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A NUMERICAL APPROACH FOR SOLVING BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS USING SHANNON WAVELET

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Abstract. In this paper, Shannon Wavelet Method (SWM) is proposed to approximate solutions for boundary value problems for fractional order differential equations. Shannon wavelets operational matrices of integration are utilized to approximate the solutions in the form of convergent series with easily computable terms. Numerical examples are provided to demonstrate the accuracy, efficiency and simplicity of the proposed Shannon wavelet method.

Keywords: Shannon wavelet; operational matrix of integration; boundary value problems; fractional differential equations.

2010 AMS Subject Classification: 42C40, 65L10, 34A08.

1. Introduction

In the past several decades, the study of fractional calculus has turned to practical application from pure mathematical theory. Compared to integer order differential equation, fractional differential equation has the advantage that it can better describe some natural physical processes

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and dynamic system processes [2, 9] because the fractional order differential operators are non - local operators. In general, it is not easy to derive the analytical solutions to most of the fractional differential equations. Particularly, there is no known method for solving fractional boundary value problems exactly. Therefore several methods for the approximate solutions to classical differential equations are extended to solve differential equations of fractional order numerically. These methods include, Adomian decomposition method [3], homotopy - perturbation method [1], homotopy analysis method [6], variational iteration method [14], generalized differential transform method [10], finite difference method [15], fractional linear multi - step method [8], extrapolation method [4] and predictor - corrector method [5].

It should be noted that compare to the initial value problems, the numerical solutions of boundary value problems for fractional differential equations have received much less attention. In this work, we focus on providing a numerical scheme based on Shannon wavelet operational matrices of integration, to solve various types of boundary value problems for linear fractional differential equations. The main characteristic of the method is that, it converts the linear fractional boundary value problem into system of linear algebraic equations.

2. Preliminaries and Basic Definitions

Here, let us start with recalling the essentials of the fractional calculus. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer - order differentiation and n - fold integration.

2.1 Riemann-Liouville Fractional derivative and integral. Let $\alpha > 0, n-1 < \alpha \leq n, n \in \mathbb{N}$ and $h : (0,\infty) \to \mathbb{R}$ be continuous, then the Riemann - Liouville fractional derivative of order $\alpha > 0$, is defined by $\mathfrak{D}^{\alpha}_{RL} h(t) = \left(\frac{d}{dt}\right)^{n} I^{n-\alpha} h(t)$, where

$$I^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds$$

is the Riemann - Liouville fractional integral operator of order $\alpha > 0$.

For $\alpha, \beta > 0$, the Riemann - Liouville fractional order integral and derivative have following important properties:

(I) $I^{\alpha}\left(I^{\beta}h(t)\right) = I^{\alpha+\beta}h(t)$

(II) $I^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}$

(III) $\mathfrak{D}^{\alpha}_{RL}(I^{\alpha})h(t) = h(t)$

One of the drawbacks of Riemann - Liouville approach is that it cannot incorporate the non-zero initial condition at lower limit. For boundary value problems with non-zero initial conditions the Caputo approach is suitable.

2.2 Caputo Fractional Derivative. The Caputo definition of fractional order derivative is defined as,

$$\mathfrak{D}^{\alpha}_{C}h(t) = \frac{1}{\Gamma(n+1-\alpha)}\int_0^t \frac{h^{(n+1)}(s)}{(t-s)^{\alpha-n}}ds,\ \ n<\alpha\le n+1,\ \ n\in\mathbb{N}$$

where $\alpha > 0$ is the order of the derivative and n is the smallest integer greater than α.

2.3 Theorem. For $n = \lceil\alpha\rceil$

(I) $I^{\alpha}\left(\mathfrak{D}^{\alpha}_{RL}h(t)\right) = h(t) - \sum_{k=0}^{n-1}\frac{t^{\alpha-k-1}}{\Gamma(\alpha-k)}\lim_{s\to 0^+}\mathfrak{D}^{n-k-1}_{RL}I^{n-\alpha}h(s)$

(II) $I^{\alpha}\left(\mathfrak{D}^{\alpha}_{C}h(t)\right) = h(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}\left[\mathfrak{D}^{k}_{RL}h(t)\right]_{t=0}$

