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COMPLETENESS AND COMPACTNESS OF THE CARTESIAN PRODUCT OF TWO SOFT METRIC SPACES

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Abstract. In this paper, we prove that the Cartesian product of two soft metric spaces is a soft metric space then we consider the completeness and compactness of the Cartesian product space.

Keywords: Soft metric space; Cartesian product; Cauchy sequence; Complete soft metric space; soft sequential compact metric space.

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1. Introduction

Because of various uncertainties arising in our real world, methods of classical mathematics may not be successfully applied to solve them. Thus, new mathematical theories such as probability theory, interval mathematics and fuzzy set theory have been introduced by researchers. But each of these theories has its certain inherent difficulties as mentioned by Molodtsov [1]. In 1999, Molodtsov [1] initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties and studied some properties of this theory. From then on, many researchers started to study the theory of soft sets and a lot of activities have been shown in soft sets theory.

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In 2010, in order to introduce soft set relations, Babitha and Sunil gave a definition of the Cartesian product of two soft sets. In 2013, Das and Samanta introduced the notion of soft metric space by using a different concept of soft point and investigated some basic properties of soft metric space. In this paper, our purpose is to study the completeness and compactness of the Cartesian product space of two soft metric spaces corresponding to a common finite parameter set.

2. Preliminaries

In this section, we will recall some basic notions and examples about soft set theory.

Definition 2.1[1] Let U be an initial universe set, E be a set of parameters and $P(U)$ denotes the power set of U. A pair (F,E) is called a soft set (over U), where F is a mapping given by $F: E \to P(U)$.

Example 1[1] Soft set (F,E) describes the attractiveness of the houses which Mr. X is going to buy. U is the set of houses under consideration where $U = \{h_1,h_2,h_3,h_4,h_5,h_6\}$, E is the set of parameters. Each parameter is a word or a sentence. $E = \{beautiful, wooden, in\ the\ green\ surroundings, in\ good\ repair\} = \{e_1,e_2,e_3,e_4\}$. The mapping $F: E \to P(U)$ be defined by $F(e_1) = \{h_1,h_4,h_5\}$, $F(e_2) = \{h_1,h_2,h_3\}$, $F(e_3) = \{h_4,h_5,h_6\}$, $F(e_4) = \{h_2,h_3,h_4,h_5,h_6\}$. Then the soft set $(F,E) = \{beautiful\ houses = \{h_1,h_4,h_5\}, wooden\ houses = \{h_1,h_2,h_3\}, in\ the\ green\ surroundings\ houses = \{h_4,h_5,h_6\}, in\ good\ repair\ houses = \{h_2,h_3,h_4,h_5,h_6\}\}$.

Definition 2.2[1] A soft set (F,E) over U is said to be a null soft set denoted by Φ if for all $e \in E$, $F(e) = \phi$.

Definition 2.3[1] A soft set (F,E) over U is said to be an absolute soft set denoted by $\widetilde{U}$ if for all $e \in E$, $F(e) = U$.

Definition 2.4[10] For two soft sets (F,A) and (G,B) over a common universe U, we say that (F,A) is a soft subset of (G,B) if $A \subset B$ and $\forall e \in A,\ F(e) \subset G(e)$.

Definition 2.5[6] A union of two soft sets (F,A) and (G,B) over a common universe U, is the soft set (H,C) where $C = A \cup B$, and $\forall e \in C$,

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ G(e) & if\ e \in B - A \\ F(e) \cup G(e) & if\ e \in A \cap B \end{cases}$$

We write $(F,A)\widetilde{\cup}(G,B) = (H,C)$.

Definition 2.6[8] An intersection of two soft sets (F,A) and (G,B) over a common universe U, is the soft set (H,C) where $C = A \cap B$, and $\forall e \in C$, $H(e) = F(e) \cap G(e)$, we write $(F,A)\widetilde{\cap}(G,B) = (H,C)$.

Definition 2.7[2] Let R be the set of real numbers, $B(R)$ the collection of all non-empty bounded subsets of R and A taken as a set of parameters. Then a mapping $\widetilde{f} : A \to B(R)$ is called a soft real set. If specifically $\widetilde{f}$ is a single valued mapping on A taking values in the set R, then the pair $(\widetilde{f},A)$ or simply $\widetilde{f}$ is called a soft real number. If $\widetilde{f}$ is a single valued mapping on A taking values in the set R^+ of nonnegative real numbers, then the pair $(\widetilde{f},A)$ or simply $\widetilde{f}$ is called a nonnegative soft real number. We shall denote the set of nonnegative soft real numbers (corresponding to A) by $R(A)^*$. We will denote a particular type of soft real numbers by $\bar{r}$, $\bar{s}$, $\bar{t}$, Such as $\bar{0}$ is a soft real number where $\bar{0}(\lambda) = 0$, for all $\lambda \in A$.

