NTS-2 | 2 | 0.391846907002648 | | 4.84163169476304e-005 |
| $f_4$, $x_0$=1 | | | | |
| NM | 4 | 1.39058983057821 | 2.44249065417534e-015 | 1.39888101102769e-014 |
| SM | 3 | 1.39058983057821 | | 1.83251690488717e-008 |
| NTS-1 | 3 | 1.39058983057821 | | 2.07635105997639e-007 |
| NTS-2 | 3 | 1.39058983057821 | | 1.02673092250428e-009 |
| $f_5$, $x_0$=1 | | | | |
| NM | 5 | 1.31716296100603 | 9.76996261670138e-015 | 3.32861537577501e-005 |
| SM | 7 | 1.31716296100603 | | 7.86037901434611e-014 |
| NTS-1 | 11 | 1.31716296100603 | | 1.48738417138361e-008 |
| NTS-2 | 4 | 1.31716296100603 | | 5.11298368088831e-007 |

| *method* | *IT* | $x_n$ | $f(x_n)$ | $\delta$ |
|---|---|---|---|---|
| $f_6$, $x_0$=1 | | | | |
| NM | 5 | 0.767653266201279 | 0 | 2.27928786955545e-012 |
| SM | 6 | 0.767653266201279 | | 1.24644738974666e-012 |
| NTS-1 | 5 | 0.767653266201279 | | 1.45067710066726e-008 |
| NTS-2 | 4 | 0.767653266201279 | | 1.66533453693773e-015 |
| $f_7$, $x_0$=2 | | | | |
| NM | 4 | 2.74064609597369 | -3.10862446895044e-015 | 1.70161289503313e-008 |
| SM | 5 | 2.74064609597369 | | 1.16928688953521e-012 |
| NTS-1 | 5 | 2.74064609597369 | | 3.6804337355338e-011 |
| NTS-2 | 3 | 2.74064609597369 | | 4.51509829524355e-009 |
| $f_8$, $x_0$=1 | | | | |
| NM | 4 | 0.510973429388569 | -1.11022302462516e-016 | 3.78060912575862e-008 |
| SM | 5 | 0.510973429388569 | | 4.42400560629608e-010 |
| NTS-1 | 6 | 0.510973429388569 | | 9.30553412104019e-011 |
| NTS-2 | 3 | 0.510973429388569 | | 1.52370558392789e-008 |
| $f_9$, $x_0$=2.5 | | | | |
| NM | 3 | 2.7983604578389 | -7.16049482103465e-005 | 4.63531457597366e-005 |
| SM | 6 | 2.7983604578389 | | 7.59549133810795e-007 |
| NTS-1 | 5 | 2.7983604578389 | | 1.19236434058967e-008 |
| NTS-2 | 3 | 2.7983604578389 | | 6.77020575017285e-005 |
| $f_{10}$, $x_0$=2.5 | | | | |
| NM | 4 | 3.0000000000000 | 0.0000000000000 | 1.67942641127183e-007 |
| SM | 7 | 3.0000000000000 | | 4.97667596022211e-008 |
| NTS-1 | 5 | 3.0000000000000 | | 6.03961325396085e-014 |
| NTS-2 | 4 | 3.0000000000000 | | 2.40573058363225e-007 |

## 5. Conclusion

With the comparative study of newly developed technique (NTS-1 and NTS-2) is faster than Newton's and Steffensen's method. Our method can be considered as significant improvement of Newton's and Steffensen's method and can be considered as alternative method of solving nonlinear equations.

## Conflict of Interests

The authors declare that there is no conflict of interests.

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