2.4 The Shannon Wavelet. The Shannon scaling function $\phi(t)$ defined on R is the sinc function which is given as

$$\phi(t) = sinc(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t\neq 0 \\ 1, & t=0 \end{cases}$$

The corresponding mother wavelet is defined as

$$\psi\left(t+\frac{1}{2}\right) = 2\phi(2t) - \phi(t)$$

Shannon wavelet $\psi_{j,k}(t)$ is a family of the functions constructed from dilation and translation of mother wavelet $\psi(t)$, i. e. $\left\{\psi_{j,k}(t) = 2^{j/2}\psi(2^jt-k)\right\}_{j,k=0}^{\infty}$ where j,k are non - negative integers which are called dilation and translation parameters respectively. The Shannon wavelet system forms an orthonormal bases of $L^2(\mathbb{R})$.

3. Function approximation and operational matrices of Shannon wavelet

3.1 Function approximation. Suppose $y(x)$ is of boundary variation on every bounded interval $y(x) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ then the wavelet series

$$y_j(x) = \sum_k \langle y, \psi_{j,k} \rangle \psi_{j,k}(x)$$

converges to $y(x)$ as $j \to \infty$, at every point of continuity of $y(x)$. Hence, every square integrable function $y(x)$ can be expanded into Shannon wavelet series as

$$y(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{j,k} \psi_{j,k}(x) \tag{3.1}$$

where the Shannon wavelet coefficients $a_{j,k}$ for $j,k = 0,1,2,---$ are given by

$$a_{j,k} = \langle y(x), \psi_{j,k}(x) \rangle$$

If the infinite series in equation (3.1) is truncated then equation (1) can be written as,

$$y(x) \approx y_m(x) = \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} a_{j,k} \psi_{j,k}(x) \tag{3.2}$$

In matrix form,

$$y(x) \approx A^T_{m^2\times 1} \Psi_{m^2\times 1}(x) \tag{3.3}$$

where $A^T_{m^2\times 1}$ and $\Psi_{m^2\times 1}(x)$ are $m^2 \times 1$ matrices given by

$A_{m^2\times 1} = [a_{0,0}, a_{0,1}, ..., a_{0,m-1}, a_{1,0}, a_{1,1}, ..., a_{1,m-1}, ..., a_{m-1,0}, a_{m-1,1}, ..., a_{m-1,m-1}]^T$ and

$\Psi_{m^2\times 1} = \left[\psi_{0,0}(x), \psi_{0,1}(x), ..., \psi_{0,m-1}(x), ..., \psi_{m-1,0}(x), \psi_{m-1,1}(x), ..., \psi_{m-1,m-1}(x)\right]^T$

Now at collocation points $x_i = \frac{i}{m^2-1}, i = mj+k$ where $j,k = 0,1,---,m-1$ we can define $m^2 \times m^2$ Shannon matrix as

$\Psi_{m^2\times m^2} = \left[\Psi_{m^2\times 1}(0), \Psi_{m^2\times 1}\left(\frac{1}{m^2-1}\right), \Psi_{m^2\times 1}\left(\frac{2}{m^2-1}\right), ---, \Psi_{m^2\times 1}\left(\frac{m^2-2}{m^2-1}\right), \Psi_{m^2\times 1}(1)\right]$

For instance, when $m = 2$, the Shannon matrix is given by

$$\Psi_{4\times 4} = \begin{bmatrix} 1 & 0.826993 & 0.413497 & 0 \\ -0.636620 & 0.699057 & 0.699057 & -0.63662 \\ 0.212207 & 0.372702 & -0.521783 & -0.63662 \\ -0.900316 & 0.988616 & -0.737913 & 0.300105 \end{bmatrix}$$

The Shannon wavelet coefficients $a_{j,k}, j,k = 0,1,---,m-1$ can be determined by matrix inversion.

$$A^T_{1\times m^2} = Y_{1\times m^2}(\Psi_{m^2\times m^2})^{-1} \tag{3.4}$$

where

$$Y_{1\times m^2} = [y_m(x_0), y_m(x_1), ---, y_m(x_{m^2-1})]$$

3.2 Shannon wavelet operational matrix of integration. The integration of the Shannon function vector $\Psi_{m^2\times 1}(x)$ is given by

$$\int_0^x \Psi_{m^2\times 1}(z)dz = S_{m^2\times m^2}\Psi_{m^2\times 1}(x)$$

where $S_{m^2\times m^2}$ is the Shannon wavelet operational matrix of integration. Now define the m^2 - set of block - pulse functions on $[0,1]$ as follows

$$b_i(x) = \begin{cases} 1, & \frac{i}{m^2} \le x \le \frac{i+1}{m^2} \\ 0, & otherwise \end{cases}$$

for $i = 0,1,---,m^2-1$. Here the functions b_i are disjoint and orthogonal.