Example 2[2](Example of soft real number) Consider the example 1. If a mapping $\widetilde{f} : E \to B(R)$ is defined by $\widetilde{f}(e) =$ the number of houses which satisfy the contition e. Then we get $\widetilde{f}(e_1) = \{3\}$, $\widetilde{f}(e_2) = \{3\}$, $\widetilde{f}(e_3) = \{3\}$, $\widetilde{f}(e_4) = \{5\}$. Then $(\widetilde{f},E)$ can be taken as a soft real number such that $(\widetilde{f},E) = \{beautiful\ houses = 3, wooden\ houses = 3, in\ the\ green\ surroundings\ houses = 3, in\ good\ repair\ houses = 5\}$.

Definition 2.8[3] For two soft real numbers $\widetilde{f}$, $\widetilde{g}$, we define

(a) $\widetilde{f} \widetilde{\leq} \widetilde{g}$ if $\widetilde{f}(\lambda) \leq \widetilde{g}(\lambda)$, for all $\lambda \in A$;
(b) $\widetilde{f} \widetilde{\geq} \widetilde{g}$ if $\widetilde{f}(\lambda) \geq \widetilde{g}(\lambda)$, for all $\lambda \in A$;
(c) $\widetilde{f} \widetilde{<} \widetilde{g}$ if $\widetilde{f}(\lambda) < \widetilde{g}(\lambda)$, for all $\lambda \in A$;
(d) $\widetilde{f} \widetilde{>} \widetilde{g}$ if $\widetilde{f}(\lambda) > \widetilde{g}(\lambda)$, for all $\lambda \in A$.

Definition 2.9[3] Let X be an initial universal set and A be the non-empty set of parameters, A soft set (P,A) over X is said to be a *soft point* if there is exactly one $\lambda \in A$, such that $P(\lambda) = \{x\}$ for some $x \in X$ and $P(\mu) = \phi$, $\forall \mu \in A \setminus \{\lambda\}$. It will be denoted by P_λ^x.

Definition 2.10[3] A soft point P_λ^x is said to belongs to a soft set (F,A) if $\lambda \in A$ and $P(\lambda) = \{x\} \subset F(\lambda)$. We write $P_\lambda^x \widetilde{\in}(F,A)$.

Definition 2.11[3] Two soft points P_λ^x, P_μ^y are said to be equal if $\lambda = \mu$ and $P(\lambda) = P(\mu)$, that is $x = y$. Thus $P_\lambda^x \neq P_\mu^y \Leftrightarrow x \neq y$ or $\lambda \neq \mu$.

Let X be an initial universal set and A be the non-empty set of parameters. Let $\widetilde{X}$ be the absolute soft set i.e. $F(\lambda) = X$, $\forall \lambda \in A$, where $(F,A) = \widetilde{X}$. Let $SP(\widetilde{X})$ denotes the collection of all soft points of $\widetilde{X}$ and $R(A)^*$ denotes the set of all non-negative soft real numbers. We get the following definition:

Definition 2.12[3] A mapping $d : SP(\widetilde{X}) \times SP(\widetilde{X}) \to R(A)^*$, is said to be a *soft metric* on the soft set $\widetilde{X}$ if d satisfies the following conditions:

$(M1)$ $d\ (P_\lambda^x, P_\mu^y) \widetilde{\geq} \widetilde{0}$, for all P_λ^x, $P_\mu^y \widetilde{\in} \widetilde{X}$;

$(M2)$ $d\ (P_\lambda^x, P_\mu^y) = \widetilde{0} \Leftrightarrow P_\lambda^x = P_\mu^y$;

$(M3)$ $d\ (P_\lambda^x, P_\mu^y) = d\ (P_\mu^y, P_\lambda^x)$, for all P_λ^x, $P_\mu^y \widetilde{\in} \widetilde{X}$;

$(M4)$ For all P_λ^x, P_μ^y, $P_\gamma^z \widetilde{\in} \widetilde{X}$, $d\ (P_\lambda^x, P_\gamma^z) \widetilde{\leq} d\ (P_\lambda^x, P_\mu^y) + d\ (P_\mu^y, P_\gamma^z)$.

