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APPROXIMATE FIXED POINT IN $G-$METRIC SPACES FOR VARIOUS TYPES OF OPERATORS

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Abstract. In this paper, we introduce the approximate fixed point property for a cyclic map T on a G-metric space. Also, we prove two general lemmas which regarding approximate fixed Point of cyclic maps on a G-metric space. Using these results, we prove several approximate fixed point theorems for various types of the well-known generalized contractions on a G-metric space.

Keywords: Fixed points; Approximate fixed points; $G-$metric; Approximation fixed point property.

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1. Introduction and preliminaries

In 2011, Mohsenalhosseini et al [11], introduced the approximate best proximity pairs and proved the approximate best proximity pairs property for it. Also, In 2012 , Mohsenalhosseini et al [12], introduced the approximate fixed point for completely norm space and map T_α and proved the approximate fixed point property for it. In 2014 , Mohsenalhosseini [13] introduced

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the Approximate best proximity pairs in metric space for contraction maps. Now we give preliminaries and basic definitions which are used throughout the paper. Also, we study some types of the well-known operators on $G-$metric spaces, and we give some qualitative and quantitative results regarding approximate fixed point by considering a cyclic map $T : A\cup B\cup C \to A\cup B\cup C$ i.e. $T(A) \subseteq B$, $T(B) \subseteq C$ and $T(C) \subseteq A$.

We begin by recalling some needed definitions and results.

Definition 1.1. [14] Let X be a nonempty set and let $G : X \times X \times X \longrightarrow R^+$ be a function satisfying the following properties:

(G1) $G(x,y,z) = 0$ *if and only if* $x = y = z$;

(G2) $0 < G(x,x,y)$ *for all* $x,y \in X$ *with* $x \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z)$ *for all* $x,y,z \in X$ *with* $z \neq y$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (*symmetry in all three variables*);

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ *for all* $x,y,z,a \in X$(*rectangle inequality*).

Then, the function G is called generalized metric or, more specifically $G-metric$ on X, and the pair (X,G) is called a $G-metric$ space.

Proposition 1.2. [14] *Every G-metric* (X,G) *defines a metric space* (X,d_G) *by*

1) $d_G(x,y) = G(x,y,y) + G(y,x,x)$.

if (X,G) *is a symmetric G- metric space. Then*

2) $d_G(x,y) = 2G(x,y,y)$.

Definition 1.3. [12] *Let* $T : X \to X$, $\varepsilon > 0$, $x_0 \in X$. *Then* $x_0 \in X$ *is an* $\varepsilon-$*fixed point for* T *if* $\|Tx_0 - x_0\| < \varepsilon$.

Remark 1.4. [12] *In this paper we will denote the set of all* $\varepsilon-$ *fixed points of* T, *for a given* ε, *by :*

$$AF(T) = \{x \in X \mid x \text{ is an } \varepsilon - \text{fixed point of } T\}.$$

Definition 1.5. [12] Let $T : X \to X$. Then T has the approximate fixed point property (a.f.p.p) if

$$\forall\varepsilon > 0,\ AF(T) \neq \varnothing.$$

Theorem 1.6. [12] *Let $(X, \|.\|)$ be a completely norm space, $T : X \to X$, $x_0 \in X$ and $\varepsilon > 0$. If $\|T^n(x_0) - T^{n+k}(x_0)\| \to 0$ as $n \to \infty$ for some $k > 0$, then T^k has an $\varepsilon-$ fixed point.*

2. Approximate fixed point in $G-$metric

We begin with two lemmas which will be used in order to prove all the results given in section 3. Let (X,G) be a $G-metric$ space.

