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CR-SUBMANIFOLDS OF A NEARLY HYPERBOLIC KENMOTSU MANIFOLD ADMITTING A QUARTER-SYMMETRIC SEMI-METRIC CONNECTION

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Abstract. We consider a nearly hyperbolic Kenmotsu manifold with a quater symmetric semi metric connection and study Cr-Submanifolds of a nearly hyperbolic Kenmotsu manifold with quater symmetric semi metric connection. We also study parallel distributions on nearly hyperbolic Kenmotsu manifold with a quater symmetric semi metric connection and find the integrability conditions of some distributions on nearly hyperbolic Kenmotsu manifold with a quater symmetric semi metric connection.

Keywords: Cr-Submanifolds; Nearly hyperbolic Kenmotsu manifold; Quater symmetric semi metric connection; Integrability conditions and parallel distribution.

2010 AMS Subject Classification: 53D05, 53D25, 53D12.

1. Introduction

The notion of CR-submanifolds of a Kaehler manifold as generalization of invariant and anti-invariant submanifolds was introduced and studied by A.Bejancu in ([1],[2]).Since then, several papers on Kaehler manifolds were published. CR-submanifolds of Sasakian manifold was studied by C.J.Hsu in [5] and M.Kobayashi in [18]. CR-submanifolds of Kenmotsu

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manifold was studied by A.Bejancu and N.Papaghuic in [4]. Later, several geometers (see, [9],[12],[13],[15],[16]) enriched the study of CR-submanifolds of almost contact manifolds. The almost hyperbolic (f,ξ,η,g)-structure was defined and studied by Upadhyay and Dube in [17].Dube and Bhatt studied CR-submanifolds of trans-hyperbolic Sasakian manifold in [10]. On the other hand, S.Golab introduced the idea of semi-symmetric and quarter symmetric connections in [8].CR-submanifolds of LP-Sasakian manifold with quarter symmetric non-metric connection were studied by the first author S.K.Lovejoy Das in [11]. CR-submanifolds of a nearly hyperbolic Sasakian manifold admitting a semi-symmetric semi-metric connections were studied by the first author, M.D.Siddiqi and S.Rizvi in [14]. M.Ahmad and Kasif Ali, studied CR-submanifolds of a nearly hyperbolic Kenmotsu manifold admitting a quarter symmetric non-metric connection in [19]. In this paper, we study some properties of CR-submanifolds of a nearly hyperbolic Kenmotsu manifold admitting a quarter symmetric semi-metric connection.

2. Preliminaries

Let $\overline{M}$ be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure (ϕ,ξ,η,g), where a tensor ϕ of type (1,1), a vector field ξ, called structure vector field, η that dual 1-form of ξ and g is Riemannian metric satisfying the following

$$\phi^2 X = X + \eta(X)\xi,\ g(X,\xi) = \eta(X) \tag{2.1}$$

$$\eta(\xi) = -1,\ \phi(\xi) = 0,\ \eta o\phi = 0 \tag{2.2}$$

$$g(\phi X,\phi Y) = -g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

for any X,Y tangent to $\overline{M}$ [17].In this case

If addition to the above condition, we have

$$g(\phi X,Y) = -g(\phi Y,X) \tag{2.4}$$

An almost hyperbolic contact metric structure (ϕ,ξ,η,g) on $\overline{M}$ is called hyperbolic Kenmotsu manifold [7] if and only if

$$(\nabla_X\phi)Y = g(\phi X,Y)\xi - \eta(Y)\phi X \tag{2.5}$$

for all X,Y tangent to $\overline{M}$.

On a hyperbolic Kenmotsu manifold $\overline{M}$, we have

$$\nabla_X\xi = X + \eta(X)\xi \tag{2.6}$$

For a Riemannian metric g and Riemannian connection ∇.