The soft set $\widetilde{X}$ with a soft metric d is called a *soft metric space* and denoted by $(\widetilde{X}, d, A)$ or $(\widetilde{X}, d)$.

Definition 2.13[3] Let $\{P_{\lambda_n}^{x_n}\}$ be a sequence of soft points in a soft metric space $(\widetilde{X}, d)$. The sequence $\{P_{\lambda_n}^{x_n}\}$ is said to be convergent in $(\widetilde{X}, d)$ if there is a soft point $P_\lambda^x \widetilde{\in} \widetilde{X}$ such that $d\ (P_{\lambda_n}^{x_n}, P_\lambda^x) \to \bar{0}$ as $n \to \infty$. This means for every $\widetilde{\varepsilon} \widetilde{\geq} \bar{0}$, chosen arbitrarily, there exists a natural number $N = N(\widetilde{\varepsilon})$ such that $\bar{0} \widetilde{\leq} d\ (P_{\lambda_n}^{x_n}, P_\lambda^x) \widetilde{\leq} \widetilde{\varepsilon}$, whenever $n > N$.

Definition 2.14[3] A sequence $\{P_{\lambda_n}^{x_n}\}$ of soft points in $(\widetilde{X}, d)$ is considered as a Cauchy sequence in $\widetilde{X}$ if corresponding to every $\widetilde{\varepsilon} \widetilde{\geq} \bar{0}$, $\exists\ m \in N$ such that $d\ (P_{\lambda_i}^{x_i}, P_{\lambda_j}^{x_j}) \widetilde{\leq} \widetilde{\varepsilon}$, $\forall\ i,\ j \geq m$, i.e. $d\ (P_{\lambda_i}^{x_i}, P_{\lambda_j}^{x_j}) \to \bar{0}$ as $i,\ j \to \infty$.

Definition 2.15[3] A soft metric space $(\widetilde{X}, d)$ is called complete if every Cauchy sequence in $\widetilde{X}$ converges to some soft point of $\widetilde{X}$.

Definition 2.16 A soft metric space $(\widetilde{X}, d, A)$ is called soft sequential compact metric space if every soft sequence has a soft subsequence that converges in $\widetilde{X}$.

3. Main results

In this section, we discuss the completeness and compactness of the cartesian product of two soft metric spaces.

Definition 3.1[4] Let (F,A) and (G,B) be two soft sets over U, then the cartesian product of (F,A) and (G,B) is defined as $(F,A)\times(G,B)=(H,A\times B)$ where $H:\ A\times B\to P(U\times U)$ and $H(a,b)=F(a)\times G(b)$, $(a,b)\in A\times B$, i.e. $H(a,b)=\{(h_i,h_j):\ h_i\in F(a),\ h_j\in F(b)\}$.

Example 3[4] Consider the soft set (F,A) which describes the ”cost of the houses” and the soft set (G,B) which describes the ”attractiveness of the houses”.
Suppose that $U=\{h_1,h_2,h_3,h_4,h_5,h_6\}$, $A=\{very\ costly;\ costly;\ cheap\}$,
$B=\{beautiful, wooden, in\ the\ green\ surroundings, in\ good\ repair\}$
Let $F(very\ costly)=\{h_2,h_4\}$, $F(costly)=\{h_1\}$, $F(cheap)=\{h_3,h_5,h_6\}$,
$G(beautiful)=\{h_1,h_4,h_5\}$, $G(wooden)=\{h_1,h_2,h_3\}$,
$G(in\ the\ green\ surroundings)=\{h_4,h_5,h_6\}$, $G(in\ good\ repair)=\{h_2,h_3,h_4,h_5,h_6\}$.
Now $(F,A)\times(G,B)=(H,A\times B)$ where a typical element will look like
$H(very\ costly,\ beautiful)=\{h_2,h_4\}\times\{h_1,h_4,h_5\}=\{(h_2,h_1),(h_2,h_4),(h_2,h_5),(h_4,h_1),$
$(h_4,h_4),(h_4,h_5)\}$

