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GAUSS MAP SINGULARITIES OF INVERSION HYPERSURFACE IN R^{n+1}

S. A. HASSAN[1,*], E. DAHY[2]

[1]Department of Mathematics, Assiut University, Assiut 71516, Egypt

[2]Department of Mathematics, Al-Azhar University, Assiut 71524, Egypt

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Abstract. This paper mainly studies the singularities of Gauss map of inversion hypersurface in R^{n+1}. The geometry of inversion hypersurface in R^{n+1} and its Gauss map are given. Using the lagrangian and Legendrian singularity theory, the singularities of Gauss map of inversion hypersurface are classified and plotted.

Keywords: extrinsic differential geometry, hypersurfaces, inversion hypersurfaces, Gauss map, lagrangian and Legendrian singularities.

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1. Geometry of inversion hypersurfaces in Euclidean space

In this section we review the classical theory of differential geometry on hypersurfaces in Euclidean space R^{n+1} [1, 13].

Let $X : U \to R^{n+1}$ be an embedding open subset of the Euclidian space R^n. Identify M and U through the embedding X, i.e., $M = X(U)$, in this case M is called hypersurface in R^{n+1}. The

*Corresponding author

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tangent space of M at $p = X(u)$, $u \in U$ is

$$T_pM = \langle X_1(u), X_2(u), ..., X_n(u) \rangle. \quad , X_i = \frac{\partial X}{\partial u_i}, \tag{1.1}$$

and the unit normal vector field along $X : U \to R^{n+1}$ is given by:

$$N(u) = \frac{X_1(u) \times X_2(u) \times ... \times X_n(u)}{\|X_1(u) \times X_2(u) \times ... \times X_n(u)\|}, \tag{1.2}$$

where

$$X_1 \times X_2 \times ... \times X_n = \begin{vmatrix} e_1 & e_2 & \cdots & e_{n+1} \\ X_1^1 & X_2^1 & \cdots & X_{n+1}^1 \\ X_1^2 & X_2^2 & \cdots & X_{n+1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ X_1^n & X_2^n & \cdots & X_{n+1}^n \end{vmatrix},$$

where $\{e_1, \ldots, e_{n+1}\}$ is the canonical basis of the Euclidian space R^{n+1} and

$$X_i = (X_i^1, X_i^2, \ldots, X_i^{n+1}) \in T_pM \subset R^{n+1}.$$

A map $G : U \to S^{n+1}$ defined by $G(u) = N(u)$ is called the Gauss map of $M = X(U)$, and the derivative of the Gauss map $dG(u) : T_pM \to T_pM$ can be interpreted as a liner transformation on the tangent space T_pM .The linear transformation $S_p = -dG(u)$ is called the shape operator (or Weingarten map) of the hypersurface $M = X(U)$. The eigenvalues of S_p are called the principal curvatures, and the eigenvectors of S_p are called the principal directions on M. By definition, k_p is a principal curvature if and only if $det(S_p - k_pI) = 0$. The Gauss-Kronecker curvature of $M = X(U)$ at $p = X(u)$ is defined to be $K(u) = detS_p$. Since the set $\{X_i | (i = 1, ..., n)\}$ is

linearly independent, the Riemannian metric (first fundamental form) on $M = X(U)$ is given by $ds^2 = \sum_{i=1}^n g_{ij} du_i du_j$, where $g_{ij} = \langle X_i(u), X_j(u) \rangle$ are first fundamental coefficients for any $u \in U$. The second fundamental coefficients l_{ij} are given by $l_{ij} = \langle -N_i(u), X_j(u) \rangle = \langle N(u), X_{ij}(u) \rangle$, for any $u \in U$. Recall the following Weingarten formula [13]:

$$N_i(u) = -l_i^j(u)X_j(u), \tag{1.3}$$

where $l_i^j(u) = l_{ik}(u)g^{kj}(u)$, $g^{kj}(u) = (g_{kj}(u))^{-1}$ and $g_{ik}(u)g^{kj}(u) = \delta_i^j$.
By the Weingarten formula, the Gauss-Kronecker curvature is given by

$$K(u) = \frac{\det(l_{ij})}{\det(g_{\alpha\beta})}. \tag{1.4}$$

The point $p = X(u) \in M$ is a parabolic point if $K(u) = 0$.

The inversion hypersurface of $M = X(U)$ with respect to a point $q \in R^n$ with inversion radius ρ is the map [2]:

$$\bar{X} : U \to R^{n+1}, \quad \bar{X}(u) = q + \frac{\rho^2(X(u)-q)}{\|X(u)-q\|^2},$$

where $\bar{M} = \bar{X}(U)$. The tangent space of $\bar{M}$ at $p = \bar{X}(u)$ is

$$\begin{aligned} T_p(\bar{M}) = \frac{\rho^2}{\|X(u)-q\|^2}(\langle X_1(u) - &\frac{2\langle X_1(u), X(u)-q\rangle(X(u)-q)}{\|X(u)-q\|^2}, \\ X_2(u) - &\frac{2\langle X_2(u), X(u)-q\rangle(X(u)-q)}{\|X(u)-q\|^2}, \\ &\cdots, \\ X_n(u) - &\frac{2\langle X_n(u), X(u)-q\rangle(X(u)-q)}{\|X(u)-q\|^2}\rangle). \end{aligned} \tag{1.5}$$

The unit normal vector field along $\bar{X} : U \to R^{n+1}$ is given by :

$$\bar{N}(u) = -N(u) + \frac{2\langle X(u)-q, N(u)\rangle(X(u)-q)}{\|X(u)-q\|^2}. \tag{1.6}$$

Since the set $\{\bar{X}_i|(i = 1, ..., n)\}$ is linearly independent, the Riemannian metric (first fundamental form) on $\bar{M} = \bar{X}(U)$ is defined as:

$$ds^2 = \sum_{i=1}^{n} \bar{g}_{ij} du_i du_j = \left(\frac{\rho}{\|X(u)-q\|} \right)^4 \sum_{i=1}^{n} g_{ij} du_i du_j,$$

where $\bar{g}_{ij} = \langle \bar{X}_{u_i}(u), \bar{X}_{u_j}(u) \rangle = \left(\frac{\rho}{\|X(u)-q\|} \right)^4 g_{ij}$ are the first fundamental coefficients for inversion hypersurface, and $\bar{g}$ on inversion Hypersurface is given by the relation:

$$\bar{g} = \left(\frac{\rho}{\|X(u)-q\|} \right)^{4n} g, \quad g = \det(g_{ij}). \tag{1.7}$$

