Available online at http://scik.org
J. Math. Comput. Sci. 6 (2016), No. 6, 975-988
ISSN: 1927-5307

OPERATORS ON LATTICE-VALUED AUTOMATA

A.P. PATIL[1,*], S.R. CHAUDHARI[2]

[1]Department of Mathematics, Shivaji University, Kolhapur(M.S.) 416 004, India
[2]Department of Mathematics, School of Mathematical Sciences, North Maharashtra University, Jalgaon (M.S.) 425 001, India

Abstract. The concepts of source core and successor core for residuated lattice valued finite state machine are introduced and their properties are studied. We found the relation between the genetic subset of lattice valued finite state machine and the source core. Using the implication of complete residuated lattice we introduced the fuzzy operators namely fuzzy initial core and fuzzy final core which are generalizations of source core and successor core respectively. Also we have done comparative study of these operators with the source and successor operators.

Keywords: L-valued finite state machine; fuzzy automaton; source, successor and core operators.

2010 AMS Subject Classification: 08A72, 68Q45, 68Q70.

1. Introduction

Study of fuzzy automata theory was firstly proposed by Wee in 1967[37] and Wee and Fu in 1969 [38] as a model of learning systems. Since then many authors has been studied fuzzy automaton for characterizations of fuzzy languages [12, 27, 29, 30, 32, 36]. Algebraic study of fuzzy automata theory was started by Malik, Mordeson and Sen [19, 20] and then by many

*Corresponding author

Received July 25, 2016

others[9, 10, 11, 14, 17, 18, 21]. Topological aspects of fuzzy automata was studied by Das [3], Srivastava and Tiwari [31], Tiwari et.al [35] etc. Many researchers have studied fuzzy automata theory in allied directions to see them we refer to [5, 8, 22, 23].

Generalization of fuzzy automaton as lattice valued automaton was proposed by Qiu [24, 25]. Since then many researchers have studied fuzzy automata with membership values in complete residuated lattice, lattice ordered monoid, orthomodular lattice and other kinds of lattices [2, 4, 13, 15, 16, 26, 33, 34, 39]. Tiwari et.al [33, 34] were studied algebraic and topological characterization of L-valued finite state machines(called it has also L-fuzzy automaton) with the underlying structure of lattice ordered monoid.

The paper is arranged in six sections. Preliminary section 2 consist of basic concepts of complete residuated lattice and L-valued finite state machine. Section 3 contains the concepts of source, successor operators and their fuzzified operators. The operators source core and successor core studied in section 4. Newly introduced operators Fuzzy initial core and fuzzy final core are discussed along with their properties in section 5. Lastly, we conclude the discussion in section 6.

2. Preliminaries

In this section, we discuss complete residuated lattice and L-valued finite state machines. We begin with the definition of residuated lattice.

Definition 2.1. [1, 6, 24] *A residuated lattice is a quintuple* $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$ *,where*
(i) $(L, \wedge, \vee, 0, 1)$ *is a lattice with* 0 *and* 1 *as its least and greatest elements respectively;*
(ii) $(L, *, 1)$ *is a commutative monoid with unit 1;*
(iii) $*$ *and* $\rightarrow$ *form an adjoint pair, i.e. they satisfy the adjuction property: i.e.* $\alpha * \beta \leq \gamma$ *iff* $\alpha \leq \beta \rightarrow \gamma$ *for any* $\alpha, \beta, \gamma \in L$.

In literature the structure of residuated lattice is also called as integral, commutative, residuated, l-monoid [6]. If in addition, $(L, \wedge, \vee, 0, 1)$ is a complete lattice, the $\mathscr{L}$ is called as complete residuated lattice. Throughout this paper, L stands for complete residuated lattice.