Further, an almost hyperbolic contact metric manifold $\overline{M}$ on (ϕ,ξ,η,g) is called a nearly hyperbolic Kenmotsu manifold [7], if

$$(\nabla_X\phi)Y + (\nabla_Y\phi)X = -\eta(X)\phi Y - \eta(Y)\phi X \tag{2.7}$$

where ∇ is Riemannian connection on $\overline{M}$.

Now, Let M be a submanifold immersed in $\overline{M}$. The Riemannian metric symbol g induced on M. Let TM and $T^{\perp}M$ be the Lie algebra of vector field tangential to M and normal to M respectively and ∇^* be induced Levi-Civita connection on M then the Gauss formula and Weingarten formula are given respectively

$$\nabla_X Y = \nabla^*_X Y + h(X,Y) \tag{2.8}$$

$$\nabla_X N = -A_N X + \nabla^{\perp}_X N \tag{2.9}$$

for any $X,Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ is a connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_N is the Weingarten map associated with N as

$$g(h(X,Y),N) = g(A_N X,Y) \tag{2.10}$$

any vector X tangent to M is given as

$$X = PX + QX, \tag{2.11}$$

where $PX \in D$ and $QX \in D^{\perp}$. For any N normal to M ,we have

$$\phi N = BN + CN, \tag{2.12}$$

where $BN(resp.CN)$ is the tangential component (resp. normal component) of ϕN.

Now, we define a quarter-symmetric semi-metric connection

$$\overline{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y + g(\phi X, Y)\xi \tag{2.13}$$

such that

$$(\overline{\nabla}_X g)(Y,Z) = -\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y)$$

From (2.13) and using (2.1) and (2.3), we have

$$(\overline{\nabla}_X \phi)Y + \phi(\overline{\nabla}_X Y) = (\nabla_X \phi)Y + \phi(\nabla_X Y) - \eta(X)Y - 2\eta(X)\eta(Y)\xi - g(X,Y)\xi$$

Interchanging X and Y, we have

$$(\overline{\nabla}_Y \phi)X + \phi(\overline{\nabla}_Y X) = (\nabla_Y \phi)X + \phi(\nabla_Y X) - \eta(Y)X - 2\eta(Y)\eta(X)\xi - g(X,Y)\xi$$

Adding above two equations, we get

$$(\overline{\nabla}_X \phi)Y + (\overline{\nabla}_Y \phi)X + \phi(\overline{\nabla}_X Y - \nabla_X Y) + \phi(\overline{\nabla}_Y X - \nabla_Y X) = (\nabla_X \phi)Y + (\nabla_Y \phi)X -$$

$$\eta(X)Y - \eta(Y)X - 4\eta(Y)\eta(X)\xi - 2g(X,Y)\xi$$

Using equation (2.7) and (2.13) in above, we have

$$(\overline{\nabla}_X \phi)Y + (\overline{\nabla}_Y \phi)X = -\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi \tag{2.14}$$

$$\overline{\nabla}_X \xi = X + \eta(X)\xi \tag{2.15}$$

An almost hyperbolic contact metric manifold with almost hyperbolic contact structure (ϕ, ξ, η, g) is called nearly hyperbolic Kenmotsu manifold with quarter-symmetric semi-metric connection if it is satisfied (2.14) and (2.15).

The Gauss formula and Weingarten formula for a nearly hyperbolic Kenmotsu manifold admitting quarter symmetric semi metric connection is

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.16}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N - \eta(X)\phi N + g(\phi X, N)\xi \tag{2.17}$$

Definition 2.1. An m-dimensional sub-manifold M of an n-dimensional nearly hyperbolic Kenmotsu manifold $\overline{M}$ is called a CR- submanifold if there exist a differentiable distribution $D : x \to D_x$ on M satisfying the following conditions:

(i) The distribution D is invariant under ϕ that is $\phi D_x = D_x$, for each $x \in M$,

(ii) The complementary orthogonal distribution $D^{\perp}$ of D is anti- invariant under ϕ, that is $\phi D_x^{\perp} \subset T^{\perp}M$ for each $x \in M$.