The second fundamental invariants are given by

$$\bar{l}_{ij} = \left(\frac{\rho^2 l_{ij}}{\|X(u)-q\|^2} + \frac{2\rho^2 \langle X(u)-q, N(u) \rangle g_{ij}}{\|X(u)-q\|^4} \right), \quad \forall \quad u \in U,$$

for any $u \in U$. We have the following Weingarten formula:

$$\bar{N}_i(u) = -\bar{l}_i^j(u) \bar{X}_j(u) = \frac{1}{\rho^2} \left(\|X(u)-q\|^2 l_i^j + 2\langle X(u)-q, N(u) \rangle \delta_{ij} \right) \bar{X}_{u_j}(u), \tag{1.8}$$

where $\bar{l}_i^j(u) = \bar{l}_{ik}(u) \bar{g}^{kj}(u)$ and $\bar{g}^{kj}(u) = (\bar{g}_{kj}(u))^{-1}$. By the Weingarten formula, the Gauss-Kronecker curvature is given by:

$$\bar{K}(u) = \frac{\det(\bar{l}_{ij})}{\det(\bar{g}_{\alpha\beta})}. \tag{1.9}$$

From equation (1.8) it is easy to see that the Gauss-Kronecker curvature can be written as:

$$\bar{K}(u) = \frac{1}{\rho^2} \prod_{i=1}^{n} \left(\|X(u)-q\|^2 l_i^j + 2\langle X(u)-q, N(u) \rangle \delta_{ij} \right), \tag{1.10}$$

and the mean curveture is given by

$$\bar{H}(u) = \frac{1}{\rho^2} \left(\|X(U)-q\|^2 H(u) + 2\langle X(u)-q, N(u) \rangle \right). \tag{1.11}$$

For an inversion hypersurface $\bar{X}: U \to R^{n+1}$ the point $q = \bar{X}(u) \in \bar{M}$ is a parabolic point if $\bar{K}(u) = 0$ [13].

Proposition 1. *Let $\bar{X}: U \to R^{n+1}$ is the inversion hypersurface to M, $q = \underline{0}$, and the support function $S(u) = \langle X(u), N(u)\rangle$ to original hypersurfaces is zero then we have:*
1) $\bar{N}(u) = -N(u)$.
2) the support function of $\bar{M}$ is zero.

Proof. 1) From equation (1.6) by putting $q = \underline{0}$ we fined

$$\bar{N}(u) = -N(u) + \frac{2\langle X(u), N(u)\rangle (X(u))}{\|X(u)\|^2} = -N(u) + \frac{2S(u)X(u)}{\|X(u)\|^2},$$

and let $S(u) = \langle X(u), N(u)\rangle = 0$. so we fined

$$\bar{N}(u) = -N(u).$$

2) From above we fined :

$$\bar{N}(u) = -N(u),$$

and from the definition of the inversion hypersurface by putting $q = \underline{0}$ we fined:

$$\bar{X}(u) = \frac{\rho^2 X(u)}{\|X(u)\|^2}.$$

so

$$\bar{S}(u) = \langle \bar{X}(u), \bar{N}(u)\rangle = \langle \frac{\rho^2 X(u)}{\|X(u)\|^2}, -N(u)\rangle = \frac{-\rho^2 S(u)}{\|X(u)\|^2} = 0.$$

Proposition 2. [13]*: Suppose that $\bar{M} = \bar{X}(U)$ is totally umbilic, then $\bar{k}_p$ is constant $\bar{k}$. Under this condition, we have the following classification:*
1) If $\bar{k} \neq 0$, then $\bar{M}$ is a part of a hypersphere.
2) If $\bar{k} = 0$, then $\bar{M}$ is a part of a hyperplane.

Proposition 3. [13]*: Let $\bar{M} = \bar{X}(U)$ be a hypersurface in R^{n+1}.Then following are equivalent:*
(1) $\bar{M}$ is totally umbilic with $\bar{K} = 0$.
(2) The Gauss map is a constant map.
(3)$\bar{M}$ is a part of a hyperplane.

2. Height functions on inversion Hypersurface

In this section we discuss the properties of important family of function on the inversion hypersurface.

The Height function $\bar{h}$ is define as the following [12, 14]:

$$\bar{h}(u,v) : U \times S^n \to R$$

by $\quad \bar{h}(u,v) = \langle \bar{X}(u), v\rangle = \bar{h}_v$ on $\bar{M} = \bar{X}(u), \forall u \in U \subset R^n, v \in S^n$.

These families of functions are introduced by Thom [3, 4, 7] for the study of parabolic points and umbilical points.

Proposition 4. *: Let $\bar{X} : U \to R^{n+1}$ be an inversion hypersurface. Then*
(1) $\bar{h}(u,v) = 0$ if and only if $v = \pm\left(\bar{N}(u) - \frac{\bar{S}(u)\bar{X}_i(u)}{\langle \bar{X}(u),\bar{X}_i(u)\rangle}\right)$ where $\bar{S}(u) = \langle \bar{X}(u),\bar{N}(u)\rangle$ is the support function of inversion hypersurface.
(2) $\frac{\partial \bar{h}_v}{\partial u_i} = 0$, $(i = 1,2,\ldots,n)$ if and only if $v = \pm\bar{N}(u)$

Proof. (1) since $\{\bar{N}(u),\bar{X}_i(u)\}$, $i = 1,2,...,n$ is a basis of the vector space $\bar{T}_pR^{n+1}$ where $p = \bar{X}(u)$ then, there exist a real numbers α, α_i, $i = 1,2,...,n$ such that v can be written as linear combination of the base as the following:

$$v = \alpha\bar{N}(u) + \alpha_i\bar{X}_i,$$

and since $\bar{h}(u,v) = 0$ then:

$$\langle \bar{X}(u), v\rangle = \langle \bar{X}(u), \alpha\bar{N}(u) + \alpha_i\bar{X}_i\rangle = 0.$$

