Available online at http://scik.org

J. Math. Comput. Sci. 6 (2016), No. 6, 1145-1156

ISSN: 1927-5307

NUMERICAL ACCURACY BETWEEN RUNGE-KUTTA FEHLBERG METHOD AND ADAMS-BASHFORTH METHOD FOR FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH BOUNDARY VALUE PROBLEM

NAJMUDDIN AHMAD* AND SHIV CHARAN

Department of Mathematics, Integral University, Lucknow, Uttar Pradesh, INDIA

Abstract There are many problems in the field of science, engineering and technology which can be solved by differential equations formulation. This research will compare the accuracy of various method, the Runge-Kutta Fehlberg method and Adams-Bashforth method, in completing numerical solutions of differential equations, which is limited to ordinary differential equation of first order. Numbers of differential equations solved by the MATLAB version 7.9 to compare the accuracy between Runge-Kutta Fehlberg and Adams-Bashforth method. It can be concluded that Runge-Kutta Fehlberg method as more rigorous accuracy than the Adams-Bashforth method for solving linear ordinary differential equations of first order [1].

Keywords: differential equations formulation; Runge-Kutta Fehlberg method; Adams-Bashforth method.

2010 AMS Subject Classification: 65L03.

Introduction:

Differential equation is one of the significant topics in the field of mathematics especially in dealing with science (engineering) issues [1][8]. Considering the vast usefulness of differential equations, methods are being proposed to find the solution of differential equations, with the availability of various solutions. Ordinary differential equations can be solved by analytical and numerical methods. The solutions generated by the analytical method are generally exact values, whereas with the numerical method an approximation is given as a solution approaching the real value [1].

*Corresponding author

Received March 20, 2016

Many real-life problems in Science and Engineering are often modeled in the form of Ordinary Differential equation of Second order. Such problems may not have closed form then numerical algorithms are usually employed to solve them [10].

Effort have been made by researchers and solve higher order initial valued problems, especially Second order ODEs by a number of different method [10].

Runge – Kutta Felhberg method:

One way to guarantee accuracy in the solution of an I.V.P.is to solve the problem twice the using step size h and h/2 and compare answers at the mesh points corresponding to the large step size. But this requires a significant amount of computation for the smaller step size and must be repeated if it is determined that the agreement is not good enough.

The Runge – Kutta Felhberg method (denote RKF45) is one way to try to resolve this problem. It has a procedure to determine if the proper step size h is being used. At each step, two different approximation for the solution are made and compared. If the two answer are in close agreement, the approximation is accepted. If the two answer do not agree to specified accuracy, the step size is reduced. If the answer agrees to more significant digits than required, the step size is increased. In each step the following six values are required.

$$k_1 = hf\left(x_{i,} y_i\right)$$

$$k_2 = hf\left(x_i + \frac{h}{24}, y_i + \frac{k_1}{4}\right)$$

$$k_3 = hf\left(x_i + \frac{3h}{8}, y_i + \frac{3k_1}{32} + \frac{9k_2}{32}\right)$$

$$k_4 = hf\left(x_i + \frac{12h}{13}, y_i + \frac{1932k_1}{2197} - \frac{7200k_2}{2197} + \frac{7296k_3}{2197}\right)$$

$$k_5 = hf\left(x_i + h, y_i + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right)$$

$$k_6 = hf\left(x_i + \frac{h}{2}, y_i + \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right)$$

Then an approximation using fourth order Runge-Kutta method is

$$y_{i+1} = y_i + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5$$

It may be noted that the value of k_2 is not used in the above formula. The other value of y is determined by fifth order Runge – Kutta method as follows:

$$y_{i+1}^{*} = y_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6$$

If $\left|y_{i+1} + y_{i+1}^{*}\right|$ is small enough than the method is terminated; otherwise, the computation is repeated by reducing the step size h. The local truncation error of this method is $y_{i+1} - y_{i+1}^{*}$. It should be noted that the particular coefficient used above were developed by Cash and Karp(1990). Therefore, it is sometimes called the Cash-Karp RK method[13].

