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BOUNDS FOR THE BLOW-UP TIME AND BLOW–UP RATE ESTIMATES FOR NONLINEAR BLACK-SHOLES EQUATIONS WITH DIRICHLET OR NEUMANN BOUNDARY CONDITIONS

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Abstract: The blow-up of the solution to Black-Sholes equations with weighted nonlinear source was studied. We obtained the lower bounds for blow-up time of the solution under different assumptions. Moreover, the corresponding blow-up rate estimates was also established.

Keywords: Black-Sholes equation; lower bounds; blow-up time; blow-up rate estimates.

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1. Introduction

Blow-up solutions for nonlinear parabolic equations are discussed by many authors. The Phenomena of the blow-up for nonlinear parabolic equations have been investigated extensively by many authors. Wu [7] *et al*, Wang and He [8] studied the blow-up of the solutions for a semilinear parabolic equation involving variable source and positive initial energy. Ding (cf.[9] and [11]) and Zhang [10] studied the global existence and blow-up solutions for the parabolic problems. C. Enache [14], L.E. Payne, P.W. Schaefer [15] and L.E. Payne, J.C. Song [18] discussed the lower bounds for the blow-up time to parabolic problems under Neumann boundary conditions. L.E. Payne, P.W. Schaefer (cf.[16] and [17]) dealt with the bounds for the blow-up time of the solution. Many approaches have been developed in discussing the upper or lower bounds for the blow-up time of various nonlinear parabolic problems. However, the blow-up rate of the solution to the problem with general nonlinearity is unknown. K. Baghaei, M.B.

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Ghaemi and M. Hesaaraki [6] studied the following semilinear parabolic problem with a variable source:

$$\begin{cases} u_t = \Delta u + u^{p(x)}, x \in \Omega, t > 0, \\ u(x,t) = 0 \quad\quad x \in \partial\Omega\,, t > 0, \\ u(x,0) = u_0(x), \quad\quad x \in \Omega, \end{cases}$$

Where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is bounded domain with smooth boundary. They obtained the lower bound for the blow-up time in some appropriate measure.

In this paper we intend to study the Blow-up Phenomenon of forward parabolic PDE. Through putting forward different assumptions, we obtain the lower bounds for the blow-up time of the solution. Furthermore, we got the corresponding blow-up rate estimates. This paper is organized as follows. In section 2 we established a model and in section three we will use two methods to obtain the lower bounds for the blow-up time and blow-up rate estimates of the solution to (2.5).

2. The Model

The risk adjusted Black-Scholes equation can be viewed as an equation with a variable volatility coefficient

$$\partial_t V + \frac{\sigma^2(s,t)}{2} S^2 \left(1 - \mu(S\partial_S V)^{\frac{1}{3}}\right) \partial_S^2 V + rs\partial_S V - rV = 0, \quad (2.2)$$

where $\sigma(S,t)$ represents volatility part of the process depends on a solution $V = V(s,t)$ and $\mu = 3\left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{3}}$, μ represent a trend or drift of the process, c is the transaction cost and R the portfolio risk measure. If $\mu = 0$ we recover the equation discussed in [18].

Taking $\hat{\sigma}^2(s,t) = \sigma^2(1 - \mu(S\partial_S^2 V(S,t))^{\frac{1}{3}}$, equation (2.2) becomes

$$\partial_t V + \frac{\hat{\sigma}^2}{2} S^2 \partial_S^2 V + rS\partial_S V - rV = 0. \quad (2.3)$$

By setting $S = e^x, u(x,t) = V(e^x, t) and\ h(e^x) = g(x)$, we obtain the following parabolic PDE .

$$\frac{\partial u(x,t)}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} - (\Lambda - \alpha) \frac{\partial u(x,t)}{\partial x} + \Lambda u(x,t) = 0\,,$$

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + (\Lambda - \alpha) \frac{\partial u(x,t)}{\partial x} - \Lambda u(x,t) \quad (2.4)$$

where $g(x)$ is the pay-off function. , $\alpha = \frac{\sigma^2(1-\mu(S\partial_S^2 v(s,t))^{\frac{1}{3}}}{2}$ $and\ \Lambda = r.$

In this paper we are concerned with the blow –up phenomenon of the following problem:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \alpha \dfrac{\partial^2 u}{\partial x^2} + (\Lambda - \alpha)\dfrac{\partial u(x,t)}{\partial x} - \Lambda u(x,t) & x \in \Omega, t > 0. \\ u(x,t) = 0 \; or \; \dfrac{\partial u}{\partial n} = 0 & x \in \Omega, t > 0 \\ u(x,0) = g(x) \geq 0, & x \in \Omega, \end{cases} \tag{2.5}$$