Definition 2.1. Let A,B,C are closed subsets of a $G-metric$ space X and $T : A\cup B\cup C \to A\cup B\cup C$ be a cyclic map. Let $\varepsilon > 0$ and $x_0 \in A\cup B\cup C$. Then x_0 is an ε- fixed point of T if

$$[G(x_0,Tx_0,Tx_0)+G(Tx_0,x_0,x_0)] < \varepsilon.$$

Remark 2.2. In this paper we will denote the set of all ε-fixed points of T, for a given ε, by:

$$F_G^{\varepsilon}(T) = \{x \in A\cup B\cup C \mid x \text{ is an } \varepsilon - \text{fixed point of } T\}.$$

Definition 2.3. Let A,B,C are closed subsets of a $G-metric$ space X and $T : A\cup B\cup C \to A\cup B\cup C$ be a cyclic map. Then T has the approximate fixed point property (a.f.p.p) if $\forall\varepsilon > 0$,

$$F_G^{\varepsilon}(T) \neq \emptyset.$$

In 2015 Abbas et al (see[3]), introduced the example of G-metric spaces which it has g-best proximity point. We in the following show that it has approximate fixed point on G-metric spaces.

Example 2.4. Let $X = \{0,1,2,...,18\}$ and $G : X\times X\times X \to \mathbb{R}^+$ be defind as follows:

$$G(x,y,z) = \begin{cases} x+y+z & if\ x\neq y\neq z\neq 0, \\ x+y & if\ x=y\neq z\ x,y,z\neq 0, \\ y+z+1 & if\ x=0,\ y\neq z,\ y,z\neq 0, \\ y+2 & if\ x=0,\ y=z\neq 0, \\ z+1 & if\ x=0,\ y=0,\ z\neq 0, \\ 0 & if\ x=y=z. \end{cases}$$

Let $A=\{2,18\}$, $B=\{1,7,17\}$ and $C=\{0\}$. Obviously A,B,C are closed subsets of $G-metric$ space X. Define the mapping $T: A\cup B\cup C\to A\cup B\cup C$ by

$$Tx=\begin{cases} x-1 & if\ x\in\{1,18\}\\ 0 & if\ x\in\{1,7,17\}\\ 2 & if\ x=0.\end{cases}$$

Then $G(0,T0,T0)+G(T0,0,0)<\varepsilon$ for evrey $\varepsilon>0$. Hence $F_G^{\varepsilon}(T)\neq\emptyset$.

Definition 2.5. Let A,B,C are closed subsets of a $G-metric$ space X. A cyclic map $T: A\cup B\cup C\to A\cup B\cup C$ is said to be asymptotically regular at a point $x\in A\cup B\cup C$, if

$$\lim_{n\to\infty}\{G(T^nx,T^{n+1}x,T^{n+1}x)+G(T^{n+1}x,T^nx,T^nx)\}=0,$$

where T^n denotes the nth iterate of T at x.

Lemma 2.6. *Let A,B,C are closed subsets of a $G-metric$ space X. If $T: A\cup B\cup C\to A\cup B\cup C$ is asymptotically regular at a point $x\in A\cup B\cup C$, Then T has an approximate fixed point.*

Proof. Using Proposition 1.2 and Theorem 1.6, we find that T has an approximate fixed point. □

Lemma 2.7. *Let A,B,C are closed subsets of a $G-metric$ space X, $T: A\cup B\cup C\to A\cup B\cup C$ a cyclic map and $\varepsilon>0$. We assume that:*

a) $F_G^{\varepsilon}(T)\neq\emptyset$;

b) $\forall\xi>0\ \exists\psi(\xi)>0$ *such that*

$$[G(x,y,y)+G(y,x,x)]-[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)]<\xi\implies$$

$$G(x,y,y)+G(y,x,x)\leq\psi(\xi),\quad\forall x,y\in F_G^{\varepsilon}(T).$$

Then:

$$\delta(F_G^{\varepsilon}(T))\leq\psi(2\varepsilon).$$

Proof. Let $\varepsilon>0$ and $x,y\in F_G^{\varepsilon}(T)$. Then

$$[G(x,Tx,Tx)+G(Tx,x,x)]<\varepsilon,\ [G(y,Ty,Ty)+G(Ty,y,y)]<\varepsilon.$$

By $G5$ of Definition 1.1 we can write:

$$\begin{aligned} G(x,y,y)+G(y,x,x) &\leq G(x,Tx,Tx)+G(Tx,x,x) \\ &+G(Tx,Ty,Ty)+G(Ty,Tx,Tx) \\ &+G(Ty,y,y)+G(y,Ty,Ty) \\ &\leq 2\varepsilon+G(Tx,Ty,Ty)+G(Ty,Tx,Tx). \Longrightarrow \end{aligned}$$

$$G(x,y,y)+G(y,x,x)-[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] < 2\varepsilon.$$