Adams-Bashforth methods:

The double step method is an additional method and it is used to improve the one-step method that has deficiency in terms of error estimation. This additional method is known as Corrector Predictor, and this method cannot launch itself. Therefore, this method requires the value of previous method to launch the prediction. In addition, Adams-Bashforth method is more efficient than the one-step method, such as Runnge-Kutta method because it only performs repetitive calculations of f(x, y) in accordance to the interested earlier points, while the runge-Kutta method requires some evaluation of the f(x, y)function, which consequently becomes more time consuming.

This is the forth-order multistep method and it needs four values (x_{i-3}, y_{i-3}), (x_{i-2}, y_{i-2}), (x_{i-1}, y_{i-1}) and (x_i, y_i) to compute y_{i+1}. These values are called starting values of this method and they may be determined by using any single step method such as Euler, Runge-Kutta, etc[12].

Let consider a differential equation

$\frac{dy}{dx} = f(x, y)$ with initial conditions $x = x_0$, $y(x_0) = y_0$.

$$y_{i+1}^{p} = y_i + \frac{h}{24}\left(-9f_{i-3} + 37f_{i-2} - 59f_{i-1} + 55f_i\right)$$

This formula is known as Adams – Bashforth predictor formula.

$$y_{i+1}^{c} = y_i + \frac{h}{24}\left(f_{i-2} - 5f_{i-1} + 19f + 9f_{i+1i}\right)$$

This formula is known as Adams – Bashforth corrector formula.

Where $f_{i+1} = f\left(x_{i+1}, y_{i+1}^{p}\right)$.

Comparing accuracy of differential equation results:

Our study will compare the accuracy of approximation between one step size method that is non-definite, which is Runge-Kutta fehlberg method, and thedouble step method that is the Adams-Bashforth method in solving an ordinary differential equation of first order and second order. The approximation accuracy defined in this study based on the percentage error in the completion of first and second order ordinary differential equation by using the Runge-Kutta fehlberg and the Adams-Bashforth method.

The Runge-Kutta fehlberg method is the method based on the calculation of two Runge-Kutta of different order, by subtracting the results to get an estimate of the error. The one step algorithm method with an adaptive step size automatically adjusts the step size as a reaction to the calculation truncations errors.

The double step method requires smaller step by step evaluation than the one step method and that almost all similar method is called as the prediction corrector method. One of the double step methods that acquire more stability properties than the other method is the double step Adams-Bashforth method. Based on the above description, the researchers want to examine which among the two methods, the Runge-Kutta fehlberg or Adams-Bashforth method would give more accurate approximation in solving linear ordinary differential equation of first and second order.

We have considered ten differential equations with initial condition. We have developed MATLAB program to solve them to get exact value and approximated value by Adams-Bashforth method and Runge-Kutta fehlberg method. We have them compared the accuracy of the result obtained by Admas-Bashforth method by Runge-kutta fehlberg method. Differential equations under study are given below:

1. $\frac{dy}{dx} = x + y;$ $\quad$ $y = 1$ when $x = 0$

2. $\frac{dy}{dx} = y - x;$ $\quad$ $y = 1$ when $x = 0$

3. $\frac{dy}{dx} = -2xy^2;$ $\quad$ $y = 1$ when $x = 0$

4. $\frac{dy}{dx} = x + y + 1;$ $\quad$ $y = 1$ when $x = 0$

5. $\frac{dy}{dx} = -x/y;$ $\quad$ $y = 1$ when $x = 0$

6. $\frac{dy}{dx} = x^2 - y;$ y =1 when x = 0

7. $\frac{dy}{dx} = 2y - 2x^2 - 3;$ y =1 when x = 0

8. $\frac{dy}{dx} = (3x + y)/2;$ y =1 when x = 0

9. $\frac{dy}{dx} = y + e^x;$ y =1 when x = 0

10. $\frac{dy}{dx} = \sin x - y;$ y =1 when x = 0

- In differential equation $\frac{dy}{dx} = x + y$; y =1 when x = 0 the average percentage error in Runge-Kutta fehlberg is 0.00000000000588, average percentage error in Adams-bashforh is 0.020832437 and the accuracy factor $(\ddot{A})$ is 354293146.3 i.e. Runge – Kutta Fehlberg method is 354293146.3 times better than Adams – Bashforth method.
- In differential equation $\frac{dy}{dx} = y - x$; y =1 when x = 0 the average percentage error in Runge-Kutta fehlberg is 0.00000000000, average percentage error in Adams-bashforh is 0.0208 and the accuracy factor $(\ddot{A})$ is ∞ i.e. Runge – Kutta Fehlberg method is ∞ times better than Adams – Bashforth method.
- In differential equation $\frac{dy}{dx} = -xy^2$; y =1 when x = 0 the average percentage error in Runge-Kutta fehlberg is 0.00000000001461, average percentage error in Adams-bashforh is 0.019967986 and the accuracy factor $(\ddot{A})$ is 1366734155 i.e. Runge – Kutta Fehlberg method is 1366734155 times better than Adams – Bashforth method.
- In differential equation $\frac{dy}{dx} = x + y + 1$; y =1 when x = 0 the average percentage error in Runge-Kutta fehlberg is 0.00000000000881, average percentage error in Adams-bashforh is 0.021648751 and the accuracy factor $(\ddot{A})$ is 245729296.3 i.e. Runge – Kutta Fehlberg method is 245729296.3 times better than Adams – Bashforth method.

- In differential equation $\frac{dy}{dx} = -x / y$; y =1 when x = 0 the average percentage error in Runge-Kutta fehlberg is 0.0004148613, average percentage error in Adams-bashforh is 0.02058702 and the accuracy factor $(\ddot{A})$ is 49.62 i.e. Runge – Kutta Fehlberg method is 49.62 times better than Adams – Bashforth method.
- In differential equation $\frac{dy}{dx} = x^2 - y$; y =1 when x = 0 the average percentage error in Runge-Kutta fehlberg is 0.00000000003463, average percentage error in Adams-bashforh is 0.019216212 and the accuracy factor $(\ddot{A})$ is 554900721.9 i.e. Runge – Kutta Fehlberg method is 554900721.9times better than Adams – Bashforth method.
- In differential equation $\frac{dy}{dx} = 2y - 2x^2 - 3$; y =1 when x = 0 the average percentage error in Runge-Kutta fehlberg is 0.00000061309481, average percentage error in Adams-bashforh is 0.019166055 and the accuracy factor $(\ddot{A})$ is 31261.1600 i.e. Runge – Kutta Fehlberg method is 31261.1600 times better than Adams – Bashforth method.
- In differential equation $\frac{dy}{dx} = 3x + y$; y =1 when x = 0 the average percentage error in Runge-Kutta fehlberg is 0.00000000000022, average percentage error in Adams-bashforh is 0.020428188 and the accuracy factor $(\ddot{A})$ is 92855400000.598 i.e. Runge – Kutta Fehlberg method is 92855400000.598 times better than Adams – Bashforth method.
- In differential equation $\frac{dy}{dx} = y + e^x$; y =1 when x = 0 the average percentage error in Runge-Kutta fehlberg is 0.000000042216928, average percentage error in Adams-bashforh is 0.000001086999688 and the accuracy factor $(\ddot{A})$ is 25.74795 i.e. Runge – Kutta Fehlberg method is 25.74795 times better than Adams – Bashforth method.
- In differential equation $\frac{dy}{dx} = \sin x - y$; y =1 when x = 0 the average percentage error in Runge-Kutta fehlberg is 0.000000000270642, average percentage error in Adams-bashforh is 0.0000000721509 and the accuracy factor $(\ddot{A})$ is 266.5916598 i.e. Runge – Kutta Fehlberg method is 266.5916598 times better than Adams – Bashforth method[11].