Thus we have:

$$\alpha_i = -\alpha\left(\frac{\langle \bar{X}(u),\bar{N}(u)\rangle}{\langle \bar{X}(u),\bar{X}_i(u)\rangle}\right) = -\alpha\left(\frac{\bar{S}(u)}{\langle \bar{X}(u),\bar{X}_i(u)\rangle}\right)$$

$$v = \alpha\left(\bar{N}(u) - \frac{\bar{S}(u)\bar{X}_i(u)}{\langle \bar{X}(u),\bar{X}_i(u)\rangle}\right) = \pm\left(\bar{N}(u) - \frac{\bar{S}(u)\bar{X}_i(u)}{\langle \bar{X}(u),\bar{X}_i(u)\rangle}\right)$$

(2) since $\frac{\partial \bar{h}_v}{\partial u_i} = 0$ and $v = \alpha\bar{N}(u) + \alpha_j\bar{X}_j$ then:

$$\langle \bar{X}_{u_i}(u), v\rangle = \langle \bar{X}_i(u), \alpha\bar{N}(u) + \alpha_j\bar{X}_j\rangle = 0$$

$$\alpha\langle \bar{X}_i(u),\bar{N}(u)\rangle + \alpha_j\langle \bar{X}_i(u),\bar{X}_j\rangle = 0$$

$$\alpha_j g_{ij} = 0 \qquad \Rightarrow \alpha_j = 0 \qquad\qquad \text{So}$$

$$v = \alpha \bar{N}(u) = \pm \bar{N}(u).$$

Proposition 5. *: Let $\bar{X} : U \times R^{n+1}$ be an inversion hypersurface, and $\bar{h}(u,v) : U \times S^n \to R$ be the hight function on inversion hypersurface, suppose $\bar{h}(u,v) = 0$, Then we have :*
(1) $v = \pm\left(\bar{N}(u) - \frac{\bar{S}(u)\bar{X}_i(u)}{\langle \bar{X}(u), \bar{X}_i(u)\rangle}\right)$ for $S(u) \neq 0 \quad on \quad M$.
(2) $v = \pm\bar{N}(u)$ for $S(u) = 0 \quad on \quad M$.

Proof. From proposition (4), as section 1.

We assume that the support function is not zero, From Proposition 4, the catastrophe set $C(\bar{h})$ of $\bar{h}$ is given as follows: [18, 19]

$$C(\bar{h}) = \{(u,v) \in U \times S^n | v = \pm\bar{N}(u)\}.$$

For $v = \pm\bar{N}(u)\}$ we have:

$$\frac{\partial^2}{\partial u_i \partial u_j}\bar{h}(u,v) = \mp\bar{l}_{ij}(u).$$

Therefore, for any $v = \bar{N}(u)$, $det(\mathscr{H}(\bar{h}_v)(u)) = det(\frac{\partial^2}{\partial u_i \partial u_j}\bar{h}(u,v))(u,v)) = 0$ if and only if $\bar{K}(p) = 0$ (i.e;$p = \bar{X}(u)$ is a parabolic point).

Proposition 6. *For any $p = \bar{X}(u)$ we have the following assertions, let $v = \bar{N}(u)$ then:*
(1) p is a parabolic point if and only if $det\mathscr{H}((\bar{h}_v)(u)) = 0$.
(2) p is flat point if rank $\quad \mathscr{H}(\bar{h}_v(u)) = 0$.

Corollary 7. *Let $\bar{h}(u,v) : U \times S^n \to R$ be the hight function on an inversion hypersurface $\bar{M} = \bar{X}(u)$ and $\bar{G}(u)$ is the Gauss map, $p = \bar{X}(u)$ and let $v = \pm\bar{N}(u) = \pm\bar{G}(u)$ then the following statements are equivalent:*
(1) $p \in \bar{M}$ is a degenerate singular point of $\bar{h}_v(u)$.
(2) $p \in \bar{M}$ is a singular point of $\pm\bar{G}(u)$.
(3) $\bar{K}(u) = 0$.

Also the family of functions [12, 14]

$$\widetilde{h} : U \times (S^n \times R) \to R, \quad \widetilde{h}(u,v,r) = \langle \bar{X}(u), v\rangle - r$$

is called the extended hight function of $\bar{M} = \bar{X}(u)$.

The catastrophe map of $\bar{h}(u,v)$ is define by:

$$\pi_{C(\bar{h})}(u, \pm\bar{N}(u)) = \pm\bar{N}(u) = \pm\bar{G}(u).$$

So we can identify the Gauss map of $\bar{M} = \bar{X}(u)$ with the plus component of the catastrophe map $\pi_{C(H)}$.

3. **Gauss Map of inversion hypersurface as Lagrangian and Legendrian maps**

In this section the singularities of Gauss map of inversion hypersurface using hight function (extended hight function) using the theory of Lagrangian (Legendrian) singularities as a caustics (a wave front) in the framework of symplectic (contact) geometry are obtained [11].

For the height function $\bar{h}$ of the inversion hypersurface $\bar{X} : U \to R^{n+1}$, We have the following:

Proposition 8. *The hight function $\bar{h} : U \times S^n \to R$ of $\bar{M} = \bar{X}(u)$ is Morse families of function.*

Proof. for any $v = (v_1, v_2, ..., v_{n+1}) \in S^n$ we have $v_1^2 + v_2^2 + ... + v_{n+1}^2 = 1$ let $v_{n+1} > 0$ then we have $v_{n+1} = \sqrt{1 - (v_1^2 + v_2^2 + ... + v_n^2)}$ so the hight function becomes

$$\bar{h}(u,v) = x_1(u)v_1 + x_2(u)v_2 + ... + x_n(u)v_n + x_{n+1}(u)\sqrt{1 - (v_1^2 + v_2^2 + ... + v_n^2)}$$

now should be prove the mapping

$$\triangle\bar{h} = \left(\frac{\partial\bar{h}}{\partial u_1}, \frac{\partial\bar{h}}{\partial u_2}, ..., \frac{\partial\bar{h}}{\partial u_n}\right)$$

is non singular at any point.

