Available online at http://scik.org
J. Math. Comput. Sci. 6 (2016), No. 6, 1074-1084
ISSN: 1927-5307

THE ALEKSANDROV-RASSIAS PROBLEM ON QUASI CONVEX N-NORMED LINEAR SPACES

XINKUN WANG*, MEIMEI SONG

College of Science, Tianjin University of Technology, Tianjin 300384, China

Abstract. We proved that the Aleksandrov-Rassias problem holds repalced the condition "$\| x_1 - y_1, \ldots, x_n - y_n \| \geq 1$ if and only if $\| f(x_1) - f(y_1), f(x_2) - f(y_2), \cdots, f(x_n) - f(y_n) \| \geq 1$ " in [7] by "$\| f(x_1) - f(y_1), f(x_2) - f(y_2), \cdots, f(x_n) - f(y_n) \| \leq \| x_1 - y_1, \ldots, x_n - y_n \|$ while $\| x_1 - y_1, \ldots, x_n - y_n \| \leq 1$" on Quasi Convex n-normed linear Spaces.

Keywords: Aleksandrov-Rassias problem; Mazur-Ulam theorem; generalized n-isometry.

2010 AMS Subject Classification: 46B20, 46B04.

1. Introduction

Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an isometry if f satisfies

$$d_Y(f(x), f(y)) = d_X(x, y)$$

for all $x, y \in X$, where $d_X(,)$ and $d_Y(,)$ denote the metric in the space X and Y, respectively. For some fixed number $r > 0$, suppose that f preserves distance r; ie, for all $x, y \in X$ with $d_X(x, y) = r$, we have $d_Y(f(x), f(y)) = r$. Then r is called a conservative distance for the mapping f. The

*Corresponding author

Received May 30, 2016

classical Mazur-Ulam theorem states that every surjective isometry between normed spaces is a linear mapping up to translation. In 1970, Aleksandrov [1] posed the following question : "Whether or not a mapping with distance one preserving property is an isometry? " It is called the *Aleksandrov problem*. The Aleksandrov problem has been investigated in several papers [2]-[15].

Rassias, ŠEMRL [9-13] proved a series of results on Aleksandrov problem on normed spaces. Chu et al., Park et al.[2-4] in linear n-normed spaces, defined the concept of an n-isometry that are suitable to represent the notion of a volumepreserving mapping, and generalized the Aleksandrov problem to n-normed spaces. Yumei Ma [7] proved the following theorem:

Theorem 1.1.[7] Let X and Y be two real n-normed linear spaces such that $dimX > n$. Suppose that $f: X \to Y$ is a surjective mapping satisfies n-SDOPP with $\| x_1 - y_1, \ldots, x_n - y_n \| \geq 1$ if and only if $\| f(x_1) - f(y_1), f(x_2) - f(y_2), \cdots, f(x_n) - f(y_n) \| \geq 1$. Then f preserves any integer k in two direction.

In this paper, We proved that the Aleksandrov-Rassias problem holds repalced the condition "$\| x_1 - y_1, \ldots, x_n - y_n \| \geq 1$ if and only if $\| f(x_1) - f(y_1), f(x_2) - f(y_2), \cdots, f(x_n) - f(y_n) \| \geq 1$ " in [13] by "$\| f(x_1) - f(y_1), f(x_2) - f(y_2), \cdots, f(x_n) - f(y_n) \| \leq \| x_1 - y_1, \ldots, x_n - y_n \|$ while $\| x_1 - y_1, \ldots, x_n - y_n \| \leq 1$" on Quasi Convex n-normed linear Spaces.

2. Preliminaries

In the remainder of this introduction, we will recall some definitions and give some Lemmas about them in quasi convex n-normed linear space.

Definition 2.1. Let X be a real linear space that has dimension greater than one and $\| \cdot, \ldots, \cdot \|$ be a function from X^n into R. Then ($X, \| \cdot, \ldots, \cdot \|$) is called a quasi convex n-normed linear space if

(a) $\| x_1, \ldots, x_n \| = 0 \Leftrightarrow x_1, \ldots, x_n$ are linearly dependent.