If dim $D_x^{\perp} = 0 (resp., dim D_x = 0)$, then the CR- Submanifold is called an invariant (resp., anti-invariant) submanifold. The distribution $D(resp., D^{\perp})$ is called the horizontal (resp., vertical) distribution. Also, the pair $(D, D^{\perp})$ is called ξ- horizontal (resp., vertical) if $\xi_x \in D_x (resp., \xi_x \in D_x^{\perp})$.

3. Some Basic Lemmas

Lemma 3.1. *If M be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection. Then*

$$-\eta(X)\phi PY - \eta(Y)\phi PX - 2\eta(X)\eta(Y)P\xi - 2g(X,Y)P\xi + \phi P(\nabla_X Y) \tag{3.1}$$

$$+\phi P(\nabla_Y X) = P\nabla_X(\phi PY) + P\nabla_Y(\phi PX) - PA_{\phi QY}X - PA_{\phi QX}Y$$

$$-g(X,QY)P\xi - g(Y,QX)P\xi - 2\eta(X)\eta(QY)P\xi - 2\eta(Y)\eta(QX)P\xi$$

$$-2\eta(X)\eta(Y)Q\xi - 2g(X,Y)Q\xi + 2Bh(X,Y) = Q\nabla_X(\phi PY) \tag{3.2}$$

$$+Q\nabla_Y(\phi PX) - QA_{\phi QY}X - QA_{\phi QX}Y - \eta(X)QY - \eta(Y)QX$$

$$-g(X,QY)Q\xi - g(Y,QX)Q\xi - 2\eta(X)\eta(QY)Q\xi - 2\eta(Y)\eta(QX)Q\xi$$

$$-\eta(X)\phi QY - \eta(Y)\phi QX + \phi Q(\nabla_X Y) + \phi Q(\nabla_Y X) + 2Ch(X,Y) = \tag{3.3}$$

$$h(X,\phi PY) + h(Y,\phi PX) + \nabla_X^{\perp}(\phi QY) + \nabla_Y^{\perp}(\phi QX)$$

for any $X,Y \in TM$.

Proof. From (2.11), we have

$$\phi Y = \phi PY + \phi QY.$$

Differentiating covariantly and using equation (2.16) and (2.17), we have

$$(\overline{\nabla}_X \phi)Y + \phi(\nabla_X Y) + \phi h(X,Y) = \nabla_X(\phi PY) + h(X,\phi PY)$$

$$-A_{\phi QY}X + \nabla^{\perp}_X(\phi QY) - \eta(X)QY - g(X,QY)\xi - 2\eta(X)\eta(QY)\xi$$

Interchanging X and Y in above equation, we have

$$(\overline{\nabla}_Y \phi)X + \phi(\nabla_Y X) + \phi h(Y,X) = \nabla_Y(\phi PX) + h(Y,\phi PX)$$

$$-A_{\phi QX}Y + \nabla^{\perp}_Y(\phi QX) - \eta(Y)QX - g(Y,QX)\xi - 2\eta(Y)\eta(QX)\xi$$

Adding above two equations, we obtain

$$(\overline{\nabla}_X \phi)Y + (\overline{\nabla}_Y \phi)X + \phi(\nabla_X Y) + \phi(\nabla_Y X) + 2\phi h(X,Y) =$$

$$\nabla_X(\phi PY) + \nabla_Y(\phi PX) + h(X,\phi PY) + h(Y,\phi PX) - A_{\phi QY}X$$

$$-A_{\phi QX}Y + \nabla^{\perp}_X(\phi QY) + \nabla^{\perp}_Y(\phi QX) - \eta(X)QY - \eta(Y)QX$$