We give an account of some properties of complete residuated lattices as follows: For any $a, b, x_i \in L$

(P_1) $(\bigvee_i x_i) * a = \bigvee_i (x_i * a)$,

(P_2) $a \leq b \Leftrightarrow a \to b = 1$,

(P_3) $a \leq b \;\Rightarrow\; c \to a \leq c \to b$ and $b \to c \leq a \to c$,

(P_4) $a \to (\bigwedge_i x_i) = \bigwedge_i (a \to x_i)$,

(P_5) $(\bigvee_i x_i) \to a = \bigwedge_i (a \to x_i)$,

(P_6) $1 \to a = a$,

(P_7) $(a * b) \to c = a \to (b \to c)$.

Additionally, some derived formulae are also necessary for us. We just report them as follows:

i) $\neg a = a \to 0,\ a \in L$

ii) If A is L-fuzzy subset of Q and $q \in Q$, then $Q - A(q) = \neg A(q) = A(q) \to 0$.

Definition 2.2. *A residuated lattice $\mathscr{L} = (L, \wedge, \vee, *, \to, 0, 1)$ is said to be a residuated lattice without zero divisiors if for all $a, b \in L$ $a \neq 0, b \neq 0 \Rightarrow\; a * b \neq 0$.*

Definition 2.3. [24] *An L-valued finite state machine(In short, L-VFSM) or L-valued semi-automaton is a triple $\mathscr{M} = (Q, \Sigma, \delta)$, where Q and Σ are non empty finite sets of states and input symbols respectively, and δ is a L-fuzzy subset of $Q \times \Sigma \times Q$ i.e. δ is a function from $Q \times \Sigma \times Q$ into L, called transition function.*

Intuitively, $\delta(p, \sigma, q)$ stands for the membership degree of the transition from state $p \in Q$ to the state $q \in Q$, when the input is $\sigma \in \Sigma$.

Note 2.1. *The collection of all finite sequence of elements of Σ (called, tapes or string) denoted by Σ^*, and ε denotes the empty string in Σ^*. For any $x \in \Sigma^*$, $|x|$ denotes the length of the string x. Clearly, $|\varepsilon| = 0$.*

Definition 2.4. [24] *Let $\mathscr{M} = (Q, \Sigma, \delta)$ be L-VFSM. Extend $\delta : Q \times \Sigma \times Q \to L$ to $\delta^* : Q \times \Sigma^* \times Q \to L$ inductively as follows:*

for any $p, q \in Q,\ \sigma \in \Sigma$ and $x \in \Sigma^*$, *(1)* $\delta^*(p,\varepsilon,q) = \begin{cases} 1 & if\ p = q \\ 0 & otherwise \end{cases}$

(2) $\delta^*(p,x\sigma,q) = \bigvee_{r\in Q}\{\delta^*(p,x,r) * \delta(r,\sigma,q)\}$.

Then δ^* is called extended transition function with respect to $*$.

If we replace $*$ by $\wedge$ in the Definition 2.4, then it reduces to.

Definition 2.5. *Let $\mathscr{M} = (Q,\Sigma,\delta)$ be L-VFSM. We extend $\delta : Q\times\Sigma\times Q \to L$ to $\delta^\wedge : Q\times\Sigma^*\times Q \to L$ as follows:*
for any $p,q \in Q$, $\sigma\in\Sigma$ and $x\in\Sigma^*$,

(1) $\delta^\wedge(p,\varepsilon,q) = \begin{cases} 1 & if\ p = q \\ 0 & otherwise \end{cases}$

(2) $\delta^\wedge(p,x\sigma,q) = \bigvee_{r\in Q}\{\delta^\wedge(p,x,r) \wedge \delta(r,\sigma,q)\}$.

Then $\delta^\wedge$ is called extended transition function with respect to $\wedge$.

Theorem 2.1. [13] *Let L be a lattice and X be a finite subset of L. Then $\wedge$-semilattice of L generated by X, $X_\wedge = \{x_1\wedge\cdots\wedge x_k | k\geq 1, x_1,\ldots,x_k \in X\}\cup\{1\}$, is finite. Similarly, the $\vee$-semilattice of L generated by X, $X_\vee = \{x_1\vee\cdots\vee x_k | k\geq 1, x_1,\ldots,x_k\in X\}\cup\{0\}$, is also finite.*

The following theorem shows that the importance of complete lattice.