The jacobian matrix of $\triangle\bar{h}$ is given as follows:

$$J(\triangle\bar{h}) = D(\triangle\bar{h}) = \frac{\partial\left(\frac{\partial\bar{h}}{\partial u_1}, \frac{\partial\bar{h}}{\partial u_2}, ..., \frac{\partial\bar{h}}{\partial u_n}\right)}{\partial\,(u_1, u_2, ..., u_n, v_1, v_2, ..., v_n)} =$$

$$\begin{pmatrix} \langle \bar{X}_{11}, v\rangle & \cdots & \langle \bar{X}_{1n}, v\rangle & x_{1,1} - x_{n+1,1}\frac{v_1}{v_{n+1}} & \cdots & x_{n,1} - x_{n+1,1}\frac{v_n}{v_{n+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \bar{X}_{n1}, v\rangle & \cdots & \langle \bar{X}_{nn}, v\rangle & x_{1,n} - x_{n+1,n}\frac{v_1}{v_{n+1}} & \cdots & x_{n,n} - x_{n+1,n}\frac{v_n}{v_{n+1}} \end{pmatrix},$$

where $x_{i,\alpha} = \frac{\partial x_i}{\partial u_\alpha}$

will shown the rank of below matrix is n at $(u,v) \in C(\bar{h})$

$$A = \begin{pmatrix} x_{1,1} - x_{n+1,1}\frac{v_1}{v_{n+1}} & \cdots & x_{n,1} - x_{n+1,1}\frac{v_n}{v_{n+1}} \\ \vdots & \vdots & \vdots \\ x_{1,n} - x_{n+1,n}\frac{v_1}{v_{n+1}} & \cdots & x_{n,n} - x_{n+1,n}\frac{v_n}{v_{n+1}} \end{pmatrix}.$$

let $C_i = \begin{pmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,n} \end{pmatrix}, i = 1, 2, ..., n+1$

i.e., it should be prove that the rank of matrix

$$\tilde{A} = \left(C_1 - C_{n+1}\frac{v_1}{v_{n+1}}, C_2 - C_{n+1}\frac{v_2}{v_{n+1}}, ..., C_n - C_{n+1}\frac{v_n}{v_{n+1}} \right)$$

is n at $(u,v) \in C(\bar{h})$ so

$$det(\tilde{A}) = \begin{vmatrix} x_{1,1} - x_{n+1,1}\frac{v_1}{v_{n+1}} & \cdots & x_{n,1} - x_{n+1,1}\frac{v_n}{v_{n+1}} \\ \vdots & \vdots & \vdots \\ x_{1,n} - x_{n+1,n}\frac{v_1}{v_{n+1}} & \cdots & x_{n,n} - x_{n+1,n}\frac{v_n}{v_{n+1}} \end{vmatrix}$$

$$= \begin{vmatrix} x_{1,1} & \cdots & x_{n,1} & x_{n+1,1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,n} & \cdots & x_{n,n} & x_{n+1,1} \\ \frac{v_1}{v_{n+1}} & \cdots & \frac{v_n}{v_{n+1}} & \frac{v_{n+1}}{v_{n+1}} \end{vmatrix} = \frac{(-1)^n}{v_{n+1}} \begin{vmatrix} x_{1,1} & \cdots & x_{n,1} & x_{n+1,1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,n} & \cdots & x_{n,n} & x_{n+1,1} \\ v_1 & \cdots & v_n & v_{n+1} \end{vmatrix}$$

$$= \frac{(-1)^n}{v_{n+1}} \langle v, \bar{X}_1 \times \bar{X}_2 \times ... \times \bar{X}_n \rangle$$

$$= \frac{(-1)^n}{v_{n+1}} \langle \pm\bar{N}(u), \bar{N}(u)\sqrt{\bar{g}} \rangle = \frac{\pm\sqrt{\bar{g}}}{v_{n+1}} \neq 0$$

For $(u,v) = (u,\bar{N}(u)) \in C(\bar{h})$. This completes the proof of the proposition.

We can define a Lagrangian immersion germ whose generating family is the height function of $\bar{M} = \bar{X}(U)$ as follows [13, 18]:

For the $n-sphere \quad S^n$, we consider the local coordinate $U_i = v = (v_1, v_2, ..., v_{n+1}) \in S^n | v_i \neq 0$. Since $T^*S^n|U_i$ is a trivial bundle, we define a map

$$L_i(\bar{h}) : C(\bar{h}) \to T^*S^n|U_i (i = 1,2,...,n)$$

by

$$L_i(\bar{h})(u,v) = (v, x_1(u) - x_i(u)\frac{v_1}{v_i}, ..., \widehat{x_i(u) - x_i(u)}\frac{v_i}{v_i}, ..., x_n(u) - x_i(u)\frac{v_n}{v_i}),$$

where $(v_1, v_2, ..., v_{n+1}) \in S^n$ we denote $(x_1, ..., x_i, ..., x_{n+1})$ as a point in the n- dimensional space such that the i-th component x_i is removed.

By definition, and by analogous to the results in [12, 13, 14, 18] we have the following corollary of the above proposition:

Corollary 9. *Under the above notations, $L(\bar{h})$is a Lagrangian immersion such that the height function $\bar{h} : U \times S^n \to R$ of $\bar{M} = \bar{X}(U)$ is a generating family $L(\bar{h})$.*

Therefore, the plus component of the Lagrangian map $\pi \circ L(\bar{h})$ can be identified with the Gauss map of $\bar{M} = \bar{X}(U)$. We also call $L(\bar{h})$ the Lagrangian lift of the Gauss map $G : U \to S^n$ of $\bar{M} = \bar{X}(U)$.

On the other hand, we consider the extended height function $\widetilde{h} : U \times (S^n \times R) \to R$ of $\bar{M} = \bar{X}(U)$, We have the following proposition.