Definition 3.2 Let $(\widetilde{X},\ d_1,\ A)$ and $(\widetilde{Y},\ d_2,\ A)$ be two soft metric spaces, A is a finite parameter set. The cartesian product of $(\widetilde{X},\ d_1,\ A)$ and $(\widetilde{Y},\ d_2,\ A)$ is the product space $(\widetilde{X}\times\widetilde{Y},\ d,\ A)$ where $\widetilde{X}\times\widetilde{Y}$ is the cartesian product of $\widetilde{X}$ and $\widetilde{Y}$, d is a mapping from $SP(\widetilde{X}\times\widetilde{Y})\ \times\ SP(\widetilde{X}\times\widetilde{Y})$ into $R(A)^*$ given by: $d\ ((P^{x_1}_{\lambda_1},\ P^{y_1}_{\mu_1}),\ (P^{x_2}_{\lambda_2},\ P^{y_2}_{\mu_2}))=max\{d_1(P^{x_1}_{\lambda_1},\ P^{x_2}_{\lambda_2}),d_2(P^{y_1}_{\mu_1},\ P^{y_2}_{\mu_2})\}$, where $(P^{x_1}_{\lambda_1},\ P^{y_1}_{\mu_1}),\ (P^{x_2}_{\lambda_2},\ P^{y_2}_{\mu_2})\in SP(\widetilde{X}\times\widetilde{Y})$.

Lemma 3.1 Let $(\widetilde{X},\ d_1,\ A)$ and $(\widetilde{Y},\ d_2,\ A)$ be two soft metric spaces, then the cartesian product space $(\widetilde{X}\times\widetilde{Y},\ d,\ A)$ defined as above is a soft metric space.

Proof.$(M1)$ is obviously established by its definition;

$$(M2)\ d\ ((P^{x_1}_{\lambda_1},\ P^{y_1}_{\mu_1}),\ (P^{x_2}_{\lambda_2},\ P^{y_2}_{\mu_2})) = \bar{0} \Leftrightarrow\ max\{d_1(P^{x_1}_{\lambda_1},\ P^{x_2}_{\lambda_2}), d_2(P^{y_1}_{\mu_1},\ P^{y_2}_{\mu_2})\} = \bar{0}$$
$$\Leftrightarrow d_1(P^{x_1}_{\lambda_1},\ P^{x_2}_{\lambda_2}) = d_2(P^{y_1}_{\mu_1},\ P^{y_2}_{\mu_2}) = \bar{0}$$
$$\Leftrightarrow P^{x_1}_{\lambda_1} = P^{x_2}_{\lambda_2}\ and\ P^{y_1}_{\mu_1} = P^{y_2}_{\mu_2}$$
$$\Leftrightarrow (P^{x_1}_{\lambda_1},\ P^{y_1}_{\mu_1}) = (P^{x_2}_{\lambda_2},\ P^{y_2}_{\mu_2})$$

$(M3)$ is obviously established by the symmetry of d_1 and d_2;

$(M4)$ For all $(P^{x_i}_{\lambda_i},\ P^{y_i}_{\mu_i}) \widetilde{\in} SP(\widetilde{X} \times \widetilde{Y})\ (i = 1,2,3)$

$$d\ ((P^{x_1}_{\lambda_1},\ P^{y_1}_{\mu_1}),\ (P^{x_3}_{\lambda_3},\ P^{y_3}_{\mu_3}))$$
$$= max\{d_1(P^{x_1}_{\lambda_1},\ P^{x_3}_{\lambda_3}),\ d_2(P^{y_1}_{\mu_1},\ P^{y_3}_{\mu_3})\}$$
$$\widetilde{\leq} max\{d_1(P^{x_1}_{\lambda_1},\ P^{x_2}_{\lambda_2}) + d_1(P^{x_2}_{\lambda_2},\ P^{x_3}_{\lambda_3}),\ d_2(P^{y_1}_{\mu_1},\ P^{y_2}_{\mu_2}) + d_2(P^{y_2}_{\mu_2},\ P^{y_3}_{\mu_3})\}$$
$$\widetilde{\leq} \max\{\ d_1(P^{x_1}_{\lambda_1},\ P^{x_2}_{\lambda_2}),\ d_2(P^{y_1}_{\mu_1},\ P^{y_2}_{\mu_2})\} + max\{d_1(P^{x_2}_{\lambda_2},\ P^{x_3}_{\lambda_3}),\ d_2(P^{y_2}_{\mu_2},\ P^{y_3}_{\mu_3})\}$$
$$= d\ ((P^{x_1}_{\lambda_1},\ P^{y_1}_{\mu_1}),\ (P^{x_2}_{\lambda_2},\ P^{y_2}_{\mu_2})) + d\ ((P^{x_2}_{\lambda_2},\ P^{y_2}_{\mu_2}),\ (P^{x_3}_{\lambda_3},\ P^{y_3}_{\mu_3}))$$

Then $(\widetilde{X} \times \widetilde{Y},\ d,\ A)$ is a soft metric space.