Theorem 2.2. *Let $\mathscr{M} = (Q,\Sigma,\delta)$ be L-VFSM. Then the image set of the extension transition function $\delta^\wedge : Q\times\Sigma^*\times Q\to L$ is finite. i.e. $Im(\delta^\wedge) = \{\delta^\wedge(p,x,q) | p,q\in Q, x\in\Sigma^*\}$ is a finite subset of L.*

Proof. Let $X = Im(\delta)$. As, Q and Σ are finite, we have X is finite. We have to prove that $Im(\delta^\wedge)$ is finite, for this we first show that, $Im(\delta^\wedge)\subseteq (X_\wedge)_\vee$.
We prove this, by induction on length of x denoted by $|x|$. When $|x| = 0$, then $\delta^\wedge = 0$ or $1\in (X_\wedge)_\vee$ and if $|x| = 1$, then $\delta^\wedge = \delta\in X\subseteq (X_\wedge)_\vee$.
Now suppose it hold for $|x| = k$. We want to prove it for $|x| = k+1$. Let $x = \sigma_1\sigma_2\ldots\sigma_k\sigma_{k+1}$, then we have for any $p,q\in Q$, $\delta^\wedge(p,\sigma_1\sigma_2\ldots\sigma_k\sigma_{k+1},q) = \bigvee_{r\in Q}\{\delta^\wedge(p,\sigma_1\sigma_2\ldots\sigma_k,r)\wedge\delta(r,\sigma_{k+1},q)\}$.
By induction step, we have for any $r\in Q$, $\delta^\wedge(p,\sigma_1\sigma_2\ldots\sigma_k,r)\in (X_\wedge)_\vee$ and thus $\delta^\wedge(p,\sigma_1\sigma_2\ldots\sigma_k,r)\wedge$

$\delta(r,\sigma_{k+1},q) \in ((X_\wedge)_\vee)_\wedge = (X_\wedge)_\vee$ and therefore, $\delta^\wedge(p,\sigma_1\sigma_2\ldots\sigma_k\sigma_{k+1},q) \in ((X_\wedge)_\vee)_\vee = (X_\wedge)_\vee$. Hence $Im(\delta^\wedge) \subseteq (X_\wedge)_\vee$. By Theorem 2.1, $(X_\wedge)_\vee$ is finite subset of L. Thus $Im(\delta^\wedge)$, as a subset of $(X_\wedge)_\vee$, is also finite subset of L.

Remark 2.1. *The above Theorem 2.2 shows that,if we use the definition 2.5, we do not need completeness of the lattice, but we need the definition 2.4 have we use complete residuated lattice throughout this paper.*

3. Source, successor operators and their fuzzified operators

The study of source and successors operators can be found in Tiwari et.al.[33] and their fuzzified operators in Tiwari et.al.[34], but their underlying lattice structure was lattice ordered monoid. Here, we use complete residuated lattice as the structure of membership values, since structure of residuated lattice, $(L,*,1)$ is a one kind of lattice ordered monoid structure. So we collect some of the definitions and results from [33, 34].

Definition 3.1. *Let $\mathscr{M} = (Q,\Sigma,\delta)$ be an L-valued finite state machine and $A \subseteq Q$. The source and the successors of A are respectively the sets*

$$\sigma(A) = \{p \in Q | \delta^*(p,x,q) > 0, \text{ for some } x \in \Sigma^* \text{and } q \in A\}$$

$$s(A) = \{q \in Q | \delta^*(p,x,q) > 0, \text{ for some } x \in \Sigma^* \text{and } p \in A\}$$