$$-g(X,QY)\xi - g(Y,QX)\xi - 2\eta(X)\eta(QY)\xi - 2\eta(Y)\eta(QX)\xi$$

Adding (2.14) in above equation and using equations (2.11) and (2.12), we have

$$-\eta(X)\phi PY - \eta(X)\phi QY - \eta(Y)\phi PX - \eta(Y)\phi QX - 2\eta(X)\eta(Y)P\xi \tag{3.4}$$

$$-2\eta(X)\eta(Y)Q\xi - 2g(X,Y)P\xi - 2g(X,Y)Q\xi + \phi P(\nabla_X Y) + \phi Q(\nabla_X Y)$$

$$+\phi P(\nabla_Y X) + \phi Q(\nabla_Y X) + 2Bh(X,Y) + 2Ch(Y,X) = P\nabla_X(\phi PY)$$

$$+Q\nabla_X(\phi PY) + P\nabla_Y(\phi PX) + Q\nabla_Y(\phi PX) + h(X,\phi PY) + h(Y,\phi PX) - PA_{\phi QY}X$$

$$-QA_{\phi QY}X - PA_{\phi QX}Y - QA_{\phi QX}Y + \nabla^{\perp}_X(\phi QY) + \nabla^{\perp}_Y(\phi QX) - \eta(X)QY - \eta(Y)QX$$

$$-g(X,QY)P\xi - g(X,QY)Q\xi - g(Y,QX)P\xi - g(Y,QX)Q\xi - 2\eta(X)\eta(QY)P\xi$$

$$-2\eta(X)\eta(QY)Q\xi - 2\eta(Y)\eta(QX)P\xi - 2\eta(Y)\eta(QX)Q\xi$$

Compairing tangential, vertical and normal components in (3.4), we get desired results. Hence the lemma is proved.

Lemma 3.2. *If M be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection. Then*

$$2(\overline{\nabla}_X\phi)Y = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y] \tag{3.5}$$

$$-\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi$$

$$2(\overline{\nabla}_Y\phi)X = \nabla_Y\phi X - \nabla_X\phi Y + h(Y,\phi X) - h(X,\phi Y) + \phi[X,Y] \tag{3.6}$$

$$-\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi,$$

for any $X,Y \in D$.

Proof. from Gauss formula (2.16), we have

$$\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) \tag{3.7}$$

Also by covariant differentiation, we have

$$\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = (\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X + \phi[X,Y] \tag{3.8}$$

From (3.7) and (3.8), we have

$$(\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y] \tag{3.9}$$

Adding (3.9) and (2.14), we have

$$2(\overline{\nabla}_X\phi)Y = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y]$$

$$-\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi$$

Subtracting (3.9) and (2.14), we have

$$2(\overline{\nabla}_Y\phi)X = \nabla_Y\phi X - \nabla_X\phi Y + h(Y,\phi X) - h(X,\phi Y) + \phi[X,Y]$$

$$-\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi$$

for any $X,Y \in D$.

Hence lemma is proved.

Corollary 3.1. *If M be a $\xi-$vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection. Then*

$$2(\overline{\nabla}_X\phi)Y = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y] - 2g(X,Y)\xi$$

$$2(\overline{\nabla}_Y\phi)X = \nabla_Y\phi X - \nabla_X\phi Y + h(Y,\phi X) - h(X,\phi Y) - \phi[X,Y] - 2g(X,Y)\xi,$$

for any $X,Y \in D$.

Lemma 3.3. *If M be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection. Then*

$$2(\overline{\nabla}_X\phi)Y = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X + \eta(Y)X - \eta(X)Y - \phi[X,Y] \tag{3.10}$$
$$-\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi$$

$$2(\overline{\nabla}_Y\phi)X = A_{\phi Y}X - A_{\phi X}Y + \nabla^{\perp}_Y\phi X - \nabla^{\perp}_X\phi Y + \eta(X)Y - \eta(Y)X + \phi[X,Y] \tag{3.11}$$
$$-\eta(X)\phi Y - \eta(Y)\phi X - 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi,$$

for any $X,Y \in D^{\perp}$.