(b) $\| x_1, \ldots, x_n \| = \| x_{j_1}, \ldots, x_{j_n} \|$ for every permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$.

(c) $\| \alpha x_1, \ldots, x_n \| = | \alpha | \| x_1, \ldots, x_n \|$.

(d) $\| tx + (1-t)y, x_2, \ldots, x_n \| \leq max\{\| x, x_2, \ldots, x_n \|, \| y, x_2, \ldots, x_n \|\}$.

for any $\alpha \in R, t \in [0,1]$ and $x, y, x_1, \ldots, x_n \in X$. The function $\| \cdot, \ldots, \cdot \|$ is called the quasi convex n-norm on X.

From now on, let X and Y be quasi convex n-normed linear space and the mapping $f : X \to Y$.

Definition 2.2.[12] A mapping $f : X \to Y$ is said to be an n-isometry if for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$, it satifies $\| x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n \| = \| f(x_1) - f(y_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n) \|$.

Definition 2.3.[12] A mapping $f : X \to Y$ satisfies the n-distance one preserving property (briefly n-DOPP), if for all $x_i, y_i \in X$, $i = 1, 2, \ldots, n$, $\| x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n \| = 1$ implies $\| f(x_1) - f(y_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n) \| = 1$.

Definition 2.4.[12] A mapping $f : X \to Y$ satisfies the n-strong distance one preserving property (briefly n-SDOPP), if for all $x_i, y_i \in X$, $i = 1, 2, \ldots, n$, $\| x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n \| = 1$ implies $\| f(x_1) - f(y_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n) \| = 1$ and conversely.

Definition 2.5.[3] The points $x_0, x_1, \ldots, x_n$ of E are said to be n-collinear, if $\{x_i - x_j \mid 0 \leq i \neq j \leq n\}$ is linearly dependent.

Definition 2.6.[4] We say that a mapping $f : X \to Y$ preserves 2-collinearity, if $x, y, z \in X$ are collinear, then $f(x), f(y), f(z)$ are collinear.

Lemma 2.7. Let X be a quasi convex n-nomed linear space with $dimX > n$, for $y_i, x_i \in X, t_i > 0, \sum_{i=1}^n t_i = 1 (i = 1, 2, \cdots, n)$, we have $\| \sum_{i=1}^n t_i y_i, x_2, \ldots, x_n \| \leq max\{\|y_i, x_2, \ldots, x_n\| : i = 1, 2, \cdots, n\}$.

Proof. If $n = 2$, then $\|t_1 y_1 + t_2 y_2, x_2, \ldots, x_n\| \leq \max\{\|y_1, x_2, \ldots, x_n\|, \|y_2, x_2, \ldots, x_n\|\}$.
Assume that

$$\| \sum_{i=1}^{k-1} t_i y_i, x_2, \ldots, x_n \| \leq max\{\| y_1, x_2, \ldots, x_n \|, \| y_2, x_2, \ldots, x_n \|, \cdots, \| y_{k-1}, x_2, \ldots, x_n \|\}.$$

Let $n = k$, we can obtain

$$\begin{aligned} \| \sum_{i=1}^{k} t_i y_i, x_2, \ldots, x_n \| &= \| \sum_{i=1}^{k-1} t_i y_i + t_k y_k, x_2, \ldots, x_n \| \\ &= \| \sum_{i=1}^{k-1} t_i (\frac{\sum_{i=1}^{k-1} t_i y_i}{\sum_{i=1}^{k-1} t_i}) + t_k y_k, x_2, \ldots, x_n \| \end{aligned}$$

$$
\begin{aligned}
&\leq \quad max\{\| \frac{\sum_{i=1}^{k-1} t_i y_i}{\sum_{i=1}^{k-1} t_i}, x_2, \ldots, x_n \|, \| y_k, x_2, \ldots, x_n \|\} \\
&\leq \quad max\{\| y_1, x_2, \ldots, x_n \|, \| y_2, x_2, \ldots, x_n \|, \\
&\cdots \quad , \| y_{k-1}, x_2, \ldots, x_n \|, \| y_k, x_2, \ldots, x_n \|\}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\| \sum_{i=1}^{n} t_i y_i, x_2, \ldots, x_n \| \leq max\{\| y_1, x_2, \ldots, x_n \|, \| y_2, x_2, \ldots, x_n \|, \\
\cdots, \| y_{n-1}, x_2, \ldots, x_n \|, \| y_n, x_2, \ldots, x_n \|\}
\end{aligned}
$$

i.e.