Definition 3.2. *An L-VFSM $\mathscr{N} = (R,\Sigma,\lambda)$ is called an lattice valued sub finite state machine(In short, L-VSFSM) of L-VFSM $\mathscr{M} = (L,Q,\Sigma,\delta)$, if $R \subseteq Q, s(R) = R$ and $\delta|_{R\times\Sigma\times R} = \lambda$. Further, this L-VSFSM is called separated if $s(Q-R) = Q-R$.*

Definition 3.3. *An L-VFSM $\mathscr{M} = (L,Q,\Sigma,\delta)$ is called connected if $\mathscr{M}$ has no separated proper L-VSFSM.*

Definition 3.4. *Let $\mathscr{M} = (Q,\Sigma,\delta)$ be an L-valued finite state machine and $A \in \mathscr{F}_L(Q)$. The fuzzy source and the fuzzy successors of A are respectively defined as:*

$$so(A)(q) = \bigvee_{\substack{x\in\Sigma^*\\ p\in Q}} \{\delta^*(q,x,p) * A(p)\}$$

$$su(A)(q) = \bigvee_{\substack{x\in\Sigma^* \\ p\in Q}} \{A(p) * \delta^*(p,x,q)\}$$

Theorem 3.1. *Let L be a complete residuated lattice without zero divisors and $A \subseteq Q$. Then for all $p \in Q$,*

i) $so(1_A)(p) > 0$ iff $p \in \sigma(A)$

ii) $su(1_A)(p) > 0$ iff $p \in s(A)$

Proof. Proof is same as in [34].

Definition 3.5. *Let $\mathscr{M} = (Q, \Sigma, \delta)$ be an L-VFSM. Then $R \in \mathscr{F}_L(Q)$ is called an L-fuzzy lattice valued sub finite state machine(In short, L-fuzzy L-VSFSM) of $\mathscr{M}$, if $su(R)(q) \leq R(q), \forall q \in Q$. Further, this L-fuzzy L-VSFSM is called separated, if $su(Q-R)(q) = Q-R(q), \forall q \in Q$.*

Definition 3.6. *Let $\mathscr{M} = (Q, \Sigma, \delta)$ be an L-VFSM. Then an L-VFSM $\mathscr{M}$ is called ℓ-connected, if $\mathscr{M}$ has no non-constant separated L-fuzzy L-VSFSM.*

Theorem 3.2. *Let $\mathscr{M} = (Q, \Sigma, \delta)$ be an L-VFSM. If $\mathscr{M}$ is ℓ-connected, then $\mathscr{M}$ is connected.*

Proof. Let $\mathscr{M}$ be ℓ-connected and $\mathscr{N} = (R, \Sigma, \lambda)$ be separated L-VSFSM of $\mathscr{M}$. Then $s(R) = R$ and $s(Q-R) = Q-R$.

Now, if $p \in R$, then $1_R(p) = 1$ and $su(1_R)(p) = \bigvee_{\substack{x\in\Sigma^* \\ q\in Q}} \{1_R(q) * \delta^*(q,x,p) \geq 1_R(p) * \delta^*(p,\varepsilon,p) = 1 * 1 = 1 \implies su(1_R)(p) = 1$

and if $p \notin R$, then $1_R(p) = 0$ and $su(1_R)(p) = \bigvee_{\substack{x\in\Sigma^* \\ q\in Q}} \{1_R(q) * \delta^*(q,x,p)$

if $q \notin R$, then $1_R(q) = 0$ and if $q \in R$ and $\delta^*(q,x,p) > 0$, then $p \in s(R)$, but $s(R) = R \implies p \in R$, which is contradiction and thus $\delta^*(q,x,p) = 0$. Therefore, for any $q \in Q$, either $1_R(q) = 0$ or $\delta^*(q,x,p) = 0$ and thus $su(1_R)(p) = 0$.

Therefore, for any $p \in Q,\ su(1_R)(p) = 1_R(p)$.

Also, since $Q - 1_R = 1_{Q-R}$, similarly for any $p \in Q$, we have $su(Q-1_R)(p) = su(1_{Q-R})(p) = 1_{Q-R}(p) = Q - 1_R(p)$.