Lemma 3.2 $\{(P^{x_n}_{\lambda_n},\ P^{y_n}_{\mu_n})\}$ is a Cauchy sequence in $(\widetilde{X} \times \widetilde{Y},\ d,\ A)$, if and only if $\{P^{x_n}_{\lambda_n}\}$ is a Cauchy sequence in $(\widetilde{X},\ d_1,\ A)$ and $\{P^{y_n}_{\mu_n}\}$ is a Cauchy sequence in $(\widetilde{Y},\ d_2,\ A)$.

Proof. "$\Rightarrow$" $\{(P^{x_n}_{\lambda_n},\ P^{y_n}_{\mu_n})\}$ is a Cauchy sequence in $(\widetilde{X} \times \widetilde{Y},\ d,\ A)$,
then for any $\widetilde{\varepsilon} \widetilde{\geq} \bar{0}$, $\exists\ m \in N$ such that $d\ ((P^{x_i}_{\lambda_i},\ P^{y_i}_{\mu_i}),\ (P^{x_j}_{\lambda_j},\ P^{y_j}_{\mu_j})) \widetilde{\leq} \widetilde{\varepsilon}$, $\forall\ i,\ j\ \geq\ m$,
That is $max\{d_1(P^{x_i}_{\lambda_i},\ P^{x_j}_{\lambda_j}),\ d_2(P^{y_i}_{\mu_i},\ P^{y_j}_{\mu_j}) \widetilde{\leq} \widetilde{\varepsilon}$, $\forall\ i,\ j\ \geq\ m$.
So we have $d_1(P^{x_i}_{\lambda_i},\ P^{x_j}_{\lambda_j}) \widetilde{\leq} \widetilde{\varepsilon}$ and $d_2(P^{y_i}_{\mu_i},\ P^{y_j}_{\mu_i}) \widetilde{\leq} \widetilde{\varepsilon}$ as $i,\ j\ \geq\ m$,
then $\{P^{x_n}_{\lambda_n}\}$ is a Cauchy sequence in $\widetilde{X}$ and $\{P^{y_n}_{\mu_n}\}$ is a Cauchy sequence in $\widetilde{Y}$.
"$\Leftarrow$" Let $\{P^{x_n}_{\lambda_n}\}$ is a Cauchy sequence in $\widetilde{X}$ and $\{P^{y_n}_{\mu_n}\}$ is a Cauchy sequence in $\widetilde{Y}$.
Then for any $\widetilde{\varepsilon} \widetilde{\geq} \bar{0}$, there exists $m_1 \in N$, such that $d_1(P^{x_i}_{\lambda_i},\ P^{x_j}_{\lambda_j}) \widetilde{\leq} \widetilde{\varepsilon}$ whereas $i,\ j \geq m_1$,
in the similar way, we can find a $m_2 \in N$, such that $d_2(P^{y_i}_{\mu_i},\ P^{y_j}_{\mu_i}) \widetilde{\leq} \widetilde{\varepsilon}$ as $i,\ j\ \geq\ m$.
Let $m = max\{m_1, m_2\}$, then when $i,\ j \geq m$, we have
$d\ ((P^{x_i}_{\lambda_i},\ P^{y_i}_{\mu_i}),\ (P^{x_j}_{\lambda_j},\ P^{y_j}_{\mu_j}))\ = max\{d_1(P^{x_i}_{\lambda_i},\ P^{x_j}_{\lambda_j}), d_2(P^{y_i}_{\mu_i},\ P^{y_j}_{\mu_i})\} \widetilde{\leq} \widetilde{\varepsilon}$,
Then we complete the proof.

Theorem 3.1 $(\widetilde{X}\times\widetilde{Y},\ d,\ A)$ defined as above is a complete soft metric space if and only if $(\widetilde{X},\ d_1,\ A)$ and $(\widetilde{Y},\ d_2,\ A)$ are two complete soft metric spaces.

Proof. "$\Rightarrow$" Let $\{P_{\lambda_n}^{x_n}\}$ is a Cauchy sequence in $(\widetilde{X},\ d_1,\ A)$, $\{P_{\mu_n}^{y_n}\}$ is a Cauchy sequence in $(\widetilde{Y},\ d_2,\ A)$.

Then by Lemma 3.2, we have $\{(P_{\lambda_n}^{x_n},P_{\mu_n}^{y_n})\}$ is a Cauchy sequence in $(\widetilde{X}\times\widetilde{Y},\ d,\ A)$.