Proof. For any $X,Y \in D^{\perp}$, from Weingarten formula (2.17),we have

$$\overline{\nabla}_X\phi Y = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \eta(X)Y - 2\eta(X)\eta(Y)\xi - g(X,Y)\xi$$

Interchanging X and Y in above, we have

$$\overline{\nabla}_Y\phi X = -A_{\phi X}Y + \nabla^{\perp}_Y\phi X - \eta(Y)X - 2\eta(Y)\eta(X)\xi - g(X,Y)\xi$$

From above two equations , we have

$$\overline{\nabla}_X\phi Y - \overline{\nabla}_Y\phi X = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X + \eta(Y)X - \eta(X)Y \tag{3.12}$$

Compairing equations (3.12) and (3.8), we have

$$(\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X + \eta(Y)X \tag{3.13}$$
$$-\eta(X)Y - \phi[X,Y]$$

Adding (3.13) and (2.14), we get

$$2(\overline{\nabla}_X\phi)Y = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla^{\perp}_Y\phi X + \eta(Y)X - \eta(X)Y - \phi[X,Y]$$

$$-\eta(X)\phi Y-\eta(Y)\phi X-2\eta(X)\eta(Y)\xi-2g(X,Y)\xi$$

Subtracting (3.13) from (2.14), we get

$$2(\overline{\nabla}_Y\phi)X=A_{\phi Y}X-A_{\phi X}Y+\nabla^{\perp}_Y\phi X-\nabla^{\perp}_X\phi Y+\eta(X)Y-\eta(Y)X+\phi[X,Y]$$

$$-\eta(X)\phi Y-\eta(Y)\phi X-2\eta(X)\eta(Y)\xi-2g(X,Y)\xi$$

for all $X,Y\in D^{\perp}$. Hence the Lemma is proved.

Corollary 3.2. *If M be a ξ- horizontal CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection. Then*

$$2(\overline{\nabla}_X\phi)Y=A_{\phi X}Y-A_{\phi Y}X+\nabla^{\perp}_X\phi Y-\nabla^{\perp}_Y\phi X-\phi[X,Y]-2g(X,Y)\xi$$

$$2(\overline{\nabla}_Y\phi)X=A_{\phi Y}X-A_{\phi X}Y+\nabla^{\perp}_Y\phi X-\nabla^{\perp}_X\phi Y+\phi[X,Y]-2g(X,Y)\xi,$$

for all $X,Y\in D^{\perp}$.

Lemma 3.4. *If M be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection. Then*

$$2(\overline{\nabla}_X\phi)Y=-A_{\phi Y}X+\nabla^{\perp}_X\phi Y-\nabla_Y\phi X-h(Y,\phi X)-\eta(X)Y-\phi[X,Y] \tag{3.16}$$

$$-\eta(Y)\phi X-\eta(X)\phi Y-4\eta(X)\eta(Y)\xi-3g(X,Y)\xi$$

$$2(\overline{\nabla}_Y\phi)X=A_{\phi Y}X-\nabla^{\perp}_X\phi Y+\nabla_Y\phi X+h(Y,\phi X)+\eta(X)Y+\phi[X,Y] \tag{3.17}$$

$$-\eta(X)\phi Y-\eta(Y)\phi X-g(X,Y)\xi,$$

for any $X\in D$ and $Y\in D^{\perp}$.