$$\int_0^1 b_i(x)b_j(x)dx = \begin{cases} 0, & i \neq j \\ \frac{1}{m^2}, & i = j \end{cases}$$

The Shannon wavelet can be expanded into m^2 - set of block - pulse functions as

$$\Psi_{m^2\times 1}(x) = \Psi_{m^2\times m^2}B_{m^2\times 1}(x) \tag{3.5}$$

where the block pulse function vector $B_{m^2\times 1}(x)$ is defined as

$$B_{m^2\times 1}(x) = [b_0(x), b_1(x), ---, b_{m^2-1}(x)]^T$$

Fractional integration of the block - pulse function vector is given as

$$\left(I^\alpha B_{m^2\times 1}\right)(x) = F^\alpha_{m^2\times m^2}B_{m^2\times 1}(x) \tag{3.6}$$

where $F^{\alpha}_{m^2\times m^2}$ is the block - pulse operational matrix of the fractional order integration defined as [7]

$$F^{\alpha}_{m^2\times m^2} = \left(\frac{1}{m^2}\right)^{\alpha}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m^2-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m^2-2} \\ 0 & 0 & 1 & \cdots & \xi_{m^2-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

where $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$, for $k = 1,2,---,m^2-1$.

Now to find the Shannon wavelet operational matrix of the fractional integration, we suppose

$$I^{\alpha}\left(\Psi_{m^2\times 1}(x)\right) \approx S^{\alpha}_{m^2\times m^2}\Psi_{m^2\times 1}(x) \tag{3.7}$$

By using equation (3.5) and (3.6)

$$\begin{aligned} S^{\alpha}_{m^2\times m^2}\Psi_{m^2\times 1}(x) &\approx I^{\alpha}\Psi_{m^2\times m^2}B_{m^2\times 1}(x) \\ &\approx \Psi_{m^2\times m^2}F^{\alpha}_{m^2\times m^2}B_{m^2\times 1}(x) \end{aligned}$$

So

$$S^{\alpha}_{m^2\times m^2} \approx \Psi_{m^2\times m^2}F^{\alpha}_{m^2\times m^2}\left(\Psi_{m^2\times m^2}\right)^{-1}$$

For $m = 2, \alpha = 1.5$ the Shannon wavelet operational matrix of integration is given by

$$S^{1.5}_{4\times 4} = \begin{bmatrix} 0.344784 & 0.029417 & -0.648696 & 0.167481 \\ -0.017471 & -0.045238 & -0.152993 & 0.003118 \\ 0.049062 & 0.062126 & -0.048808 & -0.009804 \\ -0.125848 & 0.032364 & 0.105076 & -0.100288 \end{bmatrix}$$

4 Proposed Method

In this section, the implementation of the proposed method has been explained for solving Riemann - Liouville and Caputo fractional differential equations.

4.1 Riemann - Liouville Fractional differential equations. Consider the Riemann - Liouville fractional differential equation with boundary value problem given by,

$$\mathfrak{D}^{\alpha}_{RL}y(x) = f\left(x, y(x), \mathfrak{D}^{\beta}_{RL}y(x)\right), 0 \leq x \leq 1 \tag{4.1}$$

with

$y(0) = y_0, y(1) = y_1$

where $1 < \alpha \leq 2, 0 \leq \beta \leq 1$. Now on applying the integral operator I^{α} to both sides of (4.1) and on using the theorem 2.3, we have

$$y(x) = I^{\alpha} f\left(x, y(x), \mathfrak{D}^{\beta}_{RL}y(x)\right) + c_0 x^{\alpha-1} + c_1 x^{\alpha-2} \tag{4.2}$$

For $\alpha = 2$, we have

$$c_1 = y_0, c_0 = y_1 - y_0 - I^{\alpha} f\left(1, y(1), \mathfrak{D}^{\beta}_{RL}y(1)\right) \tag{4.3}$$

For $1 < \alpha < 2$, we have (4.3) with $y_0 = 0$. Now on substituting c_0 and c_1 in (4.2) and on using the relations (3.3) and (3.7), the fractional differential equation of Riemann - Liouville type reduces to system of algebraic equations. Then solve this system to obtain the numerical solution of boundary value problem for Riemann - Liouville fractional differential equation.

