Available online at http://scik.org
J. Math. Comput. Sci. 6 (2016), No. 6, 965-974
ISSN: 1927-5307

σ-CONVERGENT SEQUENCE SPACES DEFINED BY ORLICZ FUNCTION

KHALID EBADULLAH

College of Science and Theoretical Studies, Saudi Electronic Universiy, Kingdom of Saudi Arabia

Abstract. In this paper, we introduce the sequence space $V_\sigma(M,p,r)$, where M is an Orlicz function, $p=(p_m)$ is any sequence of strictly positive real numbers and $r \geq 0$ and study some of the properties and inclusion relations that arise on the said space.

Keywords: invariant mean; paranorm; Orlicz function.

2010 AMS Subject Classification: 40F05, 40C05, 46A45.

1. Introduction

Let N, R and C be the sets of all natural, real and complex numbers respectively.

We write

$$\omega = \{x = (x_k) : x_k \in R \; or \; C\},$$

the space of all real or complex sequences.

Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively.

The following subspaces of ω were first introduced and discussed by Maddox [5-6].

Received April 20, 2016

$\ell(p) = \{x \in \omega : \sum_k |x_k|^{p_k} < \infty\}$,
$\ell_\infty(p) = \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\}$,
$c(p) = \{x \in \omega : \lim_k |x_k - l|^{p_k} = 0, \text{ for some } l \in C\ \}$,
$c_0(p) = \{x \in \omega : \lim_k |x_k|^{p_k} = 0\}$,
where $p = (p_k)$ is a sequence of striclty positive real numbers.

The concept of paranorm is closely related to linear metric spaces.It is a generalization of that of absolute value.(see[5-6])

Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if for all $x, y, z \in X$,

(PI) $g(x) = 0\ if\ x = \theta$,

(P2) $g(-x) = g(x)$,

(P3) $g(x+y) \leq g(x) + g(y)$,

(P4) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda\ (n \to \infty)$ and $x_n, a \in X$ with $x_n \to a\ (n \to \infty)$, in the sense that $g(x_n - a) \to 0\ (n \to \infty)$, in the sense that $g(\lambda_n x_n - \lambda a) \to 0\ (n \to \infty)$.

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.(see[2],[9],[12])

Lindenstrauss and Tzafriri[3] used the idea of Orlicz functions to construct the sequence space

$$\ell_M = \{x \in \omega : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty, \text{for some } \rho > 0\}$$

The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1\}$$

The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

An Orlicz function M is said to satisfy $\triangle_2$ condition for all values of x if there exists a constant $K > 0$ such that $M(Lx) \leq KLM(x)$ for all values of $L > 1$.

A sequence space E is said to be solid or normal if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequence of scalars (α_k) with $|\alpha_k| < 1$ for all $k \in N$.

Let σ be an injection on the set of positive integers N into itself having no finite orbits and T be the operator defined on ℓ_∞ by $T(x_k) = (x_{\sigma(k)})$.
A positive linear functional Φ, with $||\Phi|| = 1$, is called a σ-mean or an invariant mean if $\Phi(x) = \Phi(Tx)$ for all $x \in \ell_\infty$.
A sequence x is said to be σ-convergent, denoted by $x \in V_\sigma$, if $\Phi(x)$ takes the same value, called $\sigma - \lim x$, for all σ-means Φ. We have

$$V_\sigma = \{x = (x_k) : \sum_{m=1}^{\infty} t_{m,n}(x) = L \text{ uniformly in n, L = } \sigma - \lim x\},$$

where for $m \geq 0, n > 0$.

$$t_{m,n}(x) = \frac{x_k + x_{\sigma(k)} + ... + x_{\sigma^m(k)}}{m+1}, \text{and } t_{-1,n} = 0.$$

where $\sigma^m(k)$ denotes the mth iterate of σ at n. In particular, if σ is the translation, a σ-mean is often called a Banach limit and V_σ reduces to f, the set of almost convergent sequences.(see[1],[4],[7],[8],[10],[11])

Subsequently the spaces of invariant mean and Orlicz function have been studied by various authors.(See [1-13]).

2. Main results

In this article we introduce the sequence space

$$V_\sigma(M, p, r) = \{x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} [M(\frac{|t_{m,n}(x)|}{\rho})]^{p_m} < \infty \text{ uniformly in n, } \rho > 0\}.$$

Where M is an Orlicz function, $p = (p_m)$ is any sequence of strictly positive real numbers and $r \geq 0$.