Proof. Let $X\in D,Y\in D^{\perp}$, from Gauss formula (2.16), we have

$$\overline{\nabla}_Y\phi X=\nabla_Y\phi X+h(Y,\phi X)$$

From Weingarten formula (2.17), we have

$$\overline{\nabla}_X\phi Y=-A_{\phi Y}X+\nabla^{\perp}_X\phi Y-\eta(X)Y-2\eta(X)\eta(Y)\xi-g(X,Y)\xi$$

Now, from Gauss and Weingarten formula,we have

$$\overline{\nabla}_X\phi Y-\overline{\nabla}_Y\phi X=-A_{\phi Y}X+\nabla^{\perp}_X\phi Y-\nabla_Y\phi X-h(Y,\phi X)-\eta(X)Y \tag{3.18}$$

$$-g(X,Y)\xi - 2\eta(X)\eta(Y)\xi$$

Compairing equations (3.18) and (3.8), we have

$$(\overline{\nabla}_X\phi)Y - (\overline{\nabla}_Y\phi)X = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \eta(X)Y \tag{3.19}$$

$$-\phi[X,Y] - 2\eta(X)\eta(Y)\xi - g(X,Y)\xi$$

Adding (3.19) and (2.14), we have

$$2(\overline{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \eta(X)Y - \phi[X,Y]$$

$$-\eta(X)\phi Y - \eta(Y)\phi X - 4\eta(X)\eta(Y)\xi - 3g(X,Y)\xi$$

Subtracting (3.19) from (2.14), we find

$$2(\overline{\nabla}_Y\phi)X = A_{\phi Y}X - \nabla^{\perp}_X\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \eta(X)Y + \phi[X,Y] - \eta(X)\phi Y$$

$$-\eta(Y)\phi X - g(X,Y)\xi$$

for any $X \in D$ and $Y \in D^{\perp}$. Hence the Lemma is proved.

Corollary 3.3. *If M be a ξ- horizontal CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection. Then*

$$2(\overline{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \eta(X)Y - \phi[X,Y]$$

$$-\eta(X)\phi Y - 3g(X,Y)\xi$$

$$2(\overline{\nabla}_Y\phi)X = A_{\phi Y}X - \nabla^{\perp}_X\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \eta(X)Y + \phi[X,Y] - \eta(X)\phi Y - g(X,Y)\xi,$$

for any $X \in D$ and $Y \in D^{\perp}$.

Corollary 3.4. *If M be a ξ- vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection. Then*

$$2(\overline{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y] - \eta(Y)\phi X - 3g(X,Y)\xi$$

$$2(\overline{\nabla}_Y\phi)X = A_{\phi Y}X - \nabla^{\perp}_X\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y] - \eta(Y)\phi X - g(X,Y)\xi,$$

for any $X \in D$ and $Y \in D^{\perp}$.

3. Parallel Distribution

Definition 4.1. The horizontal (resp., vertical) distribution $D(resp.,D^{\perp})$ is said to be parallel [3] with respect to the connection on M if $\bigtriangledown_X Y \in D(resp., \bigtriangledown_Z W \in D^{\perp})$ for any vector field $X,Y \in D(resp.,W,Z \in D^{\perp})$

Theorem 4.1. *If M be a ξ- vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection. Then*

$$h(X,\phi Y) = h(Y,\phi X), \tag{4.1}$$

for any $X,Y \in D$.

Proof. Using parallelism of horizontal distribution D ,we have

$$\bigtriangledown_X(\phi Y) \in D and \bigtriangledown_Y(\phi X) \in D$$

for any $X,Y \in D$.