4.2 Caputo fractional differential equation. Consider the Caputo fractional differential equation with boundary value problem given by

$$\mathfrak{D}^{\alpha}_{C}y(x) = f\left(x, y(x), \mathfrak{D}^{\beta}_{C}y(x)\right),\ 0 \leq x \leq 1 \tag{4.4}$$

with

$$y(0) = y_0,\ y(1) = y_1 \tag{4.5}$$

where $1 < \alpha \leq 2, 0 \leq \beta \leq 1$. Now on applying the integral operator I^{α} to both sides of (4.4) and on using theorem 2.3, we have

$$y(x) = I^{\alpha} f\left(x, y(x), \mathfrak{D}^{\beta}_{C}y(x)\right) + c_0 + c_1 x \tag{4.6}$$

Using the boundary condition (4.5), find c_0 and c_1. On substituting c_0 and c_1 into equation (4.6), we have

$$y(x) = I^{\alpha} f\Big(x, y(x), \mathfrak{D}_C^{\beta} y(x)\Big) - x I^{\alpha} f\Big(1, y(1), \mathfrak{D}_C^{\beta} y(1)\Big) + x(y_1 - y_0) + y_0 \tag{4.7}$$

On using the relations (3.3) and (3.7), the solution of (4.7) can be approximated. By this process, the boundary value problem for fractional differential equation of Caputo type reduces to a system of algebraic equations. Then solve this system to obtain the numerical solution of boundary value problem.

5. Numerical Examples

In order to show the efficiency of the proposed method, for solving boundary value problems for fractional order differential equations, we apply it to solve different types of linear fractional differential equations where exact solutions are known.

Example 5.1. Consider the boundary value problem for inhomogeneous linear fractional differential equation

$$\mathfrak{D}_{RL}^{\alpha} y(x) + a y(x) = g(x) \tag{5.1}$$

with $y(0) = 0, y(1) = y_0$ where $1 < \alpha \leq 2, a \in \mathbb{R}, x \in [0,1]$. For $g(x) = x + \frac{a x^{\alpha+1}}{\Gamma(\alpha+2)}$ and $y_0 = \frac{1}{\Gamma(\alpha+2)}$, the exact solution of boundary value problem is $y(x) = \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}$. The integral representation of (5.1) is given by

$$y(x) = -a I^{\alpha} y(x) + a x^{\alpha-1} I^{\alpha} y(1) + f(x) \tag{5.2}$$

where

$$f(x) = I^{\alpha} g(x) - x^{\alpha-1} I^{\alpha} g(1) + \frac{x^{\alpha-1}}{\Gamma(\alpha+2)}$$

We approximate the solution $y(x)$ as

$$y(x) = A_{m^2\times 1}^{T} \Psi_{m^2\times 1}(x) \tag{5.3}$$

then

$$I^{\alpha}y(x) = A^T_{m^2\times 1} I^{\alpha}\Psi_{m^2\times 1}(x)$$

$$= A^T_{m^2\times 1} S^{\alpha}_{m^2\times m^2}\Psi_{m^2\times 1}(x) \tag{5.4}$$

On using (5.3) and (5.4) in (5.2), we have

$$A^T_{m^2\times 1}\Psi_{m^2\times 1}(x) = -aA^T_{m^2\times 1}S^{\alpha}_{m^2\times m^2}\Psi_{m^2\times 1}(x) + ax^{\alpha-1}A^T_{m^2\times 1}S^{\alpha}_{m^2\times m^2}\Psi_{m^2\times 1}(1)$$

$$+ f(x) \tag{5.5}$$

We solve (5.5) for Shannon coefficient vector, for $a = \frac{3}{57}, m = 5$ and $\alpha = 1.2, 1.4, 1.6, 1.8, 2$. The numerical solutions for different values of α are shown in figure 1. The absolute error is given in the table 1.