Thus, 1_R is separated L-fuzzy L-VSFSM, but since $\mathscr{M}$ is ℓ-connected, we have 1_R is constant i.e. $1_R = 1$ or 0 i.e. $R = Q$ or ϕ.

Hence, there does not exist any separated proper L-VSFSM and thus $\mathscr{M}$ is connected.

4. Source core and successor core operators

We are chiefly motivated from the concept of core (we say here, source core) defined in the article [33], which is depends on source operator. In this section, we introduce one more core operator, which will depends on successor operator and so we called it successor core.

Definition 4.1. *Let $\mathscr{M} = (Q, \Sigma, \delta)$ be an L-VFSM and $A \subseteq Q$. Then source core and successor core of A are respectively defined as*

$$\mu_\sigma(A) = \{q \in A / \sigma(q) \subseteq A\} \text{ and}$$
$$\mu_s(A) = \{q \in A / s(q) \subseteq A\}.$$

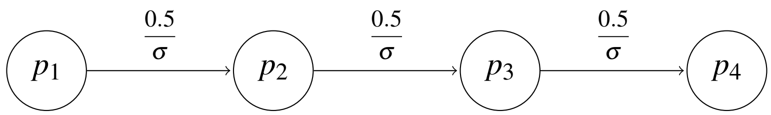
We frequently write here, $\sigma(q)$ and $s(q)$ for $\sigma(\{q\})$ and $s(\{q\})$ respectively. Similarly, we shall frequently write $\mu_\sigma(\{q\})$ and $\mu_s(\{q\})$ as just $\mu_\sigma(q)$ and $\mu_s(q)$.

Theorem 4.1. *Let $\mathscr{M} = (Q, \Sigma, \delta)$ be an L-VFSM. Then for $A, B \subseteq Q$*
i) if $A \subseteq B$, then $\mu_\sigma(A) \subseteq \mu_\sigma(B)$ and $\mu_s(A) \subseteq \mu_s(B)$
ii) $\mu_\sigma(A) \subseteq A$ and $\mu_s(A) \subseteq A$
iii) $\mu_\sigma(A \cap B) = \mu_\sigma(A) \cap \mu_\sigma(B)$ and $\mu_s(A \cap B) = \mu_s(A) \cap \mu_s(B)$.

Proof. We only prove the properties of μ_σ, because the properties of μ_s are similar.
(i) Let $q \in \mu_\sigma(A)$. Then $\sigma(q) \subseteq A$. But $A \subseteq B \implies \sigma(q) \subseteq B \implies q \in \mu_\sigma(B)$. Thus, $\mu_\sigma(A) \subseteq \mu_\sigma(B)$.
(ii) Let $q \in \mu_\sigma(A)$. Then $\sigma(q) \subseteq A$. But $q \in \sigma(q) always$ and hence, $q \in A$. Thus, $\mu_\sigma(A) \subseteq A$.
(iii) $q \in \mu_\sigma(A \cap B) \Leftrightarrow \sigma(q) \subseteq A \cap B \Leftrightarrow \sigma(q) \subseteq A$ and $\sigma(q) \subseteq B \Leftrightarrow q \in \mu_\sigma(A)$ and $q \in \mu_\sigma(A) \Leftrightarrow q \in \mu_\sigma(A) \cap \mu_\sigma(B)$. Thus, $\mu_\sigma(A \cap B) = \mu_\sigma(A) \cap \mu_\sigma(B)$.