From (3.2), we have

$$Bh(X,Y) = g(X,Y)\xi \tag{4.2}$$

From (2.12) and (4.2), we have

$$Ch(X,Y) = \phi h(X,Y) - g(X,Y)\xi \tag{4.3}$$

Now,from (3.3), we have

$$h(X,\phi Y) + h(Y,\phi X) = 2Ch(X,Y)$$

Using (4.3) in above , we have

$$h(X,\phi Y) + h(Y,\phi X) = 2\phi h(X,Y) - 2g(X,Y)\xi \tag{4.4}$$

Replacing Y by ϕY in (4.4) and using (2.1), we have

$$h(X,Y) + h(\phi Y,\phi X) = 2\phi h(X,\phi Y) - 2g(X,\phi Y)\xi \tag{4.5}$$

Similarly ,replacing X by ϕX in (4.4) and using (2.1), we have

$$h(\phi X,\phi Y) + h(Y,X) = 2\phi h(\phi X,Y) - 2g(\phi X,Y)\xi \tag{4.6}$$

Compairing (4.5) and (4.6), we have

$$\phi h(X,\phi Y) - g(X,\phi Y)\xi = \phi h(\phi X,Y) - g(\phi X,Y)\xi$$

$$\phi^2 h(X,\phi Y) - g(X,\phi Y)\phi\xi = \phi^2 h(\phi X,Y) - g(\phi X,Y)\phi\xi$$

Using (2.2), we have

$$h(X,\phi Y) = h(\phi X,Y)$$

for any $X,Y \in D$. Hence theorem is proved.

Theorem 4.2. *If M be a ξ- vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection. If the distribution $D^\perp$ is parallel with respect to the connection on M, then*

$$A_{\phi X}Y + A_{\phi Y}X \in D^\perp, \tag{4.7}$$

for any $X,Y \in D^\perp$.

Proof. Let $X,Y \in D^\perp$, then from Weingarten formula (2.17), we have

$$(\overline{\nabla}_X\phi)Y + \phi(\overline{\nabla}_X Y) = -A_{\phi Y}X + \nabla^\perp_X \phi Y - \eta(X)Y - 2\eta(X)\eta(Y)\xi - g(X,Y)\xi \tag{4.8}$$

Using Gauss equation (2.16) in (4.8), we have

$$(\overline{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla^\perp_X \phi Y - \phi(\nabla_X Y) - \phi h(X,Y) - \eta(X)Y - 2\eta(X)\eta(Y)\xi \tag{4.9}$$

$$-g(X,Y)\xi$$

Interchanging X and Y, we have

$$(\overline{\nabla}_Y\phi)X = -A_{\phi X}Y + \nabla^\perp_Y \phi X - \phi(\nabla_Y X) - \phi h(X,Y) - \eta(Y)X - 2\eta(X)\eta(Y)\xi \tag{4.10}$$

$$-g(X,Y)\xi$$

Adding (4.9) and (4.10), we get

$$(\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X = -A_{\phi Y}X - A_{\phi X}Y + \nabla^\perp_X \phi Y + \nabla^\perp_Y \phi X - \phi(\nabla_X Y) \tag{4.11}$$

$$-\phi(\nabla_Y X) - 2\phi h(X,Y) - \eta(X)Y - \eta(Y)X$$

$$-4\eta(X)\eta(Y)\xi - 2g(X,Y)\xi$$

Using (2.14) in (4.11), we have

$$-\eta(X)\phi Y-\eta(Y)\phi X=-A_{\phi Y}X-A_{\phi X}Y+\nabla^{\perp}_{X}\phi Y+\nabla^{\perp}_{Y}\phi X-\phi(\nabla_{X}Y) \tag{4.12}$$

$$-\phi(\nabla_{Y}X)-2\phi h(X,Y)-\eta(X)Y-\eta(Y)X-2\eta(X)\eta(Y)\xi$$

Taking inner product with $Z\in D$ in (4.12), we have

$$-\eta(X)g(\phi Y,Z)-\eta(Y)g(\phi X,Z)=-g(A_{\phi Y}X,Z)-g(A_{\phi X}Y,Z)+g(\nabla^{\perp}_{X}\phi Y,Z)$$

$$+g(\nabla^{\perp}_{Y}\phi X,Z)-g(\phi(\nabla_{X}Y),Z)-g(\phi(\nabla_{Y}X),Z)-2g(\phi h(X,Y),Z)$$

$$-\eta(X)g(Y,Z)-\eta(Y)g(X,Z)-2\eta(X)\eta(Y)g(\xi,Z)$$

If $D^{\perp}$ is parallel then $\nabla_{X}Y\in D^{\perp}$ and $\nabla_{Y}X\in D^{\perp}$, so that from above

$$g(A_{\phi Y}X+A_{\phi X}Y,Z)=0 \tag{4.13}$$

Consequently, we have

$$A_{\phi Y}X+A_{\phi X}Y\in D^{\perp} \tag{4.14}$$

for any$X,Y\in D^{\perp}$. Hence theorem is proved.