Where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $\frac{\partial}{\partial n}$ represents the outward normal derivative on $\partial\Omega, g(x)$,is the initial value, $1 < p \leq 2.$ Set $\mathcal{R}^+ = (0, \infty)$.We assume throughout the work, that $(F1)$: $f(x,s)$ is a nonnegative $C^1(\overline{\Omega} \times [0,\infty))$ function, and $(F2)$: $\int_s^{+\infty} \frac{d\eta}{f(.,\eta)}$ is bounded for $s > 0, b \; is \; a \; C^2(\mathcal{R}^+)$ function satisfying $1 \leq b'_m \leq b'(s) \leq b'_m, b''(s) \leq 0 \; for \; all \; s > 0.$

The following condition will be required in our results:

(F3) There exist positive constants C_1, C_2, M, k,a nonnegative constant r and a positive function $m(x) \in C(\Omega; \mathbb{R}^+) \; satisfy \;\; 0 \leq r \leq 1 < m_{\_}: inf_{x\in\Omega} m(x) \leq m(x) \leq m_+ \coloneqq sup_{x\in\Omega} m(x) \leq k + 1$ such that

$$f(x,s) \leq C_1 + C_2 s^r \left(\int_\Omega s^{m(x)} dx \right)^M, for \; all \; s \geq 0;$$

(F4) There exist positive constants C_3, C_4, k and a positive function $m(x) \in C(\Omega; \mathbb{R}^+)$ satisfy $\frac{3}{4} < m \leq m(x) \leq m_+ < \infty, k > max\{(n-1)(4m_+ - 3), 1\} \; such \; that$

$$f(x,s) \leq C_3 + C_4 s^{m(x)};$$

(F5) There exist positive constant α such that

$$sf(x,s) \geq 2(1+\alpha)F(x,s),$$

where $F(x,s) = \int_\Omega f(x,\zeta)d\zeta$;

(G1) $for \; 1 < p \leq 2,$

$$\int_\Omega |\nabla g|^p dx \leq p \int_\Omega f(x,g)dx$$

3. Lower bounds for the blow-up time of the solution to equation (2.5)

In this section we will use two different methods to establish the lower bound for the blow-up time and blow-up rate of the solution to equation (2.5) under different assumptions.

Defined

$$G(s) = (k+1)\int_0^s \eta^k b'(\eta)d\eta, \;\; A(t) = \int_\Omega G\big(u(x,t)\big)dx \tag{3.1}$$

Where k is a positive constant?

Theorem (3.1) Let u be a nonnegative solution of (1.5) subject to Dirichlet (or Nenmann) boundary condition,$A(t)$ be defined as (3.1).Assume that f satisfies (F1),(F2) and (F3),then the blow-up time t^* is bounded from below by

$$t^* \geq \int_{A(0)}^{+\infty} \frac{d\eta}{K_1\eta^{r_1} + K_2\eta^{r_2}(1+\eta^{r_3})^M}.$$

Moreover, we have the following blow –up rate estimate

$$\|u(.,t)\|_{L^{k+1}} \geq S_1^{\frac{1}{k+1}}(t^*-t)^{-\frac{1}{r+m_+M-1}},$$

Where $K_1, K_2, r_1, r_2, r_3 and\ S_1$ are positive constants which will be determined later.

Proof. Applying the divergence theorem and taking into account assumption (F3), we obtain

$$A' = \int_\Omega G'\big(u(x,t)\big)u_t dx$$

$$= (k+1)\int_\Omega u^k b'(u)u_t dx$$

$$= (k+1)\int_\Omega u^k[div(|\nabla u|^{p-2}\nabla u) + f(x,u)]dx$$

(3.2)

$$= -k(k+1)\int_\Omega u^{k-1}|\nabla u|^p dx + \int_\Omega u^k f(x,u)dx$$

$$\leq C_1(k+1)\int_\Omega u^k\, dx + C_2(k+1)\int_\Omega u^{k+r}dx\left(\int_\Omega u^{m(x)}\, dx\right)^M.$$

For each $t > 0$,we divide Ω into two sets,

$$\Omega_{\{<1\}} = \{x \in \Omega: u(x,t) < 1\}, \Omega_{\{\geq 1\}} = \{x \in \Omega: u(x,t) \geq 1\}$$