Table 1. Absolute error for $m = 5$ and $\alpha = 1.2, 1.4, 1.6, 1.8, 2$.

x	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$	$\alpha = 2$
0.1	1.07803×10^{-6}	2.94185×10^{-6}	1.09847×10^{-7}	3.27468×10^{-7}	4.74058×10^{-8}
0.2	1.49112×10^{-6}	5.47506×10^{-6}	2.66526×10^{-7}	2.76180×10^{-7}	5.92076×10^{-8}
0.3	4.62302×10^{-6}	2.70694×10^{-6}	4.92701×10^{-7}	4.65791×10^{-7}	6.41560×10^{-8}
0.4	2.16000×10^{-6}	9.57346×10^{-6}	1.94025×10^{-7}	5.14032×10^{-7}	8.98904×10^{-8}
0.5	9.52422×10^{-5}	7.19076×10^{-6}	7.11286×10^{-7}	1.19507×10^{-7}	9.46057×10^{-8}
0.6	6.98702×10^{-5}	2.95114×10^{-5}	5.49705×10^{-6}	7.10406×10^{-7}	1.24581×10^{-7}
0.7	5.73057×10^{-5}	4.80884×10^{-5}	6.70458×10^{-6}	6.42601×10^{-6}	3.17245×10^{-7}
0.8	6.31371×10^{-5}	3.78079×10^{-6}	9.87045×10^{-7}	8.98204×10^{-7}	5.42806×10^{-7}
0.9	3.29047×10^{-6}	1.25431×10^{-6}	3.32547×10^{-7}	1.58431×10^{-7}	4.84625×10^{-8}

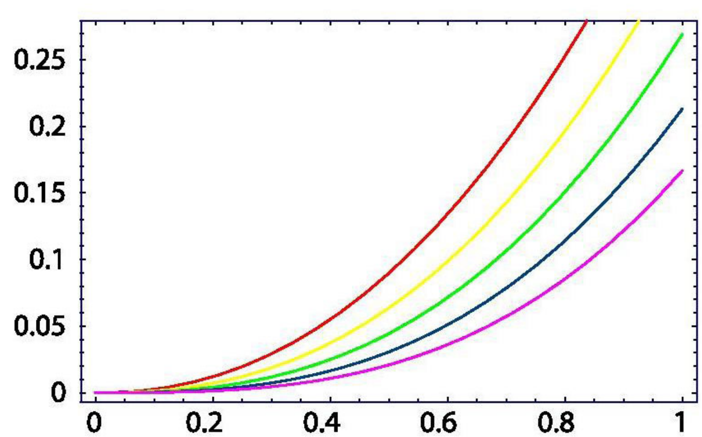

Figure 1. The numerical solutions for different values of α($\alpha = 1.2$ —, $\alpha = 1.4$ —, $\alpha = 1.6$ —, $\alpha = 1.8$ —, $\alpha = 2$ —)

Example 5.2. Consider the boundary value problem

$$\mathfrak{D}_C^{\alpha} y(x) = \mathfrak{D}_C^{\beta} y(x) - e^{x-1} - 1,\ 1 < \alpha \leq 2,\ 0 < \beta \leq 1$$

with

$$y(0) = 0, y(1) = 0$$

In general, the exact solution of the problem is not known. However, for $\alpha = 2, \beta = 1$, the problem has exact solution $y(x) = x(1 - e^{x-1})$. For integer order case the above problem of example 5.2, is solved numerically in [13] using combined homotopy perturbation method and Greens function method. We solve this problem by the proposed method. The numerical results are presented in Table 2. Computer plots for $\beta = 1$ and different values of α given in Fig. 2 show that as α approaches to 2, the corresponding solutions of fractional order differential equation approach to the solutions of integer order differential equation. Results in Table 2 show that the Shannon wavelet method agrees with the results obtained in [13] by using combination of Greens function and Homotopy perturbation method.