Definition 4.2. A CR-submanifold is said to be mixed-totally geodesic if $h(X,Y)=0$ for all $X\in D$ and $Y\in D^{\perp}$.

Definition 4.3.A Normal vector field $N\neq 0$ is called $D-parallel$ normal section if $\nabla^{\perp}_{X}N=0$ for all $X\in D$.

Theorem 4.3. *Let M be a mixed totally geodesic ξ- vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection. Then the normal section $N\in\phi D^{\perp}$ is $D-parallel$ if and only if $\nabla_{X}\phi N\in D$, for all $X\in D$.*

Proof. Let $N\in\phi D^{\perp}$, for all $X\in D$ and $Y\in D^{\perp}$ then from (3.2), we have

$$-2\eta(X)\eta(Y)Q\xi-2g(X,Y)Q\xi+2Bh(X,Y)=Q\nabla_{X}(\phi PY)+Q\nabla_{Y}(\phi PX)$$

$$-QA_{\phi QY}X-QA_{\phi QX}Y-\eta(X)QY-\eta(Y)QX-g(X,QY)Q\xi-g(Y,QX)Q\xi$$

$$-2\eta(X)\eta(QY)Q\xi-2\eta(Y)\eta(QX)Q\xi$$

As M is a ξ- vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with quarter symmetric semi metric connection, so we have from above

$$2Bh(X,Y) = Q\bigtriangledown_Y(\phi X) - QA_{\phi Y}X \tag{4.15}$$

Using definition of mixed geodesic CR-submanifold, we have

$$Q\bigtriangledown_Y(\phi X) - QA_{\phi Y}X = 0 \tag{4.16}$$

$$Q\bigtriangledown_Y(\phi X) = QA_{\phi Y}X \tag{4.17}$$

As $Q\bigtriangledown_Y(\phi X) = 0$, for $X \in D$.

In particular, we have

$$Q\bigtriangledown_Y X = 0 \tag{4.18}$$

From (3.3), we have

$$-\eta(X)\phi QY - \eta(Y)\phi QX + \phi Q(\bigtriangledown_X Y) + \phi Q(\bigtriangledown_Y X) + 2Ch(X,Y) =$$

$$h(X,\phi PY) + h(Y,\phi PX) + \bigtriangledown_X^{\perp}(\phi QY) + \bigtriangledown_Y^{\perp}(\phi QX)$$

Using (4.18) in above, we have

$$\phi Q\bigtriangledown_X Y = \bigtriangledown_X^{\perp}(\phi Y)$$

That is

$$\phi Q\bigtriangledown_X(\phi N) = \bigtriangledown_X^{\perp}(\phi^2 N)$$

$$\phi Q\bigtriangledown_X(\phi N) = \bigtriangledown_X^{\perp}(N + \eta(N)\xi)$$

$$\phi Q\bigtriangledown_X(\phi N) = \bigtriangledown_X^{\perp}(N)$$

$$\phi Q\bigtriangledown_X(\phi N) = \bigtriangledown_X^{\perp} N \tag{4.19}$$

Then by definition of parallelism of N , we have

$$\phi Q\bigtriangledown_X(\phi N) = 0$$

Consequently, we have

$$\bigtriangledown_X(\phi N) \in D \qquad (4.20.)$$

for all $X \in D$.

Converse part is easy consequence of (4.20).

Conflict of Interests

The authors declare that there is no conflict of interests.

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