Proposition 10. *: The extended hight function $\widetilde{h} : U \times (S^n \times R) \to R$ of $\bar{M} = \bar{X}(u)$ is Morse families of function.*

Proof. The proof is the similar calculation as the case for the height function.

for any $v = (v_1, v_2, ..., v_{n+1}) \in S^n$ we have $v_1^2 + v_2^2 + ... + v_{n+1}^2 = 1$ let $v_{n+1} > 0$ then the extended hight function takes the form:

$$\widetilde{h}(u,v,r) = x_1(u)v_1 + x_2(u)v_2 + ... + x_n(u)v_n + x_{n+1}(u)\sqrt{1-(v_1^2+v_2^2+...+v_n^2)} - r$$

From Legendrian singularities then, one can prove that mapping

$$\triangle^*\widetilde{h} = \left(\widetilde{h}, \frac{\partial \widetilde{h}}{\partial u_1}, \frac{\partial \widetilde{h}}{\partial u_2}, ..., \frac{\partial \widetilde{h}}{\partial u_n}\right)$$

is non-singular at any point in $\sum_*(\widetilde{h}) = \triangle^*\widetilde{h}(0)$. The Jacobian matrix of $\triangle^*\widetilde{h}$ is given as follows:

$$\begin{pmatrix} \langle \bar{X}_1, v\rangle & \cdots & \langle \bar{X}_n, v\rangle & x_1 - x_{n+1}\frac{v_1}{v_{n+1}} & \cdots & x_n - x_{n+1}\frac{v_n}{v_{n+1}} & -1 \\ \langle \bar{X}_{11}, v\rangle & \cdots & \langle \bar{X}_{1n}, v\rangle & x_{1,1} - x_{n+1,1}\frac{v_1}{v_{n+1}} & \cdots & x_{n,1} - x_{n+1,1}\frac{v_n}{v_{n+1}} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \bar{X}_{n1}, v\rangle & \cdots & \langle \bar{X}_{nn}, v\rangle & x_{1,n} - x_{n+1,n}\frac{v_1}{v_{n+1}} & \cdots & x_{n,n} - x_{n+1,n}\frac{v_n}{v_{n+1}} & 0 \end{pmatrix}.$$

Using the some terminology used in proposition (8) it is easy to show that the rank of the matrix:

$$A = \begin{pmatrix} x_{1,1} - x_{n+1,1}\frac{v_1}{v_{n+1}} & \cdots & x_{n,1} - x_{n+1,1}\frac{v_n}{v_{n+1}} \\ \vdots & \vdots & \vdots \\ x_{1,n} - x_{n+1,n}\frac{v_1}{v_{n+1}} & \cdots & x_{n,n} - x_{n+1,n}\frac{v_n}{v_{n+1}} \end{pmatrix}$$

equal n at $(u,v,r) \in \sum_*(\widetilde{h})$. thus, we have the proof.

We can also define a Legendrian immersion germ whose generating family is the extended height function of $\bar{M} = \bar{X}(U)$ as follows (see [13, 18]). For the $n-sphere \quad S^n$, we we consider the local coordinate $U_i = v = (v_1, v_2, ..., v_{n+1}) \in S^n | v_i \neq 0$. Since $PT^*(S^n \times R)|(U_i \times R)$ is a trivial bundle, we define a map

$$\ell_i(\widetilde{h}) : \textstyle\sum_*(\widetilde{h})|U \times (U_i \times R) \to PT^*(S^n \times R)|(U_i \times R)(i = 1,2,...,n)$$

by

$$\ell_i(\widetilde{h})(u,v,r)(v,r,[x_1(u) - x_i(u)\frac{v_1}{v_i} : ... : \widehat{x_i(u) - x_i(u)}\frac{v_i}{v_i} : ... : x_n(u) - x_i(u)\frac{v_n}{v_i}) : -1]),$$

where $(v_1, v_2, ..., v_{n+1}) \in S^n$ we denote $(x_1, ..., x_i, ..., x_{n+1})$ as a point in the n- dimensional space such that the i-th component x_i is removed.

Therefore we can define a global Legendrian immersion, $\ell_i(\widetilde{h}) : \sum_*(\widetilde{h}) \to PT^*(S^n \times R)$.

By definition, and by analogous to the results in [12, 13, 14, 18] we have the following corollary of the above proposition:

Corollary 11. *Under the above notations, $L(\widetilde{h})$is a Legendrian immersion such that the extended height function $\widetilde{h}: U \times (S^n \times R) \to R$ of $\bar{M} = \bar{X}(U)$ is a generating family $\ell(\widetilde{h})$.*

Therefore, we have the Legendrian immersion $\ell(\widetilde{h})$ whose wave front is the cylindrical pedal of $\bar{M} = \bar{X}(U)$ We call $\ell(\widetilde{h})$ the Legendrian lift of the cylindrical pedal $CPe_{\bar{M}}$ of $\bar{M} = \bar{X}(U)$.

4. Contact with hypersurfaces

We start to review the theory of contact due to Montaldi [5, 6].

Let X_i, Y_i ,$(i = 1,2)$ be submanifolds of R^n with $dimX_1 = dimX_2$ and $dimY_1 = dimY_2$. We say that the contact of X_1 and Y_1 at y_1 is of the same type as the contact of X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\phi : (R^n, y_1) \to (R^n, y^2)$ such that $\phi(X_1) = X_2$ and $\phi(Y_1) = Y_2$. In this case we write $K(X_1,Y_1;y_1) = K(X_2,Y_2;y_2)$. It is clear that in the definition R^n could be replaced by any manifold.

Two function germs $g_1; g_2 : (R^n; a_i) \to (R; 0)(i = 1; 2)$ are K-equivalent if there are a diffeomorphism germ $\phi : (R^n; a_1) \to (R^n; a_2)$, and a function germ ?$\beta : (R^n; a_1) \to R$ with $\beta(a_1) \neq 0$ such that $g_1 = \beta.(g_2 \circ \phi)$.

In [5] Montaldi has shown the following theorem.