Now applying Holder inequality, we have

$$\int_\Omega u^{k+r}dx \leq |\Omega|^{\frac{1-r}{k+1}}(u^{k+1}dx)^{\frac{k+r}{k+1}} \tag{3.3}$$

and

$$\int_\Omega u^{m(x)}dx \leq \int_{\Omega_{(<1)}} u^{m_-}dx + \int_{\Omega_{(\geq 1)}} u^{m_+}dx$$

$$\leq \int_{\Omega} u^{m_-}dx + \int_{\Omega} u^{m_+}dx \tag{3.4}$$

$$\leq \left(\int_{\Omega} u^{k+1}dx\right)^{\frac{m_-}{k+1}} |\Omega|^{1-\frac{m_-}{k+1}} + \left(\int_{\Omega} u^{k+1}dx\right)^{\frac{m_+}{k+1}} |\Omega|^{1-\frac{m_+}{k+1}}$$

Substituting (3.3),(3.4) into (3.2),we obtain

$$A'(t) \leq C_1(k+1)|\Omega|^{\frac{1}{k+1}}\left(\int_{\Omega} u^{k+1}dx\right)^{\frac{k}{k+1}} + C_2(k+1)|\Omega|^{\frac{1-r}{k+1}}\left(\int_{\Omega} u^{k+1}dx\right)^{\frac{k+r}{k+1}}$$

$$\left[\left(\int_{\Omega} u^{k+1}dx\right)^{\frac{m_-}{k+1}} |\Omega|^{1-\frac{m_-}{k+1}} + \left(\int_{\Omega} u^{k+1}dx\right)^{\frac{m_+}{k+1}} |\Omega|^{1-\frac{m_+}{k+1}}\right]^{M} \tag{3.5}$$

$$\leq K_1\left(\int_{\Omega} u^{k+1}dx\right)^{\frac{k}{k+1}} + K_2\left(\int_{\Omega} u^{k+1}dx\right)^{\frac{k+r+m_-M}{k+1}}\left[1+\left(\int_{\Omega} u^{k+1}dx\right)^{\frac{m_+-m_-}{k+1}}\right]^{M},$$

where $K_1 = C_1(k+1)|\Omega|^{\frac{1}{k+1}}\left(\int_{\Omega} u^{k+1}dx\right)^{\frac{k}{k+1}}, K_2 = C_2(k+1)|\Omega|^{\frac{1-r}{k+1}} max\left\{|\Omega|^{\frac{M(k+1-m_-)}{k+1}}, |\Omega|^{\frac{M(k+1-m_+)}{k+1}}\right\}$.

On the other hand.

$$A(t) = \int_{\Omega} G\big(u(x,t)\big)dx \geq \int_{\Omega} u^{k+1}dx, \tag{3.6}$$

Combing with (3.5),we have

$$A'(t) \leq K_1\big(A(t)\big)^{\frac{k}{k+1}} + K_2\big(A(t)\big)^{\frac{k+r+m_-M}{k+1}}\left[1+\big(A(t)\big)^{\frac{m_+-m_-}{k+1}}\right]^{M}. \tag{3.7}$$

Integrating (3.7) from 0 to $t(t < t^*), if\ \lim_{t\to t^*} A(t) = +\infty, we\ get$

$$t^* \geq \int_{A(0)}^{+\infty} \frac{d\eta}{K_1\eta^{r_1} + K_2\eta^{r_2}(1+\eta^3)^M}, \tag{3.8}$$

Where $r_1 = \frac{k}{k+1}, r_2 = \frac{k+r+m_-M}{k+1}, r_3 = \frac{m_+-m_-}{k+1}$.

Integrating (3.7) from $t\ to\ t^*$,we obtain

$$t^* - t \geq \int_{A(t)}^{\infty} \int_{A(0)}^{+\infty} \frac{d\eta}{K_1\eta^{r_1} + K_2\eta^{r_2}(1+\eta^3)^M} = \emptyset(A)(t), \qquad (3.9)$$

Obviously, $\emptyset(A)(t)$ is a decreasing function of A which means its inverse function $\emptyset^{-1}$ exists and it is also a decreasing one .Therefore, we have

$$A(t) \geq \emptyset^{-1}(t^* - t), \qquad (3.10)$$

which gives the lower estimate of blow-up rate. In fact, if $t\ is\ close\ to\ t^*$ enough, such that