Remark 4.1. *From, the above theorem, it is obvious that $\mu_\sigma(\mu_\sigma(A)) \subseteq \mu_\sigma(A)$ and $\mu_s(\mu_s(A)) \subseteq \mu_s(A)$. In general $\mu_\sigma(\mu_\sigma(A)) \neq \mu_\sigma(A)$ and $\mu_s(\mu_s(A)) \neq \mu_s(A)$. For this, consider.*

Example 4.1. *Let $L = [0,1]$ and $(*, \to)$ be the Lukasiewicz pair. i.e. $a * b = max(0, a+b-1)$ and $a \to b = min(1, 1-a+b)$, $a, b \in L$. Then L is a residuated lattice.*
Consider the L-VFSM, $\mathscr{M} = (Q, \Sigma, \delta)$, where $Q = \{p_1, p_2, p_3, p_4\}, \Sigma = \{\sigma\}$, and δ is represented by the following diagram:



Let $A = \{p_2, p_3, p_4\}$. *Then* $\mu_\sigma(A) = \{p_3, p_4\}$ *and* $\mu_\sigma(\mu_\sigma(A)) = \{p_4\}$. *This shows that* $\mu_\sigma(\mu_\sigma(A)) \neq \mu_\sigma(A)$. *Also, if* $B = \{p_1, p_2, p_3\}$, *then* $\mu_s(B) = \{p_1, p_2\}$ *and* $\mu_s(\mu_s(B)) = \{p_1\}$. *This shows that* $\mu_s(\mu_s(B)) \neq \mu_s(B)$.

Theorem 4.2. *Let* $\mathscr{M} = (Q, \Sigma, \delta)$ *be an L-VFSM with L as complete residuated lattice without zero divisors. Then for all* $A \subseteq Q$, $\mu_\sigma(\mu_\sigma(A)) = \mu_\sigma(A)$ *and* $\mu_s(\mu_s(A)) = \mu_s(A)$.

Proof. We prove only, $\mu_\sigma(\mu_\sigma(A)) = \mu_\sigma(A)$. Let $q \in \mu_\sigma(A)$. Then $\sigma(A) \subseteq A$. In order to show that $q \in \mu_\sigma(\mu_\sigma(A))$, it is enough to show that $\sigma(q) \subseteq \mu_\sigma(A)$. So let $p \in \sigma(q)$. Then $\delta^*(p, x, q) > 0$ for some $x \in \Sigma^*$. Also, if $r \in \sigma(p)$, we have $\delta^*(r, y, p) > 0$ for some $y \in \Sigma^*$. Since, L being without zero divisors, $\delta^*(p, x, q) * \delta^*(r, y, p) > 0$. Thus, $\delta^*(r, yx, q) > 0$, whereby $r \in \sigma(q)$. But as $\sigma(q) \subseteq A$, $r \in A$. Thus $\sigma(p) \subseteq A$, implying that $\sigma(q) \subseteq \mu_\sigma(A)$. Also, as $\mu_\sigma(\mu_\sigma(A)) \subseteq \mu_\sigma(A)$, we have $\mu_\sigma(\mu_\sigma(A)) = \mu_\sigma(A)$.

Definition 4.2. [33] *Let* $\mathscr{M} = (Q, \Sigma, \delta)$ *be an L-VFSM. A subset* $R \subseteq Q$ *is called*
(i) *genetic if* $\sigma(R) \subseteq s(R)$.
(ii) *genetically closed if* $\exists P \subseteq R$ *such that* $\sigma(P) \subseteq s(P)$ *and* $s(P) = R$.

Theorem 4.3. *Let* $\mathscr{M} = (Q, \Sigma, \delta)$ *be an L-VFSM and* $R \subseteq Q$. *Then R is*
(i) *genetic iff* $R \subseteq \mu_\sigma(s(R))$.
(ii) *genetically closed iff* $\exists\ P \subseteq R$ *such that* $P \subseteq \mu_\sigma(R)$ *and* $s(P) = R$.