Since $(\widetilde{X}\times\widetilde{Y},\ d,\ A)$ is complete, so there exists a soft point $(P_{\lambda}^{x},P_{\mu}^{y})$ in $\widetilde{X}\times\widetilde{Y}$, such that

for any $\widetilde{\varepsilon}\widetilde{\geq}\bar{0}$, $\exists\, N\geq 0$, $d((P_{\lambda_n}^{x_n},P_{\mu_n}^{y_n}),(P_{\lambda}^{x},P_{\mu}^{y}))\widetilde{\leq}\widetilde{\varepsilon}$ whereas $n\geq N$.

That is $max\{d_1(P_{\lambda_n}^{x_n},P_{\lambda}^{x}),d_2(P_{\mu_n}^{y_n},P_{\mu}^{y})\}\widetilde{\leq}\widetilde{\varepsilon}$ as $n\geq N$,

then we have $d_1(P_{\lambda_n}^{x_n},P_{\lambda}^{x})\widetilde{\leq}\widetilde{\varepsilon}$ and $d_2(P_{\mu_n}^{y_n},P_{\mu}^{y})\widetilde{\leq}\widetilde{\varepsilon}$ when $n\geq N$, and $P_{\lambda}^{x}\in\widetilde{X},\ P_{\mu}^{y}\in\widetilde{Y}$,

so $(\widetilde{X},\ d_1,\ A)$ and $(\widetilde{Y},\ d_2,\ A)$ are complete.

"$\Leftarrow$" Let $\{(P_{\lambda_n}^{x_n},P_{\mu_n}^{y_n})\}$ is a Cauchy sequence in $(\widetilde{X}\times\widetilde{Y},\ d,\ A)$,

Then by Lemma 3.2, we have $\{P_{\lambda_n}^{x_n}\}$ is a Cauchy sequence in $(\widetilde{X},\ d_1,\ A)$,

$\{P_{\mu_n}^{y_n}\}$ is a Cauchy sequence in $(\widetilde{Y},\ d_2,\ A)$.

Since $(\widetilde{X},\ d_1,\ A)$ and $(\widetilde{Y},\ d_2,\ A)$ are complete, so there exist $P_{\lambda}^{x}\in\widetilde{X}$ and $P_{\mu}^{y}\in\widetilde{Y}$,

such that for any $\widetilde{\varepsilon}\widetilde{\geq}\bar{0}$, $\exists\, N_1>0$, such that $d_1(P_{\lambda_n}^{x_n},P_{\lambda}^{x})\widetilde{\leq}\widetilde{\varepsilon}$ as $n\ \geq N_1$,

$\exists\, N_2>0$, such that $d_2(P_{\mu_n}^{y_n},P_{\mu}^{y})\widetilde{\leq}\widetilde{\varepsilon}$ as $n\ \geq N_2$.

Let $N=max\{N_1,N_2\}$, then when $n\geq N$, we have

$d((P_{\lambda_n}^{x_n},P_{\mu_n}^{y_n}),(P_{\lambda}^{x},P_{\mu}^{y}))=max\{d_1(P_{\lambda_n}^{x_n},P_{\lambda}^{x}),d_2(P_{\mu_n}^{y_n},P_{\mu}^{y})\}\widetilde{\leq}\widetilde{\varepsilon}$ and $(P_{\lambda}^{x},P_{\mu}^{y})\in\widetilde{X}\times\widetilde{Y}$.

So $(\widetilde{X}\times\widetilde{Y},\ d,\ A)$ is complete.

Theorem 3.2 $(\widetilde{X}\times\widetilde{Y},\ d,\ A)$ defined as above is a soft sequential compact metric space if and only if $(\widetilde{X},\ d_1,\ A)$ and $(\widetilde{Y},\ d_2,\ A)$ are two soft sequential compact metric spaces.

Proof. "$\Rightarrow$" Let $\{P_{\lambda_n}^{x_n}\}$ is a soft points sequence in $(\widetilde{X},\ d_1,\ A)$, and $P_{\mu_n}^{y_n}$ is a soft points sequence in $(\widetilde{Y},\ d_2,\ A)$.

Then by the definition of the Cartesian product, we have $\{(P_{\lambda_n}^{x_n},P_{\mu_n}^{y_n})\}$ is a soft points sequence in $(\widetilde{X}\times\widetilde{Y},\ d,\ A)$.