Now we define the sequence spaces as follows;

For $M(x) = x$ we get

$$V_\sigma(p,r) = \{x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} |t_{m,n}(x)|^{p_m} < \infty \text{ uniformly in n}\}$$

For $p_m = 1$, for all m, we get

$$V_\sigma(M,r) = \{x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} [M(\frac{|t_{m,n}(x)|}{\rho})] < \infty \text{ uniformly in n, } \rho > 0\}$$

For r = 0 we get

$$V_\sigma(M,p) = \{x = (x_k) : \sum_{m=1}^{\infty} [M(\frac{|t_{m,n}(x)|}{\rho})]^{p_m} < \infty \text{ uniformly in n, } \rho > 0\}$$

For $M(x) = x$ and r=0 we get

$$V_\sigma(p) = \{x = (x_k) : \sum_{m=1}^{\infty} |t_{m,n}(x)|^{p_m} < \infty \text{ uniformly in n, } \rho > 0\}$$

For $p_k = 1$, for all m and r=0, we get

$$V_\sigma(M) = \{x = (x_k) : \sum_{m=1}^{\infty} [M(\frac{|t_{m,n}(x)|}{\rho})] < \infty \text{ uniformly in n, } \rho > 0\}$$

For $M(x) = x$, $p_m = 1$, for all m and r=0, we get

$$V_\sigma(x) = \{x = (x_k) : \sum_{m=1}^{\infty} |t_{m,n}(x)| < \infty \text{ uniformly in n}\}.$$

Theorem 2.1. The sequence space $V_\sigma(M,p,r)$ is a linear space over the field C of complex numbers.

Proof. Let $x, y \in V_\sigma(M,p,r)$ and $\alpha, \beta \in C$ then there exists positive numbers ρ_1 and ρ_2 such that

$$\sum_{m=1}^{\infty} \frac{1}{m^r} [M(\frac{|t_{m,n}(x)|}{\rho_1})]^{p_m} < \infty,$$

and

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|t_{m,n}(y)|}{\rho_2})]^{p_m} < \infty$$

uniformly in n.

Define ρ_3 = max$(2|\alpha|\rho_1, 2|\beta|\rho_2)$.

Since M is non decreasing and convex we have

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\alpha t_{m,n}(x)+\beta t_{m,n}(y)|}{\rho_3})]^{p_m}$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\alpha t_{m,n}(x)|}{\rho_3}+\frac{|\beta t_{m,n}(y)|}{\rho_3})]^{p_m}$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{m^r}\frac{1}{2}[M(\frac{t_{m,n}(x)}{\rho_1})+M(\frac{t_{m,n}(y)}{\rho_2})] < \infty$$

uniformly in n.

This proves that $V_\sigma(M,p,r)$ is a linear space over the field C of complex numbers.

Theorem 2.2. For any Orlicz function M and a bounded sequence $p=(p_m)$ of strictly positive real numbers, $V_\sigma(M,p,r)$ is a paranormed space with

$$g(x) = \inf_{n\geq 1}\{\rho^{\frac{p_n}{H}} : (\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|t_{m,n}(x)|}{\rho})]^{p_m})^{\frac{1}{H}} \leq 1, \text{ uniformly in n}\}$$

where H = max(1, supp_m).

Proof. It is clear that $g(x) = g(-x)$.

Since $M(0) = 0$, we get

inf$\{\rho^{\frac{p_m}{H}}\}$ = 0, for $x = 0$

Now for α=β=1, we get

$g(x+y) \leq g(x)+g(y)$.

For the continuity of scalar multiplication let $l \neq 0$ be any complex number. Then by the definition we have

$$g(lx) = \inf_{n\geq 1}\{\rho^{\frac{p_n}{H}} : (\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|t_{m,n}(lx)|}{\rho})]^{p_m})^{\frac{1}{H}} \leq 1, \text{ uniformly in n}\}$$

$$g(lx) = \inf_{n\geq 1}\{(|l|s)^{\frac{p_n}{H}} : (\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|t_{m,n}(lx)|}{(|l|s)})]^{p_m})^{\frac{1}{H}} \leq 1, \text{ uniformly in n}\}$$

where $s = \frac{\rho}{|l|}$.
Since $|l|^{p_m} \leq \max(1,|l|^H)$, we have

$$g(lx) \leq max(1,|l|^H) \inf_{n\geq 1}\{s^{\frac{p_n}{H}} : (\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|t_{m,n}(x)|}{(|l|s)})]^{p_m})^{\frac{1}{H}} \leq 1, \text{ uniformly in n}\}$$

$$g(lx) \leq max(1,|l|^H)g(x)$$

Therefore $g(lx)$ converges to zero when $g(x)$ converges to zero in $V_\sigma(M,p,r)$.