Theorem 12. *Let X_i, Y_i, $(i = 1,2)$ be submanifolds of R^n with $dimX_1 = dimX_2$ and $dimY_1 = dimY_2$. Let $g_i : (X_i, x_i) \to (R^n, y_i)$ be immersion germs and $f_i : (R_n, y_i) \to (R^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then $K(X_1,Y_1;y_1) = K(X_2,Y_2;y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are K-equivalent.*

On the other hand, we define the following functions:

$$\mathscr{H} : R^{n+1} \times S^n \to R; \qquad \mathscr{H}(x, v) = \langle x, v \rangle,$$

$$\widetilde{\mathscr{H}} : R^{n+1} \times (S^n \times R) \to R; \qquad \widetilde{\mathscr{H}}(x, v, r) = \langle x, v \rangle - r.$$

Now consider the contact of hypersurfaces with hyperplane. For any $v \in S^n$ we denote that $\mathfrak{h}_v(x) = \mathscr{H}(x, v)$ and we have a hyperplane $\mathfrak{h}_v^{-1}(r)$. We denote it as $\bar{h}(v, r)$. For any $u \in U$, we consider the unit normal vector $v=\bar{N}(u)$ and $r=\langle \bar{X}(u),\bar{N}(u)\rangle$, then we have

$$\mathfrak{h}_v \circ \bar{X}(u) = \mathscr{H} \circ (\bar{X} \times id_{S^n})(u,v) = \bar{h}(u,\bar{N}(u)) = r.$$

We have the relation

$$\frac{\partial \mathfrak{h}_v \circ \bar{X}(u)}{\partial u_i} = \frac{\partial \bar{h}}{\partial u_i}(u,\bar{N}(u)) = 0,$$

for $i=1,2,...,n$. This means that the hyperplane $\mathfrak{h}_v^{-1}(x)=\mathscr{H}(x,v)$ is tangent to $\bar{M}=\bar{X}(U)$ at $p=\bar{X}(u)$. so, $\bar{h}(v,r)$ is the tangent hyperplane of $\bar{M}=\bar{X}(U)$ at $p=\bar{X}(u)$ (or, u), which we write $\bar{h}(\bar{X}(U),u)$. Let v_1,v_2 be unit vectors. If v_1,v_2 are linearly dependent, then corresponding hyperplanes $\bar{h}(v_1,r_1),\bar{h}(v_2,r_2)$ are parallel. Then we have the following lemma:

Lemma 13. *Let $\bar{X}:U\to R^{n+1}$ be an inversion hypersurface. Consider two points $u_1,u_2\in U$. Then*
(1) $CPe\bar{M}(u_1)=CPe\bar{M}(u_2)$ if and only if $\bar{h}(\bar{X}(U),u_1)=\bar{h}(\bar{X}(U),u_2)$.
(2) $\bar{G}(u_1)=\bar{G}(u_2)$ if and only if $\bar{h}(\bar{X}(U),u_1),\bar{h}(\bar{X}(U),u_2)$ are parallel.

we call $(\bar{X}^{-1}(\bar{h}(\bar{X}(U),u)),u)$ the tangent indicatrix germ of $\bar{M}=\bar{X}(U)$ at u (or p).we can borrow some basic invariants from the singularity theory on function germs [8]. We can denote that:

$$T-ord(\bar{X}(U),u_0) = dim\frac{C^\infty_{u_0}(U)}{\langle \langle \bar{X}(u),\bar{N}(u_0)\rangle - r_0, \langle \bar{X}_i(u),\bar{N}(u_0)\rangle\rangle_{C^\infty_{u_0}}},$$

where $r_0=\langle \bar{X}(u_0),\bar{N}(u_0)\rangle$. $T-ord(\bar{X}(U),u_0)$ is called the K-codimension of $\widetilde{h}_(v_0,r_0)$. However, we call it the order of contact with the tangent hyperplane at $\bar{X}(u_0)$. We also have the notion of corank of function germs.

$$T-corank(\bar{X}(U),u_0) = n - rankHess(\bar{h}_{v_0}(u_0)),$$

where $v_0=\bar{N}(u_0)$.

By Proposition 7, $\bar{X}(u_0)$ is a parabolic point if and only if $T-corank(\bar{X}(U),u_0)\geq 1$. Moreover $\bar{X}(u_0)$ is a flat point if and only if $T-corank(\bar{X}(U),u_0)=n$.

On the other hand, a function germ $f:(R^n,a)\to R$ has the A_k-type singularity if and only if f is K-equivalent to the germ $x_1^{k+1}\pm x_2^2\pm\cdots\pm x_n^2$. If $T-corank(\bar{X}(U),u_0)=n-1$, the height function h_{v_0} has the A_k-type singularity at u_0 in generic. In this case we have

$T - ord(\bar{X}(U), u_0) = k$. This number is equal to the order of contact in the classical sense (cf., [9]). This is the reason why we call $T - ord(\bar{X}(U), u_0)$ the order of contact with the tangent hyperplane at $\bar{X}(u_0)$.

5. Inversion hypersurfaces in four space E^4

The classification of the singularities depends on the following theorem:

Theorem 14. *If the support function of the main hypersurface is not defined then the Gauss map of the main and the Inversion hypersurface has the same singular pint.*

Proof. From proposition 1 we have $\bar{N}(u) = -N(u)$.

We assume that the support function of the main hypersurface is defined,Using the classification singularities on stable Legendrian mappings which introduced by Thoms Elementary Catastrophes theorem:

Let $\bar{G} : (U, u_0) \to (R^4, v_0)$ be the Gauss map of an inversion hypersurface $\bar{X}$ and $h_{v_0} : (U, u_0) \to R$ be the height function germ at $v_0 = \bar{G}(u_0) = \bar{N}(u_0)$. Then we have the following theorems:

Theorem 15. *u_0 is a parabolic point of $\bar{X}$ if and only if $T - corank(\bar{X}(U), u_0) \geq 1$ (i.e.,u_0 is not a flat point of $\bar{X}$).*

If u_0 is a parabolic point of $\bar{X}$, then $\widetilde{h}_{(v_0,r_0)}$ has the A_k-type singularity for k = 2, 3, 4 or $D_{\pm 4}$ singularity where $\widetilde{h}_{(v_0,r_0)}(u) = h_{v_0}(u) - r_0$.