Now by (b) it follow that

$$G(x,y,y)+G(y,x,x) \leq \psi(2\varepsilon),$$

So

$$\delta(F_G^{\varepsilon}(T)) \leq \psi(2\varepsilon).$$

□

Remark 2.8. Condition (a) in Lemma 2.7 can be replaced by the condition, as, by Lemma 2.6, the latter ensures (a). So Lemma 2.7 can be given in the form:

Lemma 2.9. *Let (X,G) be a G-metric and $T : A\cup B\cup C \to A\cup B\cup C$ a cyclic map such that for $\varepsilon > 0$ the following hold:*
a) asymptotically regular at a point $x \in A\cup B\cup C$.
b) $\forall\xi > 0\ \exists\psi(\xi) > 0$ such that

$$[G(x,y,y)+G(y,x,x)]-[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] < \xi \Rightarrow$$

$$G(x,y,y)+G(y,x,x) \leq \psi(\xi),\ \forall x,y \in F_G^{\varepsilon}(T).$$

Then:

$$\delta(F_G^{\varepsilon}(T)) \leq \psi(2\varepsilon).$$

Proof. Now by Lemma 2.6 and Lemma 2.7 it find that $\delta(F_G^{\varepsilon}(T)) \leq \psi(2\varepsilon)$. □

3. Approximate fixed point in $G-$metric spaces for various types of operators

In this section be will formulate and prove, using Lemma 2.6, qualitative results for various types of operators on a $G-$metric space, results that establish the considered under which the mappings considered have the approximate fixed point property.

In 2001, Rus (see [10]) defined $\alpha-$contraction and in 2006 Berinde (see [5]) some result on $\alpha-$ contraction for approximate fixed point in metric space. We it apply on G-metric spaces for approximate fixed point.

Definition 3.1. [10] A mapping $T : X \to X$ is a $\alpha-$contraction if there exists $\alpha \in (0,1)$ such that

$$d(Tx,Ty) \leq \alpha d(x,y), \forall x,y \in X.$$

Definition 3.2. Let A,B and C be non-empty subsets of a $G-$metric space X. A mapping $T : A\cup B\cup C \to A\cup B\cup C$ is a $G_\alpha-$contraction if there exists $\alpha \in (0,1)$

$$G(Tx,Ty,Ty)+G(Ty,Tx,Tx) \leq \alpha[G(x,y,y)+G(x,y,y)], \forall x,y \in A\cup B\cup C.$$

Theorem 3.3. *Let A,B and C be non-empty subsets of a $G-$metric space X. Suppose that the mapping $T : A\cup B\cup C \to A\cup B\cup C$ satisfying $T(A) \subset B, T(B) \subset C$ and $T(C) \subset A$ is a $G_\alpha-$contraction. Then for every $\varepsilon > 0$, $F_G^\varepsilon(T) \neq \emptyset$.*

Proof. Let $\varepsilon > 0$, $x \in A\cup B\cup C$.

$$\begin{aligned}
G(T^nx,T^{n+k}x,T^{n+k}x)+G(T^{n+k}x,T^nx,T^nx) &= G(T(T^{n-1}x),T(T^{n+k-1}x),T(T^{n+k-1}x)) \\
&+G(T(T^{n+k-1}x),T(T^{n-1}x),T(T^{n-1}x)) \\
&\leq \alpha[G(T^{n-1}x,T^{n+k-1}x,T^{n+k-1}x) \\
&+G(T^{n+k-1}x,T^{n-1}x,T^{n-1}x)] \\
&\vdots \\
&\leq \alpha^n[G(x,T^kx,T^kx)+G(T^kx,x,x)].
\end{aligned}$$