$K_2\eta^{\frac{k+r+m_+M}{k+1}}, > K_1\eta^{r_1}$

using (3.9),we have

$$t^* - t \geq \frac{k+1}{2K_2(r+m_+M-1)}\left(A(t)\right)^{\frac{m_+-m_+M}{k+1}}, \qquad (3.11)$$

which means that

$$A(t) \geq \left(\frac{k+1}{2K_2(r+m_+M-1)}\right)^{\frac{k+1}{r+m_+M-1}}(t^* - t)^{-\frac{k+1}{r+m_+M-1}} \quad . \qquad (3.12)$$

Since $A(t) \leq b'_M \int_\Omega u^{k+1}dx$, combing with (3.12),we have

$$\|u(.,t)\|_{L^{k+1}} \geq S_1^{\frac{1}{k+1}}(t^* - t)^{-\frac{k+1}{r+m_+M-1}}. \qquad (3.13)$$

where $S_1 = \frac{1}{b'_M}\left[\frac{k+1}{2K_2(r+m_+M-1)}\right]^{\frac{k+1}{r+m_+M-1}}$

Remark. This method is valid for $1 < p < \infty$ and not to restrict the space dimension.

Theorem (3.2) .Let u be a non negative solution of (1.5) subject to dirichlet boundary condition,$A(t)$ be defined as (3.1) .Assume that f satisfies the condition (F1),(F2) and (F4), then the blow –up time t^* is bounded from below .We have

$$\int_{a(0)}^{+\infty} \frac{d\eta}{K_3 + k_4\eta^{\frac{k}{k+1}} + k_5\eta^{\frac{3(n-p)}{3n-4p}}},$$

And blow-up rate estimate

$$\|u(.,t)\|_{L^{k+1}} \geq S_2^{\frac{1}{k+1}}(t^* - t)^{-\frac{3n-4p}{p(k+1)}} \ ,$$

where $K_3, K_4, K_5\ and\ S_2$ are positive constant which will defined later.

Proof. From (3.2) and (F4).we know that

$$A(t)' = -k(k+1)\int_\Omega u^{k-1}\,|\nabla u|^p dx + (k+1)\int_\Omega u^k f(x,u)dx$$

$$\leq -k(k+1)\left(\frac{p}{k-1+p}\right)^p \int_\Omega \left|\nabla u^{\frac{k-1+p}{p}}\right|^p dx + C_3(k+1)\int_\Omega u^k dx \qquad (3.14)$$

$$+C_4(k+1)\int_\Omega u^{k+m(x)} dx.$$

Like (3.4).

$$\int_\Omega u^{k+m(x)} dx \leq \int_\Omega u^{k+m_-} dx + \int_\Omega u^{k+m_+} dx, \qquad (3.15)$$

By applying Holder inequality, we have

$$\int_\Omega u^{k+m_-} dx \leq |\Omega|^{M_1}\left(\int_\Omega u^{\frac{k(4n-3p)+p(n-3)+2n}{4(n-p)}} dx\right)^{m_2} \qquad (3.16)$$

and

$$\int_\Omega u^{k+m_+} dx \leq |\Omega|^{M_3}\left(\int_\Omega u^{\frac{k(4n-3p)+p(n-3)+2n}{4(n-p)}} dx\right)^{m_4}, \qquad (3.17)$$

where

$$m_1 = 1 - \frac{4(n-p)(k+m_-)}{k(4n-3p)+p(n-3)+2n}, m_2 = \frac{4(n-p)(k+m_-)}{k(4n-3p)+p(n-3)+2n},$$

$$m_3 = 1 - \frac{4(n-p)(k+m_+)}{k(4n-3p)+p(n-3)+2n}, m_4 = \frac{4(n-p)(k+m_+)}{k(4n-3p)+p(n-3)+2n}.$$

Substituting (3.16),(3.17) into (3.15) and using Young inequality, we get

$$\int_\Omega u^{k+m(x)} dx \leq l_1 + l_2\int_\Omega u^{\frac{k(4n-3p)+p(n-3)+2n}{4(n-p)}} dx, \qquad (3.18)$$

where $l_1 = (m_1+m_2)|\Omega|, l_2 = m_2 + m_4$. Substituting (3.18) into (3.14),we have

$$A'(t) \leq -k(k+1)\left(\frac{p}{k-1+p}\right)^p \int_\Omega \left|\nabla u^{\frac{p}{k-1+p}}\right|^p dx + C_3(k+1)|\Omega|^{\frac{1}{k+1}}\left(\int_\Omega u^{k+1} dx\right)^{\frac{k}{k+1}}$$

$$+ C_4 l_1(k+1) + C_4 l_2(k+1)\int_\Omega u^{\frac{k(4n-3p)+p(n-3)+2n}{4(n-p)}} dx. \qquad (3.19)$$