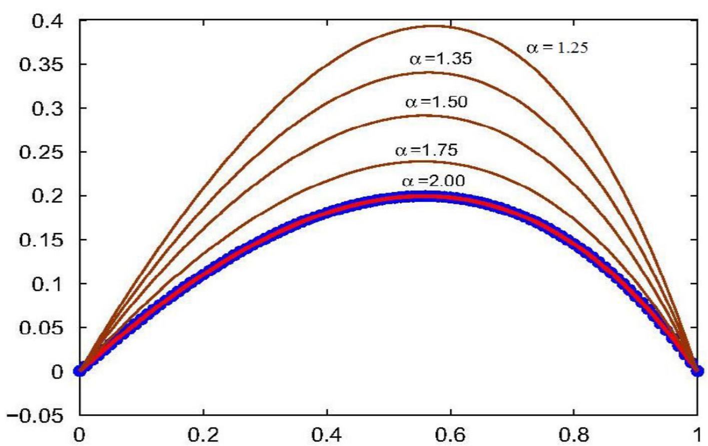

Figure 2. Shannon Wavelet solutions for $\beta = 1$ and different values of α

Table 2. Comparison of the Shannon wavelet method with HPM [13] for $m=5, \alpha=2, \beta=1$

x	Fourth order HPM [13]	Shannon wavelet method	Exact
0.1	0.05934820	0.05934312	0.05934303
0.2	0.11014318	0.11013418	0.11013421
0.3	0.15103441	0.15102443	0.15102441
0.4	0.18048329	0.18047538	0.18047535
0.5	0.19673826	0.19673462	0.19673467
0.6	0.19780653	0.19780793	0.19780797
0.7	0.18142196	0.18142729	0.18142725
0.8	0.14500893	0.14501545	0.14501540
0.9	0.08564186	0.08564626	0.08564632

Example 5.3. Consider the fractionally damped mechanical oscillator equation with boundary conditions

$$\mathfrak{D}_{RL}^{\alpha} y(x) + \lambda \mathfrak{D}_{RL}^{\beta} y(x) + v y(x) = g(x),\ x \in [0,1]$$

with

$$y(0)=0, y(1)=0$$

where $1<\alpha\leq 2, 0<\beta\leq 1, \alpha-\beta>1, \lambda, v$ are prescribed constants and $g(x)$ is the forcing function. If $\alpha=2, \beta=1$ then above problem of example 5.3, reduces to the usual differential equation of harmonic oscillator. In [11] Attila Palfalvi have applied the Adomian decomposition method on a fractionally damped mechanical oscillator for a sine excitation. We solve this equation with two - point boundary conditions using the proposed method of Shannon wavelets. For $\alpha=\frac{7}{4}, \beta=\frac{1}{2}$ and $\lambda=1, v=\frac{-1}{\sqrt{\pi}}$.

Here

$$g(x)=\frac{1}{\sqrt{\pi}}\left(\frac{16x^{3/2}p(x)}{45045}+\frac{24x^{1/4}q(x)}{9945\Gamma(5/4)}-x^2(5x-3)^2 r(x)\right)$$

where $p(x), q(x)$ and $r(x)$ are polynomials given by,

$$p(x)=28028000x^3-14620320x^2-21527571x-270270,$$

$$q(x)=6400000x^5-15360000x^4+13328000x^3-5021120x^2+757809x-29835$$

$$r(x)=25x^3-50x^2+29x-4$$

One can easily verify that the exact solution is $y(x) = 625x^7 - 2000x^6 + 2450x^5 - 1420x^4 + 381x^3 - 36x^2$. The numerical and exact solutions for $m = 6$ are shown in Fig. 3. The absolute error for different values of m is shown in the Table 3.

Table 3. For Example 5.3, absolute error for different values of m.

x	$m = 3$	$m = 4$	$m = 5$	$m = 6$
0.1	1.49560×10^{-3}	2.24307×10^{-6}	4.25106×10^{-10}	1.47215×10^{-14}
0.2	4.65813×10^{-3}	1.19045×10^{-6}	3.35971×10^{-10}	7.40692×10^{-14}
0.3	3.48062×10^{-3}	7.57403×10^{-6}	7.14602×10^{-9}	2.43158×10^{-13}
0.4	2.18467×10^{-4}	3.45815×10^{-7}	1.61034×10^{-9}	5.47025×10^{-14}
0.5	5.65049×10^{-4}	4.41687×10^{-7}	3.33245×10^{-10}	6.33470×10^{-14}
0.6	1.15406×10^{-4}	3.29706×10^{-6}	8.72045×10^{-9}	7.52014×10^{-14}
0.7	6.71240×10^{-3}	6.66740×10^{-7}	9.27406×10^{-9}	3.64280×10^{-13}
0.8	8.22161×10^{-4}	9.24812×10^{-6}	2.24250×10^{-9}	4.31154×10^{-13}
0.9	7.46590×10^{-3}	5.24267×10^{-6}	5.56207×10^{-10}	2.87084×10^{-15}