Theorem 16. *Assume u_0 is a parabolic point of $\bar{X}$. Then the following statements are equivalent:*

(a) The cylindrical pedal $CPe_{\bar{M}}$ has a cuspidaledge at u_0.

(b) $\widetilde{h}_{(v_0,r_0)}(u)$ has A_2-type singularity.

(c) $T - ord(\bar{X}(U), u_0) = 2$.

(d) Tangent indicatrix $(\bar{X}^{-1}(\bar{h}(X(U), u_0), u_0)$ is a surface $\subset R^3$, and it is diffeomorphic to the surface given by $\{(u, v, w) : u^3 \pm v^2 \pm w^2 = 0\}$.

(e) For each $\varepsilon > 0$, there exist two distinct points $u_1, u_2 \subset U$ such that $|u_0 - u_i| < \varepsilon$ for i =

1, 2, both of u_1, u_2 are not parabolic points and the tangent planes to $\bar{M} = \bar{X}(U)$ at u_1, u_2 are parallel.

(f) The Gauss map $\bar{G}$ is the fold at u_0.

Theorem 17. *Assume u_0 is a parabolic point of $\bar{X}$. Then the following statements are equivalent:*

(a) The cylindrical pedal $CPe_{\bar{M}}$ has a swallowtail. at u_0.

(b) $\widetilde{h}_{(v_0,r_0)}(u)$ has A_3-type singularity.

(c) $T-ord(\bar{X}(U), u_0) = 3$.

(d) Tangent indicatrix $(\bar{X}^{-1}(\bar{h}(X(U), u_0), u_0)$ is a surface $\subset R^3$, and it is diffeomorphic to the surface given by $\{(u,v,w) : u^4 \pm v^2 \pm w^2 = 0\}$.

(e) For each $\varepsilon > 0$, there exist two distinct points $u_1, u_2, u_3 \subset U$ such that $|u_0 - u_i| < \varepsilon$ for i = 1, 2, 3 both of u_1, u_2, u_3 are not parabolic points and the tangent planes to $\bar{M} = \bar{X}(U)$ at u_1, u_2, u_3 are parallel.

(f) The Gauss map $\bar{G}$ is the cuspidaledgeat u_0.

Theorem 18. *Assume u_0 is a parabolic point of $\bar{X}$. Then the following statements are equivalent:*

(a) The cylindrical pedal $CPe_{\bar{M}}$ is a butterfly at u_0.

(b) $\widetilde{h}_{(v_0,r_0)}(u)$ has A_4-type singularity.

(c) $T-ord(\bar{X}(U), u_0) = 4$.

(d) Tangent indicatrix $(\bar{X}^{-1}(\bar{h}(X(U), u_0), u_0)$ is a surface $\subset R^3$, and it is diffeomorphic to the surface given by $\{(u,v,w) : u^5 \pm v^2 \pm w^2 = 0\}$.

(e) For each $\varepsilon > 0$, there exist two distinct points $u_1, u_2, u_3, u_4 \subset U$ such that $|u_0 - u_i| < \varepsilon$ for i = 1, 2, 3, 4 both of u_1, u_2, u_3, u_4 are not parabolic points and the tangent planes to $\bar{M} = \bar{X}(U)$ at u_1, u_2, u_3, u_4 are parallel.

(f) The Gauss map $\bar{G}$ is the swallowtail at u_0.

Theorem 19. *Assume u_0 is a parabolic point of $\bar{X}$. Then the following statements are equivalent:*

(a) The cylindrical pedal $CPe_{\bar{M}}$ is a ell/hyp umbilic at u_0.

(b) $\widetilde{h}_{(v_0,r_0)}(u)$ *has* $D_{\pm 4}$*-type singularity.*

(c) $T-ord(\bar{X}(U),u_0)=4.$

(d) Tangent indicatrix $(\bar{X}^{-1}(\bar{h}(X(U),u_0),u_0)$ *is a surface* $\subset R^3$*, and it is diffeomorphic to the surface given by* $\{(u,v,w):\pm u^3+v^2v\pm w^2=0\}$*.*

(e) For each $\varepsilon>0$*, there exist two distinct points* $u_1,u_2,u_3,u_4\subset U$ *such that* $|u_0-u_i|<\varepsilon$ *for i = 1, 2, 3, 4 both of* u_1,u_2,u_3,u_4 *are not parabolic points and the tangent planes to* $\bar{M}=\bar{X}(U)$ *at* u_1,u_2,u_3,u_4 *are parallel.*

(f) The Gauss map $\bar{G}$ *at* u_0 *is diffeomorphic to the functions:*

$$\left\{-vu+v\sqrt{3}\sqrt{4v^2+3u^2},\frac{1}{2}\left(-4v^2-u\left(3u+\sqrt{3}\sqrt{4v^2+3u^2}\right)\right),u\right\}$$

$$\left\{-vu+v\sqrt{3}\sqrt{4v^2+3u^2},\frac{1}{2}u\left(3u+\sqrt{3}\sqrt{4v^2+3u^2}\right),u\right\}$$

corresponding to D_{+4}*,*D_{-4} *respectively.*

Proof. We have shown in section that u_0 is a parabolic point if and only if $T-corank(\bar{X}(U),u_0)\geq 1$. Since n = 4, we have $T-corank(\bar{X}(U),u_0)\leq 3$. Since the extended height function germ $\widetilde{h}:U\times(S^n\times R)\to R$ can be considered as a generating family of the Legendrian immersion germ $\ell(\widetilde{h})$,$\widetilde{h}_{(v_0,r_0)}$ has the A_k-type singularity for k = 1, 2, 3, 4 or $D_{\pm 4}$ singularity.I.e the corank of the Hessian matrix of the extend height function at (u_0,v_0)(parabolic point)equal to one.

For theorem 16 using the above same way we find; If the rank of the Hessian matrix of the extend height function is equal to two then $\widetilde{h}_{(v_0,r_0)}$ have A_2-singularity and using the theory of Legendrian singularities we find $CPe_{\bar{M}}$ has a cuspidaledge and so the $T-ord(\bar{X}(U),u_0)=2$ conditions (a),(b),(c) hold (respectively, theorem 17; (a),(b),(c), theorem 18; (a),(b),(c), theorem 19; (a),(b),(c)).