We now make use of Holder inequality to last term on right side of (3.19) to get

$$\int_\Omega u^{\frac{k(4n-3p)+p(n-3)+2n}{4(n-p)}} dx \leq \left(\int_\Omega u^{k+1} dx\right)^{\frac{3}{4}}\left(\int_\Omega \left(u^{\frac{k-1+p}{p}}\right)^{\frac{np}{n-p}} dx\right)^{\frac{1}{4}}. \qquad (3.20)$$

Note that

$$\int_{\Omega}\left(u^{\frac{k-1+p}{p}}\right)^{\frac{np}{n-p}} \leq (C_S)^{\frac{np}{n-p}}\left(\int_{\Omega}\left|\nabla u^{\frac{k-1+p}{p}}\right|^{p} dx\right)^{\frac{n}{n-p}}, \qquad (3.21)$$

here C_s is the best Sobolev constant. By inserting (3.21) in (3.20) and using the Young inequality, we have

$$\int_{\Omega} u^{\frac{k(4n-3p)+p(n-3)+2n}{4(n-p)}} dx \leq \frac{(3n-4p)(C_S)^{\frac{np}{n-p}}}{4(n-p)\epsilon^{\frac{n}{3n-4p}}}\left(\int_{\Omega} u^{k+1}dx\right)^{\frac{3(n-p)}{3n-4p}} +$$

$$\frac{n\epsilon(C_S)^{\frac{np}{4(n-p)}}}{4(n-p)}\int_{\Omega}\left|\nabla u^{\frac{k-1+p}{p}}\right|^{p} dx. \quad (3.22)$$

Where ϵ is a positive constant to be determined later. Combing with (3.22) and (3.19). we obtain

$$A'(t) \leq K_3 + K_4\left(\int_{\Omega} u^{k+1}dx\right)^{\frac{k}{k+1}} + K_5\left(\int_{\Omega} u^{k+1}dx\right)^{\frac{3(n-p)}{3n-4p}} + K_6\int_{\Omega}\left|\nabla u^{\frac{k-1+p}{p}}\right|^{p} dx$$

$$\leq K_3 + K_4\left(A(t)^{\frac{k}{k+1}} + K_5 A(t)\right)^{\frac{3(n-p)}{3n-4p}}$$

$$+ K_6\int_{\Omega}\left|\nabla u^{\frac{k-1+p}{p}}\right|^{p} dx \quad , \qquad (3.23)$$

where

$$K_3 = C_4 l_1(k+1), K_4 = C_3(k+1)|\Omega|^{\frac{1}{k+1}}, K_5 = C_4 l_2(k+1)\frac{(3n-4p)(C_S)^{\frac{np}{n-p}}}{4(n-p)^{\epsilon^{\frac{n}{3n-4p}}}},$$

$$K_6 = C_4 l_2(k+1)\frac{n\epsilon(C_s)^{\frac{np}{4(n-p)}}}{4(n-p)} - k(k+1)\left(\frac{p}{k-1+p}\right)^{p}.$$

If we choose $\epsilon > 0$ such that

$$\epsilon = \frac{4k(n-p)\left(\frac{p}{k-1+p}\right)^{p}}{C_4 l_2 n(C_s)^{\frac{np}{4(n-p)}}},$$

then,we obtain the differential inequality

$$A'(t) \leq K_3 + K_4\big(A(t)\big)^{\frac{k}{k+1}} + K_5\big(A(t)\big)^{\frac{3(n-p)}{3n-4p}}. \qquad (3.24)$$

An integrating of the differential inequality (3.24) from 0 to t $(t < t^*)$ leads to

$$t^* \geq \int_{A(0)}^{\infty}\frac{d\eta}{K_3 + K_4\eta^{\frac{k}{k+1}} + K_5\eta^{\frac{3(n-p)}{3n-4p}}}, \qquad (3.25)$$

If $\lim_{t\to t^*} A(t) = +\infty$.Similar to (3.13),we get the lower estimate of the blow-up rate

$$\|u(.,t)\|_{L^{k+1}} \geq S_2^{\frac{1}{k+1}}(t^* - t)^{\frac{3n-3p}{p(k+1)}}, \tag{3.26}$$

where $S_2 = \frac{3n-4p}{2b'_M K_5 P}$.

Conclusion

The Blow-up Phenomenon of Black-Scholes PDE was studied. We did these by putting forward different assumptions. We also obtain the lower bounds for the blow-up time of the solution and the corresponding blow-up rate estimates.

Conflict of Interests

The authors declare that there is no conflict of interests.

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