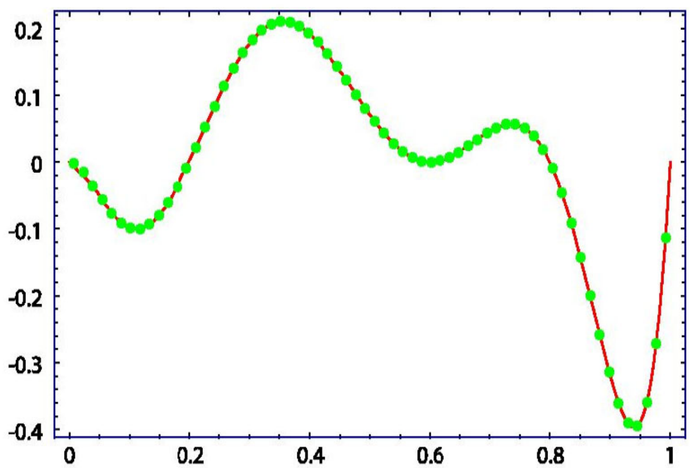

Figure 3. The numerical and exact solutions for $m = 6$ (Exact — Shannon ...)

Example 5.4. Consider the Bagley - Torvik equation [12] with boundary conditions,

$$ay''(x) + b\mathfrak{D}_{RL}^{\alpha}y(x) + cy(x) = g(x),\ x \in [0,1]$$

with

$$y(0) = 0, y(1) = y_0$$

where $a, b, c \in R$ and $a \neq 0$. The differential equation in this example is a prototype fractional differential equation with two derivatives. Bagley - Torvik equation involving fractional derivative of order $\frac{1}{2}$ or $\frac{3}{2}$ arises in the modeling of the motion of a rigid plate in a Newtonian fluid and a gas in a fluid. We solve this equation with two - point boundary conditions by using the proposed method of Shannon wavelets.

Choosing $\alpha = \frac{3}{2}, a = 1, b = \frac{8}{17}, c = \frac{13}{51}$ and $y_0 = 0, g(x) = \frac{x^{(-1/2)}}{89250\sqrt{\pi}}\left(48p(x) + 7\sqrt{x}q(x)\right)$ where

$$p(x) = 16000x^4 - 32480x^3 + 21280x^2 - 4746x + 189$$

$$q(x) = 3250x^5 - 9425x^4 + 264880x^3 - 448107x^2 + 233262x - 34578.$$

It can be easily verified that the exact solution is

$$y(x) = x^5 - \frac{29x^4}{10} + \frac{76x^3}{25} - \frac{339x^2}{250} + \frac{27x}{125}$$

The numerical and exact solutions for different values of m are shown in Fig. 4. The absolute error for different values of m is shown in the Table 4.

Table 4. For Example 5.4, absolute error for different values of m.

x	$m = 3$	$m = 4$	$m = 5$	$m = 6$
0.1	2.24508×10^{-4}	1.72084×10^{-6}	5.16405×10^{-9}	3.47310×10^{-14}
0.2	1.49176×10^{-3}	3.59217×10^{-5}	1.65721×10^{-8}	1.58046×10^{-13}
0.3	1.27680×10^{-3}	8.37916×10^{-5}	2.82671×10^{-8}	3.15058×10^{-13}
0.4	2.54952×10^{-3}	7.15162×10^{-5}	6.34085×10^{-8}	7.42075×10^{-13}
0.5	1.14207×10^{-3}	2.61807×10^{-6}	9.63205×10^{-9}	8.19172×10^{-13}
0.6	4.92904×10^{-4}	7.92786×10^{-6}	7.54861×10^{-9}	9.84175×10^{-14}
0.7	1.94357×10^{-4}	1.56738×10^{-6}	3.48197×10^{-9}	4.26810×10^{-14}
0.8	1.12159×10^{-4}	5.62047×10^{-7}	8.50547×10^{-10}	1.78404×10^{-14}
0.9	3.49180×10^{-4}	4.68914×10^{-7}	5.76482×10^{-10}	5.34283×10^{-15}