But $\alpha \in (0,1)$, therefore

$$\lim_{n \longrightarrow \infty} [G(T^n x, T^{n+k} x, T^{n+k} x) + G(T^{n+k} x, T^n x, T^n x)] = 0, \ \forall x \in A \cup B \cup C.$$

By Lemma 2.6, we find that $F_G^{\varepsilon}(T) \neq \emptyset$ for all $\varepsilon > 0$. □

In 1968, Kannan (see [4] [8]) proved a fixed point theorem for operators which need not be continuous. We it apply on G-metric space for approximate fixed point.

Definition 3.4. Let A, B and C be non-empty subsets of a $G-$metric space X. A mapping $T : A \cup B \cup C \to A \cup B \cup C$ satisfying $T(A) \subset B, T(B) \subset C$, and $T(C) \subset A$ is a G-Kannan operator if there exists $\alpha \in (0, \frac{1}{2})$ such that

$$\begin{aligned} G(Tx, Ty, Ty) + G(Ty, Tx, Tx) \leq & \alpha[G(x, Tx, Tx) + G(Tx, x, x)] \\ & + G(y, Ty, Ty) + G(Ty, y, y) \ \forall x, y \in A \cup B \cup C. \end{aligned}$$

Theorem 3.5. *Let A, B and C be non-empty subsets of a G-metric space X. Suppose that the mapping $T : A \cup B \cup C \to A \cup B \cup C$ satisfying $T(A) \subset B, T(B) \subset C$ and $T(C) \subset A$ is a G- Kannan operator. Then for every $\varepsilon > 0$, $F_G^{\varepsilon}(T) \neq \emptyset$.*

Proof. Let $\varepsilon > 0$ and $x \in A \cup B \cup C$.

$$\begin{aligned} G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x) = & \\ & G(T(T^{n-1} x), T(T^n x), T(T^n x)) \\ & + G(T(T^n x), T(T^{n-1} x), T(T^{n-1} x)) \\ & \leq \alpha[G(T^{n-1} x, T^n x, T^n x) + G(T^n x, T^{n-1} x, T^{n-1} x) \\ & + G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)]. \end{aligned}$$

Therefore,

$$\begin{aligned} (1-\alpha) G(T^n x, T^{n+1} x, T^{n+1} x) + G(T^{n+1} x, T^n x, T^n x)] \leq & \\ & \alpha[G(T^{n-1} x, T^n x, T^n x) + G(T^n x, T^{n-1} x, T^{n-1} x)]. \end{aligned}$$

So,

$$G(T^nx,T^{n+1}x,T^{n+1}x)+G(T^{n+1}x,T^nx,T^nx)] \leq$$

$$\frac{\alpha}{1-\alpha}[G(T^{n-1}x,T^nx,T^nx)+G(T^nx,T^{n-1}x,T^{n-1}x)]$$

$$\vdots$$

$$\leq (\frac{\alpha}{1-\alpha})^n[G(x,Tx,Tx)+G(Tx,x,x)].$$

But $\alpha \in (0,1)$, therefore

$$\lim_{n\longrightarrow\infty}[G(T^nx,T^{n+1}x,T^{n+1}x)+G(T^{n+1}x,T^nx,T^nx)] = 0, \forall x \in A\cup B\cup C.$$

Using Lemma 2.6 We find that $F_G^{\varepsilon}(T) \neq \emptyset$ for all $\varepsilon > 0$. □

Definition 3.6. Let (X,d) be a metric space. A mapping $T : X \to X$ is a **Mohseni operator** if there exists $\alpha \in (0,\frac{1}{2})$ such that

$$d(Tx,Ty) \leq \alpha[d(x,y)+d(Tx,Ty)].$$

Definition 3.7. Let A,B and C be non-empty subsets of a $G-$metric space X. A mapping $T : A\cup B\cup C \to A\cup B\cup C$ satisfying $T(A) \subset B, T(B) \subset C$, and $T(C) \subset A$ is a $G-$ **Mohseni operatore** if there exists $\alpha \in (0,\frac{1}{2})$ such that

$$G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \leq \alpha[G(x,y,y)+G(y,x,x)$$
$$+G(Tx,Ty,Ty)+G(Ty,Tx,Tx)].$$