Value of x, Calculated value of y, Exact value of y, Difference Between Calculated and Exact Values of y and percentage error in the value of y for the differential equation $dy/dx = x^2 - y$; x=0, y=1 using Adams - Bashforth Method

Value of x	Exact Value of y	Adams –Bashforth Value of y	Difference between Exact value and Adams Bashforh	Percentage error in Adams - Bashforth
0	1			
0.2	0.980201326693245			
0.04	0.960810560847677			
0.06	0.941835466415751			
0.08	0.923283653613364	0.923283653818395	8.43283653613364E-01	
0.1	0.905162581964041	0.905162582251057	8.05162581964041E-01	3.17088E-08
0.12	0.887479563282843	0.887479563648551	7.67479563282843E-01	4.12075E-08
0.14	0.870241764601194	0.870241765042367	-7.30241764601194E-01	5.06955E-08
0.16	0.853456211033789	0.853456211547294	-6.93456211033789E-01	6.01677E-08
0.18	0.837129788588728	0.837129789171529	-6.57129788588728E-01	6.9619E-08
0.2	0.821269246922018	0.821269247571169	-6.21269246922018E-01	7.90424E-08
0.22	0.805881202037522	0.805881202750166	-5.85881202037522E-01	8.84304E-08
0.24	0.790972138933447	0.790972139706816	-5.50972138933447E-01	9.77745E-08
0.26	0.776548414196434	0.776548415027843	-5.16548414196434E-01	1.07065E-07
0.28	0.762616258544275	0.762616259431122	-4.82616258544275E-01	1.1629E-07
0.3	0.749181779318282	0.749181780258047	-4.49181779318282E-01	1.25439E-07
0.32	0.736250962926309	0.736250963916547	-4.16250962926309E-01	1.34497E-07
0.34	0.72382967723739	0.723829678275735	-3.83829677237390E-01	1.43452E-07
0.36	0.711923673928969	0.711923675013126	-3.51923673928969E-01	1.52286E-07
0.38	0.700538590787644	0.700538591915393	-3.20538590787644E-01	1.60983E-07
0.4	0.689679953964361	0.689679955133549	-2.89679953964361E-01	1.69526E-07
0.42	0.679353180184943	0.679353181393489	-2.59353180184943E-01	1.77897E-07
0.44	0.669563578916859	0.669563580162744	-2.29563578916859E-01	1.86074E-07
0.46	0.660316354493074	0.660316355774346	-2.00316354493074E-01	1.94039E-07
0.48	0.651616608193859	0.651616609508628	-1.71616608193859E-01	2.0177E-07
0.5	0.643469340287367	0.643469341633804	-1.43469340287367E-01	2.09246E-07
0.52	0.635879452029806	0.635879453406141	-1.15879452029806E-01	2.16446E-07

0.54	0.62885174762601	0.628851749030532	-8.88517476260100E-02	2.23347E-07
0.56	0.622390936151185	0.622390937582238	-6.23909361511850E-02	2.29928E-07
0.58	0.616501633434598	0.61650163489058	-3.65016334345980E-02	2.36168E-07
0.6	0.611188363905974	0.611188365385336	-1.11883639059740E-02	2.42047E-07
0.62	0.606455562405326	0.606455563906572	1.35444375946739E-02	2.47544E-07
0.64	0.602307575956952	0.602307577478635	3.76924240430481E-02	2.52642E-07
0.66	0.598748665508301	0.598748667049024	6.12513344916991E-02	2.57324E-07
0.68	0.59578300763441	0.595783009192822	8.42169923655901E-02	2.61574E-07
0.7	0.59341469620859	0.593414697783386	1.06585303791410E-01	2.65379E-07
0.72	0.591647744040028	0.591647745629948	1.28352255959972E-01	2.68727E-07
0.74	0.590486084478966	0.590486086082793	1.49513915521034E-01	2.71611E-07
0.76	0.589933572990091	0.589933574606652	1.70066427009909E-01	2.74024E-07
0.78	0.589993988694777	0.589993990322938	1.90006011305223E-01	2.75962E-07
0.8	0.590671035882778	0.590671037521447	2.09328964117222E-01	2.77425E-07
0.82	0.591968345494001	0.591968347142122	2.28031654505999E-01	2.78414E-07
0.84	0.59388947657092	0.593889478227478	2.46110523429080E-01	2.78934E-07
0.86	0.596437917682251	0.596437919346264	2.63562082317749E-01	2.78992E-07
0.88	0.599617088318419	0.599617089988942	2.80382911681581E-01	2.78598E-07
0.9	0.603430340259401	0.603430341935525	2.96569659740599E-01	2.77766E-07
0.92	0.607880958915486	0.607880960596333	3.12119041084514E-01	2.76509E-07
0.94	0.612972164641479	0.612972166326206	3.27027835358521E-01	2.74846E-07
0.96	0.618707114024888	0.618707115712681	3.41292885975112E-01	2.72794E-07
0.98	0.625088901148601	0.625088902838679	3.54911098851399E-01	2.70374E-07
1	0.632120558828558	0.632120560520168	3.67879441171442E-01	2.67609E-07