Since $(\widetilde{X}\times\widetilde{Y},\ d,\ A)$ is a soft sequential compact metric space, so there is a subsequence $\{(P_{\lambda_{n_k}}^{x_{n_k}},P_{\mu_{n_k}}^{y_{n_k}})\}$ converges to $(P_{\lambda}^{x},P_{\mu}^{y})\,\widetilde{\in}\,\widetilde{X}\times\widetilde{Y}$, that is for any $\widetilde{\varepsilon}\widetilde{\geq}\bar{0}$, $\exists\, N>0$, such that $d((P_{\lambda_{n_k}}^{x_{n_k}},P_{\mu_{n_k}}^{y_{n_k}}),(P_{\lambda}^{x},P_{\mu}^{y}))=max\{d_1(P_{\lambda_{n_k}}^{x_{n_k}},P_{\lambda}^{x}),d_2(P_{\mu_{n_k}}^{y_{n_k}},P_{\mu}^{y})\}\widetilde{\leq}\widetilde{\varepsilon}$ whereas $n>N$.

So $d_1(P^{x_{n_k}}_{\lambda_{n_k}}, P^x_\lambda) \widetilde{\leq} \widetilde{\varepsilon}$ and $d_2(P^{y_{n_k}}_{\mu_{n_k}}, P^y_\mu)\} \widetilde{\leq} \widetilde{\varepsilon}$ as $n > N$.

Then we can get a subsequence $\{P^{x_{n_k}}_{\lambda_{n_k}}\}$ of $\{P^{x_n}_{\lambda_n}\}$ which converged in $\widetilde{X}$ and a subsequence $\{P^{y_{n_k}}_{\mu_{n_k}}\}$ of $\{P^{y_n}_{\mu_n}\}$ which converged in $\widetilde{Y}$.

"$\Leftarrow$" For any soft points sequence $\{(P^{x_n}_{\lambda_n}, P^{y_n}_{\mu_n})\}$ in $(\widetilde{X} \times \widetilde{Y},\ d,\ A)$,

$\{P^{x_n}_{\lambda_n}\}$ is a soft points sequence in $(\widetilde{X},\ d_1,\ A)$ and $\{P^{y_n}_{\mu_n}\}$ is a soft points sequence in $(\widetilde{Y},\ d_2,\ A)$.

Because of $(\widetilde{X},\ d_1,\ A)$ is a soft sequential compact metric space,

we can get a subsequence $\{P^{x_{n_k}}_{\lambda_{n_k}}\}$ of $\{P^{x_n}_{\lambda_n}\}$ which converged to $P^x_\lambda \widetilde{\in} \widetilde{X}$,

for the index n_k, $\{P^{y_{n_k}}_{\mu_{n_k}}\}$ is a subsequence of $\{P^{y_n}_{\mu_n}\}$ and also is a soft points sequence in $\widetilde{Y}$,

since $(\widetilde{Y},\ d_2,\ A)$ is a soft sequential compact metric space, so we can find a subsequence $\{P^{y_{n_{k_m}}}_{\mu_{n_{k_m}}}\}$ of $\{P^{y_{n_k}}_{\mu_{n_k}}\}$ and a soft point P^y_μ in $\widetilde{Y}$, such that $\{P^{y_{n_{k_m}}}_{\mu_{n_{k_m}}}\}$ converges to P^y_μ.

Since $\{P^{x_{n_k}}_{\lambda_{n_k}}\}$ is a convergent sequence and it converges to P^x_λ, so $\{P^{x_{n_{k_m}}}_{\lambda_{n_{k_m}}}\}$ converges to P^x_λ.

Then we have a subsequence $\{P^{x_{n_{k_m}}}_{\lambda_{n_{k_m}}}, P^{y_{n_{k_m}}}_{\mu_{n_{k_m}}}\}$ of $\{(P^{x_n}_{\lambda_n}, P^{y_n}_{\mu_n})\}$ which converged to (P^x_λ, P^y_μ) $\widetilde{\in} \widetilde{X} \times \widetilde{Y}$.

Definition 3.3 d', d are two soft metrics on $\widetilde{X} \times \widetilde{Y}$, if for any $(P^{x_1}_{\lambda_1}, P^{y_1}_{\mu_1}),\ (P^{x_2}_{\lambda_2}, P^{y_2}_{\mu_2}) \widetilde{\in} \widetilde{X} \times \widetilde{Y}$ $d'((P^{x_1}_{\lambda_1}, P^{y_1}_{\mu_1}),\ (P^{x_2}_{\lambda_2}, P^{y_2}_{\mu_2})) \widetilde{\leq} d((P^{x_1}_{\lambda_1}, P^{y_1}_{\mu_1}),\ (P^{x_2}_{\lambda_2}, P^{y_2}_{\mu_2}))$, that is for any $\eta \in A$, we have $d'((P^{x_1}_{\lambda_1}, P^{y_1}_{\mu_1}),$ $(P^{x_2}_{\lambda_2}, P^{y_2}_{\mu_2}))(\eta) \leq\ d((P^{x_1}_{\lambda_1}, P^{y_1}_{\mu_1}),\ (P^{x_2}_{\lambda_2}, P^{y_2}_{\mu_2}))(\eta)$, then we say d' is stronger than d.