Now let x be fixed element in $V_\sigma(M,p,r)$. There exists $\rho > 0$ such that

$$g(x) = \inf_{n\geq 1}\{\rho^{\frac{p_n}{H}} : (\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|t_{m,n}(x)|}{\rho})]^{p_m})^{\frac{1}{H}} \leq 1, \text{ uniformly in n}\}$$

.

Now

$$g(lx) = \inf_{n\geq 1}\{\rho^{\frac{p_n}{H}} : (\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|t_{m,n}(lx)|}{\rho})]^{p_m})^{\frac{1}{H}} \leq 1, \text{ uniformly in n}\} \to 0 \text{ as } l \to 0.$$

This completes the proof.

Theorem 2.3. The sequence space

$$V_\sigma(M,p,r)=\{x=(x_k):\sum_{m=1}^{\infty}\frac{1}{m^r}[M(\frac{|t_{m,n}(x)|}{\rho})]^{p_m}<\infty \text{ uniformly in n, } \rho>0\}.$$

is a Banach space with the norm

$$g(x)=\inf_{n\geq 1}\{\rho^{\frac{p_n}{H}}:(\sum_{m=1}^{\infty}\frac{1}{m^r}[M(\frac{|t_{m,n}(x)|}{\rho})]^{p_m})^{\frac{1}{H}}\leq 1\}.$$

Theorem 2.4. Suppose that $0<p_m<t_m<\infty$ for each $m\in N$ and $r>0$. Then

(a) $V_\sigma(M,p)\subseteq V_\sigma(M,t)$.

(b) $V_\sigma(M)\subseteq V_\sigma(M,r)$

Proof.(a) Suppose that $x\in V_\sigma(M,p)$.

This implies that $[M(\frac{|t_{i,n}(x)|}{\rho})]^{p_m})\leq 1$

for sufficiently large value of i, say $i\geq m_0$ for some fixed $m_0\in N$.

Since M is non decreasing, we have

$$\sum_{m=m_0}^{\infty}[M(\frac{|t_{i,n}(x)|}{\rho})]^{t_m}\leq\sum_{m=m_0}^{\infty}[M(\frac{|t_{i,n}(x)|}{\rho})]^{p_m}<\infty.$$

Hence $x\in V_\sigma(M,t)$.

(b) The proof is trivial.

Corollary 2.5. $0<p_m\leq 1$ for each m, then $V_\sigma(M,p)\subseteq V_\sigma(M)$

If $p_m\geq 1$ for all m , then $V_\sigma(M)\subseteq V_\sigma(M,p)$.

Theorem 2.6. The sequence space $V_\sigma(M,p,r)$ is solid.

Proof. Let $x\in V_\sigma(M,p,r)$. This implies that

$$\sum_{m=1}^{\infty}\frac{1}{m^r}[M(\frac{|t_{m,n}(x)|}{\rho})]^{p_m}<\infty.$$

Let α_m be a sequence of scalars such that $|\alpha_m| \leq 1$ for all $m \in N$. Then the result follows from the following inequality.

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\alpha_m t_{i,n}(x)|}{\rho})]^{p_m} \leq \sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|t_{i,n}(x)|}{\rho})]^{p_m} < \infty.$$

Hence $\alpha x \in V_\sigma(M,p,r)$ for all sequence of scalars (α_m) with $|\alpha_m| \leq 1$ for all $m \in N$ whenever $x \in V_\sigma(M,p,r)$.

Corollary 2.7. The sequence space $V_\sigma(M,p,r)$ is monotone.

Theorem 2.8. Let M_1, M_2 be Orlicz function satisfying $\triangle_2$ condition and
$r, r_1, r_2 \geq 0$. Then we have
(a) If $r > 1$ then $V_\sigma(M_1,p,r) \subseteq V_\sigma(M0M_1,p,r)$,
(b) $V_\sigma(M_1,p,r) \cap V_\sigma(M_2,p,r) \subseteq V_\sigma(M_1+M_2,p,r)$,
(c) If $r_1 \leq r_2$ then $V_\sigma(M,p,r_1) \subseteq V_\sigma(M,p,r_2)$.