$$\| \sum_{i=1}^{n} t_i y_i, x_2, \ldots, x_n \| \leq max\{\|y_i, x_2, \ldots, x_n\| : i = 1, 2, \cdots, n\}.$$

3. Main results

Lemma 3.1. *Let X and Y be two real quasi convex n-normed linear spaces , if $f : X \to Y$ satisfies (n-DOPP) and*

$$\| f(x_1) - f(y_1), f(x_2) - f(y_2), \cdots, f(x_n) - f(y_n) \| \leq \| x_1 - y_1, x_2 - y_2, \cdots, x_n - y_n \|$$

for $\| x_1 - y_1, x_2 - y_2, \cdots, x_n - y_n \| \leq 1$. *then f satisfies*

$$\| f(x_1) - f(y_1), f(x_2) - f(y_2), \cdots, f(x_n) - f(y_n) \| = \| x_1 - y_1, x_2 - y_2, \cdots, x_n - y_n \|$$

for $\| x_1 - y_1, x_2 - y_2, \cdots, x_n - y_n \| \leq 1$, *and for all* $x_i, y_i \in X, i = 1, 2, \cdots, n$

$$\| f(x_1) - f(y_1), f(x_2) - f(y_2), \cdots, f(x_n) - f(y_n) \| \leq \| x_1 - y_1, x_2 - y_2, \cdots, x_n - y_n \| .$$

Proof. (1) Firstly, we proof that f preserves 2-collinearity. If $\| x_1 - y_1, x_2 - y_2, \cdots, x_n - y_n \| = 0$, then

$$\| f(x_1) - f(y_1), f(x_2) - f(y_2), \cdots, f(x_n) - f(y_n) \| = 0$$

that is

$$\| f(x_1)-f(y_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| = \| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \| .$$

Let $y_1 = y_2$, then $f(x_1)-f(y_1)$ and $f(x_2)-f(y_1)$ are linearly dependent. So we acquire that f preserves 2-collinearity.

(2)Secondly, we prove that

$$\| f(x_1)-f(y_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| = \| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \|$$

for $0 < \| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \| \leq 1$.

Suppose $\| f(x_1)-f(y_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| < \| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \|$,

let $\omega = y_1 + \frac{x_1-y_1}{\|x_1-y_1, x_2-y_2, \cdots, x_n-y_n\|}$, then $\| \omega-y_1, x_2-y_2, \cdots, x_n-y_n \| = 1$ and

$$\| \omega-x_1, x_2-y_2, \cdots, x_n-y_n \| = 1 - \| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \| .$$

Hence $\| f(\omega)-f(y_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| = 1$ and

$$\| f(\omega)-f(x_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| \leq 1 - \| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \| .$$

On the other hand, since f preserves 2-collinearity, there exists a real number α such that

$$f(\omega)-f(y_1) = \alpha(f(x_1)-f(y_1))$$

and

$$f(\omega)-f(x_1) = (\alpha-1)(f(x_1)-f(y_1))$$

We have

$$\begin{aligned}
& \| f(\omega)-f(y_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| \\
= \ & |\alpha| \, \| (f(x_1)-f(y_1)), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| \\
\leq \ & |\alpha-1| \, \| (f(x_1)-f(y_1)), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| \\
& + \| (f(x_1)-f(y_1)), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| \\
= \ & \| (|\alpha-1|)(f(x_1)-f(y_1)), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| \\
& + \| (f(x_1)-f(y_1)), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| \\
= \ & \| f(\omega)-f(x_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| \\
& + \| (f(x_1)-f(y_1)), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| \\
< \ & 1- \| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \| + \| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \| = 1.
\end{aligned}$$

This contradicts the equality $\| f(\omega)-f(y_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| = 1$, Hence

$$\| f(x_1)-f(y_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| = \| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \|$$

for all $0 < \| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \| \leq 1$.