Theorem 3.8. *Let A,B and C be non-empty subsets of a G-metric space X. Suppose that the mapping $T : A\cup B\cup C \to A\cup B\cup C$ satisfying $T(A) \subset B, T(B) \subset C$ and $T(C) \subset A$ is a $G-$ Mohseni operatore. Then for every $\varepsilon > 0$,*

$$F_G^{\varepsilon}(T) \neq \emptyset.$$

Proof. Let $\varepsilon > 0$ and $x \in A\cup B\cup C$.

$$\begin{aligned}
G(T^n x, T^{n+k}x, T^{n+k}x) + G(T^{n+k}x, T^n x, T^n x) &= G(T(T^{n-1}x), T(T^{n+k-1}x), T(T^{n+k-1}x)) \\
&+ G(T(T^{n+k-1}x), T(T^{n-1}x), T(T^{n-1}x)) \\
&\leq \alpha[G(T^{n-1}x, T^{n+k-1}x, T^{n+k-1}x) \\
&+ G(T^{n+k-1}x, T^{n-1}x, T^{n-1}x) \\
&+ G(T^n x, T^{n+k}x, T^{n+k}x) + G(T^{n+k}x, T^n x, T^n x)].
\end{aligned}$$

Therefore,

$$\begin{aligned}
(1-\alpha)G(T^n x, T^{n+k}x, T^{n+k}x) + G(T^{n+k}x, T^n x, T^n x)] \leq \\
\alpha[G(T^{n-1}x, T^{n+k-1}x, T^{n+k-1}x) \\
+ G(T^{n+k-1}x, T^{n-1}x, T^{n-1}x)].
\end{aligned}$$

So,

$$\begin{aligned}
G(T^n x, T^{n+k}x, T^{n+k}x) + G(T^{n+k}x, T^n x, T^n x)] \leq \\
&\frac{\alpha}{1-\alpha}[G(T^{n-1}x, T^{n+k-1}x, T^{n+k-1}x) \\
&+ G(T^{n+k-1}x, T^{n-1}x, T^{n-1}x)] \\
&\vdots \\
&\leq (\frac{\alpha}{1-\alpha})^n[G(x, T^k x, T^k x) + G(T^k x, x, x)].
\end{aligned}$$

But $\alpha \in (0, \frac{1}{2})$, therefore $(\frac{\alpha}{1-\alpha}) \in (0,1)$. Hence

$$\lim_{n\longrightarrow\infty}[G(T^n x, T^{n+k}x, T^{n+k}x) + G(T^{n+k}x, T^n x, T^n x)] = 0, \forall x \in A\cup B\cup C.$$

Using proposition 1.2 and Theorem 1.6 , we find that $F_G^{\varepsilon}(T) \neq \emptyset$ for all $\varepsilon > 0$. □

In 1972, Chatterjea (see [6]) considered another which again dose not impose the continuity of the operator. We it apply on G-metric space for approximate fixed point.

Definition 3.9. Let A,B and C be non-empty subsets of a $G-$metric space X. A mapping $T : A\cup B\cup C \to A\cup B\cup C$ satisfying $T(A)\subset B, T(B)\subset C$ and $T(C)\subset A$ is a G- Chatterjea operator if there exists $\alpha\in(0,\frac{1}{2})$ such that

$$G(Tx,Ty,Ty)+G(Ty,Tx,Tx)]\leq \alpha[G(x,Ty,Ty)+G(Ty,x,x) + G(y,Tx,Tx)+G(Tx,y,y)], \forall x,y\in A\cup B\cup C.$$