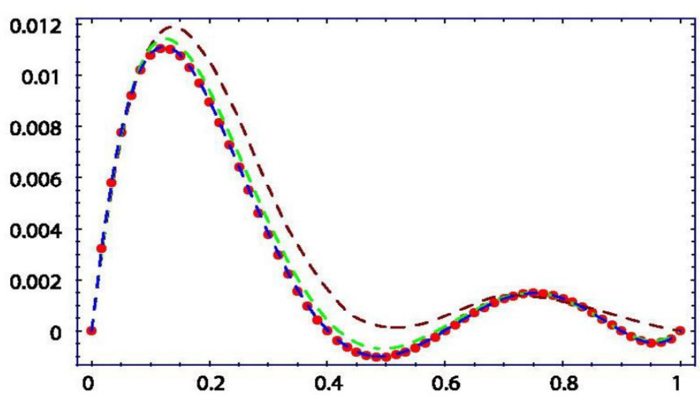

Figure 4. The numerical and exact solutions for different values of m and

$\alpha = \frac{3}{2}(m = 3 - - -\ m = 4 - - -\ m = 5 - - -\ exact = \ldots)$

6. Conclusion

In this paper, a numerical scheme based on operational matrices of integration for Shannon wavelets, is proposed to obtain approximate solutions of two - point boundary value problems for linear fractional differential equations with constant coefficients. The problem has been reduced to system of algebraic equations. The results obtained are compared with exact solutions and also with the solutions obtained by some other numerical methods in literature. It is worth mentioning that results obtained agree well with exact solutions and therefore the proposed method is very convenient, effective and reliable for obtaining approximate solutions of fractional boundary value problems.

Conflicts of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] O. Abdulaziz, I. Hashim, S. Momani, Solving systems of fractional differential equations by homotopy-perturbation method, Phys. Lett. A, 372 (2008), no. 4, 451-459.

[2] M. Ciesielski, J. Leszczynski, Numerical simulations of anomalous diffusion. In: Proceedings of the 15th Conference on Computer Methods in Mechanics, Wisla, Poland, 3-6 June, 2003.

[3] V. Daftardar-Gejji, H. Jafari, Solving a multi-order fractional differential equation using adomian decomposition, Appl. Math. Comput. 189 (2007), no. 1, 541-548.

[4] K. Diethelm, G. Walz, Numerical solution of fractional order differential equations by extrapolation, Numer. Algor. 16 (1997), no. 3 - 4, 231-253.

[5] K. Diethelm, N.J. Ford, A.D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, Nonlinear Dynamics, 29 (2002), 3-22.

[6] I. Hashim, O. Abdulaziz, S. Momani, Homotopy analysis method for fractional IVPs, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009), no. 3, 674-684.

[7] A. Kilicman, Z.A.A. Al Zhour, Kronecker operational matrices for fractional calculus and some applications, Appl. Math. Comput., 187 (2007), no. 1, 250-265.

[8] C. Lubich, Fractional linear multistep methods for Abel-Volterra integral equations of the second kind, Math. Comp., 45 (1985), 463-469.

[9] R. Metzler, J. Klafter, The random walks guide to anomalous diffusion: a fractional dynamics approach, Physics reports, 339 (2000), no. 1, 1-77.

[10] Z. Odibat, S. Momani, V. Suat Erturk, Generalized differential transform method: application to differential equations of fractional order, Appl. Math.Comput., 197 (2008), no. 2, 467-477.

[11] A. Pálfalvi, Efficient solution of a vibration equation involving fractional derivatives, Int. J. Non-Linear Mech., 45 (2010), no. 2, 169-175.

[12] P. J. Torvik, R. L. Bagley, On the appearance of the fractional derivative in the behavior of real materials, ASME. J. Appl. Mech., 51(1984), no. 2, 294-298.

[13] Y. Wang, H. Song, D. Li, Solving two-point boundary value problems using combined homotopy perturbation method and Greens function method, Appl. Math. Comput., 212 (2009), no. 2, 366-376.

[14] G. Wu, E.W.M. Lee, Fractional variational iteration method and its application, Phys. Lett. A., 374 (2010), no. 25, 2506-2509.

[15] Y. Zhang, A finite difference method for fractional partial differential equation, Appl. Math. Comput., 215 (2009), no. 2, 524-529.