(3)Finally, we prove that

$$\| f(x_1)-f(y_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| \leq \| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \|$$

for all $x_i, y_i \in X, i = 1, 2, \cdots, n$. We can find two positive integers m, n with $\| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \| \leq \frac{m}{n}$. If $m = 1$, the result is obvious. We suppose that $m \geq 2$. Define

$$z_i = y_2 + \frac{i}{m}(x_2-y_2), i = 0, 1, \cdots, m$$

then for $i = 0, 1, \cdots, m$, we have

$$\begin{aligned}
\| x_1-y_1, z_{i+1}-z_i, \cdots, x_n-y_n \| &= \| x_1-y_1, \frac{1}{m}(x_2-y_2), \cdots, x_n-y_n \| \\
&= \frac{1}{m} \| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \| \leq \frac{1}{n}
\end{aligned}$$

$$
\begin{aligned}
& \| f(x_1)-f(y_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| \\
= \; & \| f(x_1)-f(y_1), \sum_{i=0}^{m-1} f(z_{i+1})-f(z_i), \cdots, f(x_n)-f(y_n) \| \\
= \; & m \| f(x_1)-f(y_1), \sum_{i=0}^{m-1} \frac{1}{m} f(z_{i+1})-f(z_i), \cdots, f(x_n)-f(y_n) \| \\
\leq \; & m \cdot max\{\| f(x_1)-f(y_1), f(z_1)-f(z_0), \cdots, f(x_n)-f(y_n) \|, \\
& \| f(x_1)-f(y_1), f(z_2)-f(z_1), \cdots, f(x_n)-f(y_n) \|, \cdots, \\
& \| f(x_1)-f(y_1), f(z_m)-f(z_{m-1}), \cdots, f(x_n)-f(y_n) \|\} \\
= \; & m \cdot max\{\| x_1-y_1, z_1-z_0, \cdots, x_n-y_n \|, \| x_1-y_1, z_2-z_1, \cdots, x_n-y_n \|, \cdots, \\
& \| x_1-y_1, z_m-z_{m-1}, \cdots, x_n-y_n \|\} \leq \frac{m}{n},
\end{aligned}
$$

Thus we obtain

$$\| f(x_1)-f(y_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| \leq \| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \|$$

Theorem 3.2. *Let X and Y be two real quasi convex n-normed linear spaces such that one of them has dimension greater than n. Suppose that $f : X \to Y$ is a surjective mapping satisfies n-SDOPP with $\| f(x_1)-f(y_1), f(x_2)-f(y_2), \cdots, f(x_n)-f(y_n) \| \leq \| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \|$ for $\| x_1-y_1, x_2-y_2, \cdots, x_n-y_n \| \leq 1$, then f is a injective mapping satisfies*

$$(1) \qquad |\| f(x_1)-f(y_1), \cdots, f(x_n)-f(y_n) \| - \| x_1-y_1, \cdots, x_n-y_n \||< 1$$

for all $x_i, y_i \in X, i = 1, 2, \cdots, n$. Morveover, f preserves any positive integer k in two directions.

Proof. (1) Firstly, We show that both spaces have dimension greater than n, Let us first assume that $dimY > n$. It follows that there exists vector $x_i^{'}, y_i' \in Y, i = 1, 2, \cdots, n$ such that $\| x_1' - y_1', x_2' - y_2', \cdots, x_n' - y_n' \| = 1$, since f is given to be surjective and preserve distance one in both directions, thus we can find $f^{-1}(x_i') = x_i, f^{-1}(y_i') = y_i \in X, i = 1, 2, \cdots, n$ such that $\| x_1 - y_1, x_2 - y_2, \cdots, x_n - y_n \| = 1$. This implies that $dimX > n$. Similarly, one can prove that if $dimX > n$ then $dimY > n$.