Theorem 3.10. *Let A,B and C be non-empty subsets of a G-metric space X. Suppose that the mapping $T : A\cup B\cup C\to A\cup B\cup C$ satisfying $T(A)\subset B, T(B)\subset C$ and $T(C)\subset A$ is a G-Chatterjea operator. Then for every $\varepsilon>0$, $F_G^{\varepsilon}(T)\neq\emptyset$.*

Proof. Let $\varepsilon>0$ and $x\in A\cup B\cup C$.

$$\begin{aligned}G(T^nx,T^{n+1}x,T^{n+1}x)+G(T^{n+1}x,T^nx,T^nx) = \\ &G(T(T^{n-1}x),T(T^nx),T(T^nx)) \\ &+G(T(T^nx),T(T^{n-1}x),T(T^{n-1}x)) \\ &\leq \alpha[G(T^{n-1}x,T^{n+1}x,T^{n+1}x)+G(T^{n+1}x,T^{n-1}x,T^{n-1}x) \\ &+G(T^nx,T^nx,T^nx)+G(T^nx,T^nx,T^nx)] \\ &=\alpha[G(T^{n-1}x,T^{n+1}x,T^{n+1}x)+G(T^{n+1}x,T^{n-1}x,T^{n-1}x)].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}G(T^{n-1}x,T^{n+1}x,T^{n+1}x)+G(T^{n+1}x,T^{n-1}x,T^{n-1}x)\leq \\ &G(T^nx,T^{n-1}x,T^{n-1}x)+G(T^{n-1}x,T^nx,T^nx) \\ &+G(T^nx,T^{n+1}x,T^{n+1}x)+G(T^{n+1}x,T^nx,T^nx).\end{aligned}$$

Therefore,

$$\begin{aligned}(1-\alpha)G(T^nx,T^{n+1}x,T^{n+1}x)+G(T^{n+1}x,T^nx,T^nx)]\leq \\ &\alpha[G(T^nx,T^{n-1}x,T^{n-1}x)+G(T^{n-1}x,T^nx,T^nx)].\end{aligned}$$

So,

$$G(T^n x, T^{n+1}x, T^{n+1}x) + G(T^{n+1}x, T^n x, T^n x)] \leq$$
$$\frac{\alpha}{1-\alpha}[G(T^n x, T^{n-1}x, T^{n-1}x) + G(T^{n-1}x, T^n x, T^n x)]$$
$$\vdots$$
$$\leq (\frac{\alpha}{1-\alpha})^n [G(x, Tx, Tx) + G(Tx, x, x)].$$

Since $\alpha \in (0, \frac{1}{2})$, we find $(\frac{\alpha}{1-\alpha}) \in (0,1)$, therefore

$$\lim_{n \longrightarrow \infty} [G(T^n x, T^{n+1}x, T^{n+1}x) + G(T^{n+1}x, T^n x, T^n x)] = 0, \forall x \in A \cup B \cup C.$$

Using Lemma 2.6, we find that $F_G^{\varepsilon}(T) \neq \emptyset$ for all $\varepsilon > 0$. □

We, by combining the three independent contraction conditions above obtain another approximate fixed point result for operators which satisfy the following.

Definition 3.11. Let A, B and C be non-empty subsets of a $G-$metric space X. A mapping $T : A \cup B \cup C \to A \cup B \cup C$ satisfying $T(A) \subset B, T(B) \subset C$ and $T(C) \subset A$ is a $G-$Zamfiresuc operator if there exists $\alpha_1 \in [0,1)$, $\beta \in [0, \frac{1}{2}]$, $\gamma \in [0, \frac{1}{2})$ such that for all $x, y \in A \cup B \cup C$ at least one of the following is true.

$$i) G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] \leq \alpha_1 [G(x, y, y) + G(y, x, x)].$$
$$ii) G(Tx, Ty, Ty) + G(Ty, Tx, Tx) \leq \beta [G(x, Tx, Tx) + G(Tx, x, x)] + G(y, Ty, Ty) + G(Ty, y, y).$$
$$iii) G(Tx, Ty, Ty) + G(Ty, Tx, Tx)] \leq \gamma [G(x, Ty, Ty) + G(Ty, x, x) + G(y, Tx, Tx) + G(Tx, y, y)].$$