If $\widetilde{h}_{(v_0,r_0)}$ has A_2-singularity,then it is K-equivalent to the germ $u^3\pm v^2\pm w^2$, Since the K-equivalence preserves the zero level sets, the tangent indicatrix is diffeomorphic to the surface given by $u^3\pm v^2\pm w^2=0$.

Form for the A_3-singularity is given by $u^4\pm v^2\pm w^2$, so the tangent indicatrix is diffeomorphic to the surface given by $u^4\pm v^2\pm w^2=0$.

Form for the A_4-singularity is given by $u^5\pm v^2\pm w^2$, so the tangent indicatrix is diffeomorphic to the surface given by $u^5\pm v^2\pm w^2=0$.

Form for the $D_{\pm 4}$-singularity is given by $\pm u^3 + v^2 u \pm w^2$, so the tangent indicatrix is diffeomorphic to the surface given by $\pm u^3 + v^2 u \pm w^2 = 0$.

This means that the condition (d)in theorem 16 (respectively,theorems 17, 18, 19; (d)) is also equivalent to the other conditions.

The parabolic sets for this case are given as A_2, A_3, A_4, D_{+4} and D_{-4} which are equivalent to :

$u = 0$ (plane), $6u^2 + v = 0$ (fold), $10u^3 + 3uv + w = 0$ (cusp), $3u^2 + w^2 + uv = 0$ and $3u^2 - w^2 + uv = 0$ respectively as shown in the last four figure in [12,13].

According to the classification results on stable Legendrian mappings, we have also can give the classification of the singularities sets for the map germ $f : (R^3, 0) \to (R^4, 0)$ are given by:
A_2, A_3, A_4, D_{+4} and D_{-4} which are given analytically through the sets:
$\{u, v, w, 0\}$, $\left\{2u^3, 3u^2, v, w\right\}$, $\left\{3u^4 + u^2 v, 4u^3 + uv, v, w\right\}$,
$\left\{4u^5 + 2u^3 v + wu^2, 5u^4 + 3u^2 v + 2uw, v, w\right\}$, $\left\{2u^3 + 2uv^2 + v^2 w, 2uv + 2vw, 3u^2 + v^2, w\right\}$
and $\left\{2u^3 - 2uv^2 - v^2 w, 2uv + 2vw, 3u^2 - v^2, w\right\}$ respectively.

Remark 20. *The geometrical interpretation of theorems 16, 17, 18 and 20 are given though figures [1,2,3,4,5,6,7,8,9, 10,11]*

Conflict of Interests

The authors declare that there is no conflict of interests.

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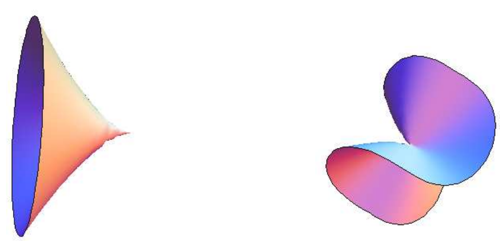

FIGURE 1. Tangent indicatrix for theorem 16

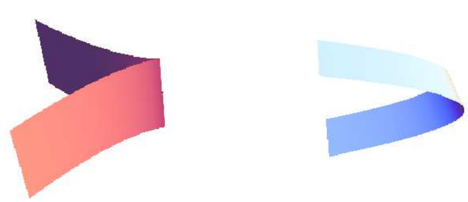

FIGURE 2. cuspidaledge and fold (theorem 16)

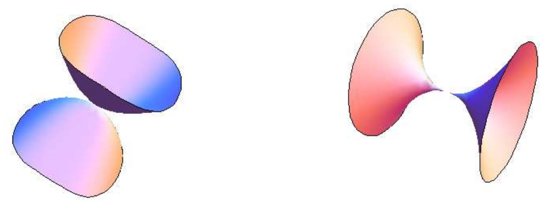

FIGURE 3. Tangent indicatrix for theorem 17

FIGURE 4. swallowtail and cuspidaledge (theorem 17)

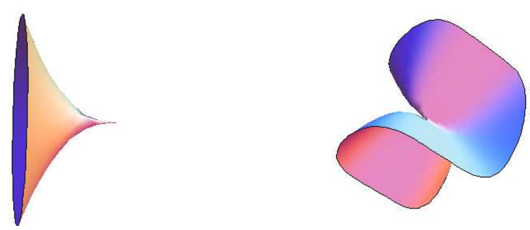

FIGURE 5. Tangent indicatrix for theorem 18

FIGURE 6. projection of butterfly on some hyperplane (theorem 18)

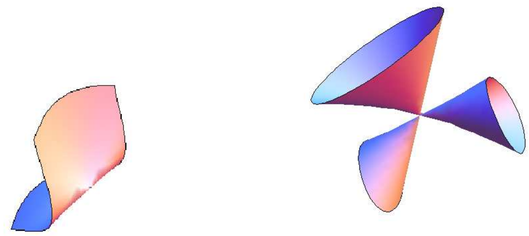

FIGURE 7. Tangent indicatrix for theorem 19

FIGURE 8. projection of D_{+4} on some hyperplane (theorem 19)

FIGURE 9. the shape of Gauss map (D_{+4}) (theorem 19)

FIGURE 10. projection of D_{-4} on some hyperplane (theorem 19)

FIGURE 11. the shape of Gauss map (D_{+4}) (theorem 19)

FIGURE 12. the shape of parabolic set in case A_3, A_4

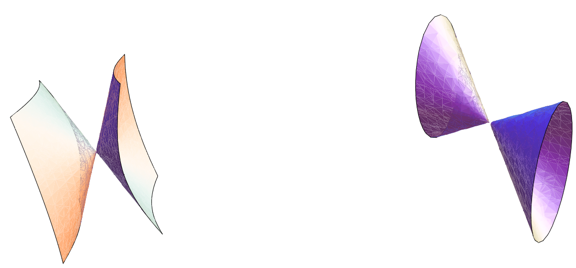

FIGURE 13. the shape of parabolic set in case D_{+4}, D_{-4}