(2)Secondly, we shall show that f is injective. Since $dimX > n$, for any $x_0, x_1 \in X$ with $x_0 \neq x_1$, it follows that exists vector $x_2, \cdots, x_n$ such that $\| x_1 - x_0, x_2 - x_0, \cdots, x_n - x_0 \| = 1$. because of f satisfies n-SDOPP, thus $\| f(x_1) - f(x_0), f(x_2) - f(x_0), \cdots, f(x_n) - f(x_0) \| = 1$. This implies

$f(x_1) \neq f(x_0)$, so we prove f is injective.

In the sequel, we shall need the following notions:

$$K(x,r) = \{(y_1,y_2,\cdots,y_n) : \| y_1-x_1, y_2-x_2, \cdots, y_n-x_n \| < r\}$$

$$\overline{K}(x,r) = \{(y_1,y_2,\cdots,y_n) : \| y_1-x_1, y_2-x_2, \cdots, y_n-x_n \| \leq r\}$$

$$C_x(k,k+1] = \{(y_1,y_2,\cdots,y_n) : k < \| y_1-x_1, y_2-x_2, \cdots, y_n-x_n \| \leq k+1\}$$

for $x = \{x_1,x_2,\cdots,x_n\}, x_i \in X$.

Let $x_1,x_2,\cdots,x_n$ be arbitrary vector in X and k be an any positive integer with $k \geq 2$. Assume that $(y_1,y_2,\cdots,y_n) \in \overline{K}(x,k)$. According to Lemma 2.1 we know $(f(y_1), f(y_2), \cdots, f(y_n)) \in \overline{K}(f(x),k)$ for $f(x) = \{f(x_1), f(x_2), \cdots, f(x_n)\}$. Therefore,

$$f(\overline{K}(x,k)) \subset \overline{K}(f(x),k).$$

The same result can be obtained for f^{-1}. Hence,

$$f(\overline{K}(x,k)) = \overline{K}(f(x),k),$$

for $x = \{x_1,x_2,\cdots,x_n\}, x_i \in X, k \in N \setminus \{1\}$.

However, f is bijective and thus

$$f(C_x(k,k+1]) = C_{f(x)}(k,k+1], \tag{2}$$

Fix $x_1,x_2,\cdots,x_n \in X$, choose $y_1,y_2,\cdots,y_n$ satisfies $(y_1,y_2,\cdots,y_n) \in C_x(1,2]$ and also we know $(f(y_1), f(y_2), \cdots, f(y_n)) \in \overline{K}(f(x),2)$.

Let $z = y_1 + \frac{y_1-x_1}{\|y_1-x_1,y_2-x_2,\cdots,y_n-x_n\|}$, then

$$\begin{aligned}
2 &< \| z-x_1, y_2-x_2, \cdots, y_n-x_n \| \\
&= (1+\frac{1}{\| y_1-x_1, y_2-x_2, \cdots, y_n-x_n \|}) \| y_1-x_1, y_2-x_2, \cdots, y_n-x_n \| \\
&= 1+ \| y_1-x_1, y_2-x_2, \cdots, y_n-x_n \| \leq 3
\end{aligned}$$

thus the vector$(z,y_2,\cdots,y_n)\in C_x(2,3]$. According to (2) we have $(f(z),f(y_2),\cdots,f(y_n))\in C_{f(x)}(2,3]$, thus

$$\| f(z)-f(x_1),f(y_2)-f(x_2),\cdots,f(y_n)-f(x_n)\| > 2. \tag{3}$$

Let us assume that

$$\| f(y_1)-f(x_1),f(y_2)-f(x_2),\cdots,f(y_n)-f(x_n)\| \leq 1$$

Then we obtain

$$\begin{aligned}
& \| f(z)-f(x_1),f(y_2)-f(x_2),\cdots,f(y_n)-f(x_n)\| \\
= \;& 2\| \frac{1}{2}(f(z)-f(y_1))+\frac{1}{2}(f(y_1)-f(x_1)),f(y_2)-f(x_2),\cdots,f(y_n)-f(x_n)\| \\
\leq \;& 2\cdot max\{\| f(z)-f(y_1),f(y_2)-f(x_2),\cdots,f(y_n)-f(x_n)\|,\| f(y_1)-f(x_1), \\
& f(y_2)-f(x_2),\cdots,f(y_n)-f(x_n)\|\}=2,
\end{aligned}$$