Theorem 3.12. *Let A, B and C be non-empty subsets of a G-metric space X. Suppose that the mapping $T : A \cup B \cup C \to A \cup B \cup C$ satisfying $T(A) \subset B, T(B) \subset C$ and $T(C) \subset A$ is a G-Zamfiresuc operator. Then for every $\varepsilon > 0$, $F_G^{\varepsilon}(T) \neq \emptyset$.*

Proof. Let $\varepsilon > 0$ and $x \in A \cup B \cup C$. Supposing (ii) hold, we have that:

$$\begin{aligned}
[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] &\leq \beta[G(x,Tx,Tx)+G(Tx,x,x)]+G(y,Ty,Ty)+G(Ty,y,y) \\
&\leq \beta[G(x,Tx,Tx)+G(Tx,x,x)]+\beta[G(x,Tx,Tx)+G(Tx,x,x) \\
&+ G(x,y,y)+G(y,x,x)+G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \\
&= 2\beta[G(x,Tx,Tx)+G(Tx,x,x)]+\beta[G(x,y,y)+G(y,x,x)] \\
&+ \beta[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)].
\end{aligned}$$

Thus,

$$\begin{aligned}
[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] &\leq \beta[G(x,Tx,Tx)+G(Tx,x,x)] \\
&+ (\frac{\beta}{1-\beta})[G(x,y,y)+G(y,x,x)]. \qquad (3.1)
\end{aligned}$$

Supposing (iii) holds, we have that:

$$\begin{aligned}
[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] &\leq \gamma[G(x,Ty,Ty)+G(Ty,x,x)+G(y,Tx,Tx)+G(Tx,y,y)]. \\
&\leq \gamma[G(x,y,y)+G(y,x,x)+G(y,Ty,Ty)+G(Ty,y,y)] \\
&+\gamma[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \\
&=\gamma[G(x,y,y)+G(y,x,x)]+2\gamma[G(y,Ty,Ty)+G(Ty,y,y)] \\
&+\gamma[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)].
\end{aligned}$$

Thus,

$$\begin{aligned}
[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] &\leq \frac{2\gamma}{1-\gamma}[G(y,Ty,Ty)+G(Ty,y,y)] \\
&+ (\frac{\gamma}{1-\gamma})[G(x,y,y)+G(y,x,x)]. \qquad (3.2)
\end{aligned}$$

Similarly,

$$\begin{aligned}[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] &\leq \gamma[G(x,Ty,Ty)+G(Ty,x,x)+G(y,Tx,Tx)+G(Tx,y,y)].\\ &\leq \gamma[G(x,Tx,Tx)+G(Tx,x,x)+G(Tx,Ty,Ty)+G(Ty,Tx,Tx)]\\ &+\gamma[G(x,y,y)+G(y,x,x)+G(x,Tx,Tx)+G(Tx,x,x)]\\ &=2\gamma[G(x,Tx,Tx)+G(Tx,x,x)]+\gamma[G(Tx,Ty,Ty)\\ &+G(Ty,Tx,Tx)]+\gamma[G(x,y,y)+G(y,x,x)].\end{aligned}$$

Then

$$\begin{aligned}[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] &\leq (\frac{2\gamma}{1-\gamma})[G(x,Tx,Tx)+G(Tx,x,x)]\\ &+ (\frac{\gamma}{1-\gamma})[G(x,y,y)+G(y,x,x)]. \qquad (3.3)\end{aligned}$$

In view of (i), (3.1),(3.1), (3.2) and (3.3), we have, $\xi = max\{\alpha_1, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$, and it is easy to see that $\xi \in [0,1)$ for T satisfying at least one of the condition (i), (ii) and (iii) we have that.