Proof. (a) Since M is continuous at 0 from right, for $\varepsilon > 0$ there exists $0 < \delta < 1$ such that $0 \leq c \leq \delta$ implies $M(c) < \varepsilon$.
If we define

$$I_1 = \{m \in N : M_1(\frac{|t_{m,n}(x)|}{\rho}) \leq \delta \text{ for some } \rho > 0\},$$

$$I_2 = \{m \in N : M_1(\frac{|t_{m,n}(x)|}{\rho}) > \delta \text{ for some } \rho > 0\},$$

when

$$M_1(\frac{|t_{m,n}(x)|}{\rho}) > \delta$$

we get

$$M(M_1(\frac{|t_{m,n}(x)|}{\rho})) \leq \{\frac{2M(1)}{\delta}\}M_1(\frac{|t_{m,n}(x)|}{\rho})$$

Hence for $x \in V_\sigma(M_1, p, r)$ and $r > 1$

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M0M_1(\frac{|t_{m,n}(x)|}{\rho})]^{p_m} = \sum_{m \in I_1} \frac{1}{m^r}[M0M_1(\frac{|t_{m,n}(x)|}{\rho})]^{p_m} + \sum_{m \in I_2} \frac{1}{m^r}[M0M_1(\frac{|t_{m,n}(x)|}{\rho})]^{p_m}.$$

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M0M_1(\frac{|t_{m,n}(x)|}{\rho})]^{p_m} \leq max(\varepsilon^h, \varepsilon^H) \sum_{m=1}^{\infty} \frac{1}{m^r} + max(\{\frac{2M_1}{\delta}\}^h, \{\frac{2M_1}{\delta}\}^H)$$

$$\text{where } 0 < h = \inf p_m \leq p_m \leq H = \sup_m p_m < \infty$$

(b)The proof follows from the following inequality

$$\frac{1}{m^r}[(M_1 + M_2)(\frac{|t_{m,n}(x)|}{\rho})]^{p_m} \leq C\frac{1}{m^r}[M_1(\frac{|t_{m,n}(x)|}{\rho})]^{p_m} + C\frac{1}{m^r}[M_2(\frac{|t_{m,n}(x)|}{\rho})]^{p_m}$$

(c)The proof is straightforward.

Corollary 2.9. Let M be an Orlicz function satisfying $\triangle_2$ condition. Then we have

(a) If $r > 1$ then $V_\sigma(p, r) \subseteq V_\sigma(M, p, r)$,

(b) $V_\sigma(M, p) \subseteq V_\sigma(M, p, r)$,

(c) $V_\sigma(p) \subseteq V_\sigma(p, r)$,

(d) $V_\sigma(M) \subseteq V_\sigma(M, r)$.

Proof. The proof is straightforward.

Conflict of Interests

The author declares that there is no conflict of interests.

REFERENCES

[1] Z.U.Ahmad, M.Mursaleen. An application of Banach limits.*Proc.Amer. Math. soc*. 103 (1983), 244-246.

[2] S.T.Chen. Geometry of Orlicz spaces.*Dissertation Math. The Institute of Mathematics, Polish Academy of Sciences*. (1996)

[3] J.Lindenstrauss, L.Tzafriri. On Orlicz sequence spaces. *Israel J. Math.,* 101 (1971), 379-390.

[4] C.G.Lorentz. A contribution to the theory of divergent series.*Acta Math.,* 80 (1948), 167-190.

[5] I.J.Maddox. *Elements of Functional Analysis*, Cambridge University Press. (1970)

[6] I.J.Maddox. Some properties of paranormed sequence spaces., *J.London. Math.Soc*. 1 (1969), 316-322.

[7] M.Mursaleen. Matrix transformation between some new sequence spaces.*Houston J. Math.,* 9 (1983), 505-509.

[8] M.Mursaleen. On some new invariant matrix methods of summability.*Quart. J. Math. Oxford,* (2)34 (1983), 77-86.

[9] J.Musielak. Orlicz spaces and Modular spaces.*Lecture notes in Math.(Springer-Verlag)*1034:(1983).

[10] R.A.Raimi. Invariant means and invariant matrix methods of summability.*Duke J. Math.,*30 (1963), 81-94.

[11] P.Schafer. Infinite matrices and Invariant means.*Proc.Amer. Math. soc.*36 (1972), 104-110.

[12] A.k.vakeel On a new sequence space defined by Orlicz Functions.*Commun.Fac.Sci Univ.Ank.Series A1*. 57 (2008), 25-33.

[13] A.K.Vakeel, K. Ebadullah. *On a new difference sequence space of invariant means defined by Orlicz functions.*Bulletin Of The Allahabad Mathematical Society. 26(2) (2011), 259-272.