Value of x, Calculated value of y, Exact value of y, Difference Between Calculated and Exact Values of y and percentage error in the value of y for the differential equation dy/dx = x^2 – y; x=0, y=1 using Runge-Kutta Fehlberg Method

Value of x	Exact Value value of y	Fehlberg value of y	Difference between Exact value and Fehlberg	Percentege error in Felhberg
0	1	1	0	0
0.2	0.980201326693245	0.98020132669321	3.4972E-14	0.0000000000036
0.04	0.960810560847677	0.960810560847608	6.90559E-14	0.0000000000072
0.06	0.941835466415751	0.941835466415647	1.04028E-13	0.0000000000110
0.08	0.923283653613364	0.923283653613224	1.39999E-13	0.0000000000152
0.1	0.905162581964041	0.905162581963865	1.7597E-13	0.0000000000194
0.12	0.887479563282843	0.88747956328263	2.13052E-13	0.0000000000240
0.14	0.870241764601194	0.870241764600945	2.48912E-13	0.0000000000286
0.16	0.853456211033789	0.853456211033503	2.85993E-13	0.0000000000335
0.18	0.837129788588728	0.837129788588405	3.22964E-13	0.0000000000386
0.2	0.821269246922018	0.821269246921658	3.59934E-13	0.0000000000438
0.22	0.805881202037522	0.805881202037124	3.98015E-13	0.0000000000494
0.24	0.790972138933447	0.790972138933011	4.35985E-13	0.0000000000551
0.26	0.776548414196434	0.77654841419596	4.73954E-13	0.0000000000610
0.28	0.762616258544275	0.762616258543762	5.12923E-13	0.0000000000673
0.3	0.749181779318282	0.749181779317732	5.50004E-13	0.0000000000734
0.32	0.736250962926309	0.73625096292572	5.88973E-13	0.0000000000800
0.34	0.72382967723739	0.723829677236763	6.27054E-13	0.0000000000866
0.36	0.711923673928969	0.711923673928303	6.66023E-13	0.0000000000936
0.38	0.700538590787644	0.70053859078694	7.03992E-13	0.0000000001005
0.4	0.689679953964361	0.689679953963618	7.42961E-13	0.0000000001077
0.42	0.679353180184943	0.679353180184161	7.8193E-13	0.0000000001151
0.44	0.669563578916859	0.669563578916038	8.2101E-13	0.0000000001226
0.46	0.660316354493074	0.660316354492215	8.5898E-13	0.0000000001301
0.48	0.651616608193859	0.651616608192962	8.9706E-13	0.0000000001377
0.5	0.643469340287367	0.64346934028643	9.37028E-13	0.0000000001456
0.52	0.635879452029806	0.635879452028831	9.74998E-13	0.0000000001533