$$[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \leq 2\xi[G(x,Tx,Tx)+G(Tx,x,x)]+\xi[G(x,y,y)+G(y,x,x)] \qquad (3.4)$$

and

$$[G(Tx,Ty,Ty)+G(Ty,Tx,Tx)] \leq 2\xi[G(y,Ty,Ty)+G(Ty,y,y)]+\xi[G(x,y,y)+G(y,x,x)]$$

hold. Using these conditions implied by i) -iii) and taking $x \in A \cup B \cup C$, we have

$$\begin{aligned}
&G(T^n x, T^{n+1}x, T^{n+1}x) + G(T^{n+1}x, T^n x, T^n x)\\
&= G(T(T^{n-1}x), T(T^n x), T(T^n x))\\
&+ G(T(T^n x), T(T^{n-1}x), T(T^{n-1}x))\\
&\leq^{(3.4)} 2\xi [G(T^{n-1}x, T(T^{n-1}x), T(T^{n-1}x))\\
&+ G(T(T^{n-1}x), T^{n-1}x, T^{n-1}x)\\
&+ \xi G(T^{n-1}x, T^n x, T^n x) + G(T^n x, T^{n-1}x, T^{n-1}x)]\\
&= 3\xi [G(T^{n-1}x, T^n x, T^n x) + G(T^n x, T^{n-1}x, T^{n-1}x)]\\
&\vdots\\
&\leq (3\xi)^n [G(x, Tx, Tx) + G(Tx, x, x)].
\end{aligned}$$

Therefore,

$$G(T^n x, T^{n+1}x, T^{n+1}x) + G(T^{n+1}x, T^n x, T^n x) \leq (3\xi)^n [G(x, Tx, Tx) + G(Tx, x, x)].$$

Then, we have

$$\lim_{n\to\infty} [G(T^n x, T^{n+1}x, T^{n+1}x) + G(T^{n+1}x, T^n x, T^n x)] = 0,\ \forall x \in A \cup B \cup C.$$

Using Lemma 2.6, we find that $F_G^{\varepsilon}(T) \neq \emptyset$ for all $\varepsilon > 0$. □

Now, we consider the contraction condition given in 2004 by V. Berinde, who also formulated a corresponding fixed point theorem, (see [4]), for example.

Definition 3.13. Let A, B and C be non-empty subsets of a G-metric space X. A mapping $T : A \cup B \cup C \to A \cup B \cup C$ satisfying $T(A) \subset B, T(B) \subset C$ and $T(C) \subset A$ is a weak contraction if there exists $a \in (0,1)$ and $L \geq 0$ such that for all $x, y \in A \cup B \cup C$,

$$G(Tx, Ty, Ty) + G(Ty, Tx, Tx) \leq a[G(x, y, y) + G(y, x, x)] + L[G(y, Tx, Tx) + G(Tx, y, y)].$$

Theorem 3.14. *Let A, B and C be non-empty subsets of a G-metric space X. Suppose that the mapping $T : A \cup B \cup C \to A \cup B \cup C$ satisfying $T(A) \subset B, T(B) \subset C$ and $T(C) \subset A$ is a weak contraction . Then for every $\varepsilon > 0$, $F_G^{\varepsilon}(T) \neq \emptyset$.*

Proof. Let $\varepsilon > 0$ and $x \in A \cup B \cup C$.

$$\begin{aligned} G(T^n x, T^{n+1}x, T^{n+1}x) + G(T^{n+1}x, T^n x, T^n x) \\ &= a[G(T^{n-1}x, T(T^n x), T(T^n x)) + G(T(T^n x), T^{n-1}x, T^{n-1}x) \\ &+ LG(T^n x, T^n x, T^n x) + G(T^n x, T^n x, T^n x)] \\ &\vdots \\ &\leq (a)^n [G(x, Tx, Tx) + G(Tx, x, x)]. \end{aligned}$$

But $a \in (0,1)$, therefore

$$\lim_{n \to \infty} [G(T^n x, T^{n+1}x, T^{n+1}x) + G(T^{n+1}x, T^n x, T^n x)] = 0, \ \forall x \in A \cup B \cup C.$$

Now by Lemma 2.6 it follow that $F_G^{\varepsilon}(T) \neq \emptyset$ for all $\varepsilon > 0$. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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