Example 4 $(\widetilde{X},\ d_1,\ A)$ and $(\widetilde{Y},\ d_2,\ A)$ are two soft metric spaces, $(\widetilde{X} \times \widetilde{Y},\ d,\ A)$ defines as definition 3.2, $(P^x_{\lambda_1}, P^y_{\mu_1}),\ (P^x_{\lambda_2}, P^y_{\mu_2}) \in \widetilde{X} \times \widetilde{Y}$, $\bar{0} \widetilde{<} \bar{t} \widetilde{<} \bar{1}$.

Let $d'((P^{x_1}_{\lambda_1}, P^{y_1}_{\mu_1}),\ (P^{x_2}_{\lambda_2}, P^{y_2}_{\mu_2})) = \bar{t}\ d_1(P^{x_1}_{\lambda_1}, P^{x_2}_{\lambda_2}) + \overline{(1-t)}\ d_2(P^{y_1}_{\mu_1}, P^{y_2}_{\mu_2})$,

we can deduce d' is a soft metric on $\widetilde{X} \times \widetilde{Y}$, and for any $\eta \in A$, we have

$$\begin{aligned} d'((P^{x_1}_{\lambda_1}, P^{y_1}_{\mu_1}),\ (P^{x_2}_{\lambda_2}, P^{y_2}_{\mu_2}))(\eta) &= [\bar{t}\ d_1(P^{x_1}_{\lambda_1}, P^{x_2}_{\lambda_2}) + \overline{1-t}\ d_2(P^{y_1}_{\mu_1}, P^{y_2}_{\mu_2})](\eta) \\ &\widetilde{\leq} max\{d_1(P^{x_1}_{\lambda_1}, P^{x_2}_{\lambda_2}), d_2(P^{y_1}_{\mu_1}, P^{y_2}_{\mu_2})\}(\eta) \\ &= d((P^{x_1}_{\lambda_1}, P^{y_1}_{\mu_1}),\ (P^{x_2}_{\lambda_2}, P^{y_2}_{\mu_2}))(\eta) \end{aligned}$$

then d' is stronger than d.

Properties Let $(\widetilde{X},\ d,\ A)$ is a soft metric space, d' is another soft metric on $\widetilde{X}$, if for any sequence of soft points $\{P^{x_n}_{\lambda_n}\}$ and a soft point P^x_λ, d' is stronger than d, then we have:

(i) if $\{P^{x_n}_{\lambda_n}\}$ is a Cauchy sequence in $(\widetilde{X},\ d,\ A)$, then it is a Cauchy sequence in $(\widetilde{X},\ d',\ A)$;

(ii) if $d(P^{x_n}_{\lambda_n}, P^{x}_{\lambda}) \to \bar{0}$, then $d^{'}(P^{x_n}_{\lambda_n}, P^{x}_{\lambda}) \to \bar{0}$.

Then proof is straightforward.

Theorem 3.3 $(\widetilde{X} \times \widetilde{Y},\ d,\ A)$ is a soft sequential compact space, then $(\widetilde{X} \times \widetilde{Y},\ d^{'},\ A)$ is also a soft sequential compact metric space, whereas $d^{'}$ is stronger than d.

Proof. For any sequence of soft points $\{(P^{x_n}_{\lambda_n},\ P^{y_n}_{\mu_n})\}$ in $\widetilde{X} \times \widetilde{Y}$, since $(\widetilde{X} \times \widetilde{Y},\ d,\ A)$ is a soft sequential compact space, there exists a subsequence $\{(P^{x_{n_k}}_{\lambda_{n_k}},\ P^{y_{n_k}}_{\mu_{n_k}})\}$ converges to $(P^{x}_{\lambda},\ P^{y}_{\mu}) \in \widetilde{X} \times \widetilde{Y}$, because of $d^{'}$ is stronger than d, by Properties(ii), we have $d^{'}((P^{x_{n_k}}_{\lambda_{n_k}},\ P^{y_{n_k}}_{\mu_{n_k}}), (P^{x}_{\lambda},\ P^{y}_{\mu})) \to \bar{0}$, then we complete the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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