Which contradicts (3), we have proved that

$$f(C_x(1,2])\subset C_{f(x)}(1,2].$$

The same result holds for the mapping f^{-1}. Consequently, the relations

$$f(C_x(1,2])=C_{f(x)}(1,2] \text{ and } f(K(x,1))=K(f(x),1)$$

for all $x=\{x_1,x_2,\cdots,x_n\},x_i\in X$. This together with (2) implies the inequality(1).

For purpose of f preserves any positive integer k in both direcions, assume that f preserves distance $k(k>1)$ in both directions. Let $x_1,x_1,\cdots,x_1,y_1,y_1,\cdots,y_1$ be vectors in X such that $\| y_1-x_1,y_2-x_2,\cdots,y_n-x_n\|=k+1$. It follows that

$$\| f(y_1)-f(x_1),f(y_2)-f(x_2),\cdots,f(y_n)-f(x_n)\| \leq k+1$$

Since f is sujective we can find $\upsilon\in X$ such that

$$f(\upsilon)=f(x_1)+\frac{f(y_1)-f(x_1)}{\| f(y_1)-f(x_1),f(y_2)-f(x_2),\cdots,f(y_n)-f(x_n)\|},$$

thus

$$\| f(\upsilon)-f(x_1),f(y_2)-f(x_2),\cdots,f(y_n)-f(x_n)\|=1,$$

$$\| \upsilon - x_1, y_2 - x_2, \cdots, y_n - x_n \| = 1.$$

Assume $\| f(\upsilon) - f(y_1), f(y_2) - f(x_2), \cdots, f(y_n) - f(x_n) \| < k$, we get $\| \upsilon - y_1, y_2 - x_2, \cdots, y_n - x_n \| < k$, this together with $\| \upsilon - x_1, y_2 - x_2, \cdots, y_n - x_n \| = 1$ we have

$$\begin{aligned} \| y_1 - x_1, y_2 - x_2, \cdots, y_n - x_n \| &= \| y_1 - \upsilon + \upsilon - x_1, y_2 - x_2, \cdots, y_n - x_n \| \\ &= 2 \cdot \| \frac{1}{2}(y_1 - \upsilon) + \frac{1}{2}(\upsilon - x_1), y_2 - x_2, \cdots, y_n - x_n \| \\ &\leq 2 \cdot max\{\| y_1 - \upsilon, y_2 - x_2, \cdots, y_n - x_n \|, \| \upsilon - x_1, \\ &\quad y_2 - x_2, \cdots, y_n - x_n \|\} < 2k, (k > 1). \end{aligned}$$

Which is a contradiction, thus $\| f(\upsilon) - f(y_1), f(y_2) - f(x_2), \cdots, f(y_n) - f(x_n) \| \geq k$. This implies

$$\begin{aligned} k &\leq \| f(\upsilon) - f(y_1), f(y_2) - f(x_2), \cdots, f(y_n) - f(x_n) \| \\ &= |\| f(x_1) - f(y_1), f(y_2) - f(x_2), \cdots, f(y_n) - f(x_n) \| - 1 | \leq k. \end{aligned}$$

Hence, $\| f(x_1) - f(y_1), f(y_2) - f(x_2), \cdots, f(y_n) - f(x_n) \| = k + 1$. The same proof shows that f^{-1} preserves distance $k+1$ as well.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgements

The authors also would like to express their appreciation to Professor Huang Xujian of Tianjin University of Technology for a careful reading and many very helpful suggestions for the improvement of the original.

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