0.54	0.62885174762601	0.628851747624997	1.01308E-12	0.0000000001611
0.56	0.622390936151185	0.622390936150134	1.05105E-12	0.0000000001689
0.58	0.616501633434598	0.616501633433508	1.09002E-12	0.0000000001768
0.6	0.611188363905974	0.611188363904846	1.12799E-12	0.0000000001846
0.62	0.606455562405326	0.60645556240416	1.16607E-12	0.0000000001923
0.64	0.602307575956952	0.602307575955747	1.20493E-12	0.0000000002001
0.66	0.598748665508301	0.598748665507059	1.24201E-12	0.0000000002074
0.68	0.59578300763441	0.595783007633131	1.27898E-12	0.0000000002147
0.7	0.59341469620859	0.593414696207274	1.31595E-12	0.0000000002218
0.72	0.591647744040028	0.591647744038674	1.35392E-12	0.0000000002288
0.74	0.590486084478966	0.590486084477574	1.392E-12	0.0000000002357
0.76	0.589933572990091	0.589933572988662	1.42897E-12	0.0000000002422
0.78	0.589993988694777	0.589993988693311	1.46605E-12	0.0000000002485
0.8	0.590671035882778	0.590671035881277	1.50102E-12	0.0000000002541
0.82	0.591968345494001	0.591968345492463	1.53799E-12	0.0000000002598
0.84	0.59388947657092	0.593889476569346	1.57396E-12	0.0000000002650
0.86	0.596437917682251	0.596437917680641	1.60993E-12	0.0000000002699
0.88	0.599617088318419	0.599617088316773	1.64602E-12	0.0000000002745
0.9	0.603430340259401	0.603430340257719	1.68199E-12	0.0000000002787
0.92	0.607880958915486	0.607880958913769	1.71696E-12	0.0000000002825
0.94	0.612972164641479	0.612972164639727	1.75193E-12	0.0000000002858
0.96	0.618707114024888	0.618707114023101	1.78701E-12	0.0000000002888
0.98	0.625088901148601	0.625088901146779	1.8221E-12	0.0000000002915
1	0.632120558828558	0.632120558826702	1.85596E-12	0.0000000002936

Conclusion:

The conclusion drawn from this results of this research are as follows:

1. The result prove that the Runge – Kutta Fehlberg method is more rigorous than the Adams – Bashforth method in solving first order ordinary differential equations.
2. Result of this research supports the research in Runge – Kutta Fehlberg method & Adams – Bashforth method.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Gerardus Polla – Comparing Accuracy of differential equation Results between Runge – Kutta Fehlberg Method and Adams – Moulton Methods, Applied mathematical Science, 7 (2013), no. 103, 5115-5127.

[2] R.Lakshmi, M.Muthuselvi ―Numerical Solution for Boundary Value Problem Using Finite Difference Method, International Journal of Innovative Research in Science, Engineering and Technology, 2 (2013), 5305- 5313.

[3] Goni, Umar M. Modu and Muktar Modu ―Numerical Solution of Ordinary Differential Equations (O.D.E) using Forth Order Runge-Kutta Method with initial Value Conditions (I.V.C), Journal of Physical Science and Innovation, ISSN: 2277-0119.

[4] P. N. Brown, G. D. Byrne, and A. C. Hindmarsh. VODE, a variable-coefficient ode solver. SIAM J. Sci. Stat. Comput., 10(5) (1989), 1038–1051.

[5] S.C. Chapra and R.P. Canale, Numerical Methods for Engineers, 2nd ed., Singapore: McGraw-Hill, Co.,1989.

[6] S.C. Chapra and R.P. Canale, Mrtode Numerik Untuk Teknik, Jakarta: Universitas Indonesia, 1991.

[7] Sankar Prasad Mondal, Sanhita Banerjee and Tapan Kumar Roy ―First Order Linear Homogeneous Ordinary Differential Equation in Fuzzy Environment, International Journal of Pure and Applied Sciences and Technology. 14(1) (2013), 16-26.

[8] Atkinson, K.E., and Han, W., and Steward, D., Numerical Solutions of Ordinary Differential equations, John Wiley & Sons, Inc., Hoboken, New Jersey, Canada, (1961).

[9] Butcher, J., C., Numerical Methods for Ordinary Differential Equations, John Wiley & Sons, The University of Auckland, New Zealand, (1933).

[10] SA Aga, FE Ekpenyong – Upgrading Runge – Kutta Fehlberg Method(RKFM) for Second Order Ordinary Differential Equations, International Journal of Science and Technology, 3 (2013), 258-263.

[11] R.B. Srivastaav, Najmuddin Khan and S. Srivastav – Comparison of Numerical Accuracy of Runge – Kutta Forth order Method and Heun's Method, Vol. 15(Poorvanchal academy of Sciences 2009) ISSN 0972 – 3498.

[12] B.S. Grewal, Numerical Methods in Engineering and Science 10th ed. Khanna Publihers – 2014.

[13] Srimanta pal Numerical Methods principal analysis and algorithm Oxford publication 1st 2009.