Proof. (i) Let R be a genetic and $r \in R$. Then $\sigma(r) \subseteq \sigma(R)$. But as R is genetic, we have $\sigma(R) \subseteq s(R) \Rightarrow \sigma(r) \subseteq s(R) \Rightarrow r \in \mu_\sigma(s(R))$ and hence $R \subseteq \mu_\sigma(s(R))$. Conversely, let $R \subseteq \mu_\sigma(s(R))$ and $q \in \sigma(R) \Rightarrow \exists\ p \in R$ such that $q \in \sigma(p)$. Since $p \in R$, we have $p \in \mu_\sigma(s(R)) \Rightarrow \sigma(p) \subseteq s(R) \Rightarrow q \in s(R)$. Thus $\sigma(R) \subseteq s(R)$. Hence, R is genetic.
(ii) Follows from (i) and the definition 4.2.

5. Fuzzy initial core and fuzzy final core operators

In this section we introduce two new fuzzy operators say fuzzy initial core and fuzzy final core. We show that these operators are the generalizations of source core and successor core operators respectively.

Definition 5.1. *Let $\mathscr{M} = (Q, \Sigma, \delta)$ be an L-VFSM and $A \in \mathscr{F}_L(Q)$. Then the fuzzy initial core and fuzzy final core of A are defined respectively as*

$$\begin{aligned} Ico(A)(q) &= \bigwedge_{\substack{x\in\Sigma^* \\ p\in Q}} \{\delta^*(p,x,q) \to A(p)\} \text{ and} \\ Fco(A)(q) &= \bigwedge_{\substack{x\in\Sigma^* \\ p\in Q}} \{\delta^*(q,x,p) \to A(p)\} \end{aligned}$$

Remark 5.1. *From the following theorem one can conclude that fuzzy initial core (Ico) and fuzzy final core (Fco) are the generalization of source core (μ_σ) and successor core (μ_s) respectively.*

Theorem 5.1. *Let $\mathscr{M} = (Q, \Sigma, \delta)$ be an L-VFSM. Then for all $q \in Q$,*
(i) $Ico(1_A)(q) = 1$ iff $q \in \mu_\sigma(A)$
(ii) $Fco(1_A)(q) = 1$ iff $q \in \mu_s(A)$.

Proof. (i) Let $A \subseteq Q$ and $q \in Q$. Suppose $Ico(1_A)(q) = 1$. Then $\bigwedge_{\substack{x\in\Sigma^* \\ p\in Q}} \{\delta^*(p,x,q) \to 1_A(p)\} = 1 \Rightarrow$ for all $p \in Q, x \in \Sigma^*, \delta^*(p,x,q) \to 1_A(p) = 1 \Rightarrow$ for all $p \in Q, x \in \Sigma^*, \delta^*(p,x,q) \leq 1_A(p)$. If, $p \in \sigma(q)$, we have $\delta^*(p,x,q) > 0$, for some $x \in \Sigma^*$. Therefore, $1_A(p) > 0$ and thus $p \in A$ Hence, $\sigma(q) \subseteq A \Rightarrow q \in \mu_\sigma(A)$.

Conversely, suppose that $q \in \mu_\sigma(A) \Rightarrow \sigma(q) \subseteq A$. For, any $p \in Q, x \in \Sigma^*$, if $\delta^*(p,x,q) = 0$ then $\delta^*(p,x,q) \leq 1_A(p)$, also if $\delta^*(p,x,q) > 0$ then $p \in \sigma(q) \subseteq A$, there fore, $1_A(p) = 1$ and thus $\delta^*(p,x,q) \leq 1_A(p) \implies$ for all $p \in Q, x \in \Sigma^*$, $\delta^*(p,x,q) \leq 1_A(p) \Rightarrow$ for all $p \in Q, x \in \Sigma^*$, $\delta^*(p,x,q) \to 1_A(p)\} = 1 \Rightarrow \bigwedge_{\substack{x\in\Sigma^* \\ p\in Q}} \{\delta^*(p,x,q) \to 1_A(p)\} = 1$ i.e. $Ico(1_A)(q) = 1$.

(ii) On the same line we can prove.

We now depict few properties of these operators.

Theorem 5.2. *Let $\mathscr{M} = (Q, \Sigma, \delta)$ be an L-VFSM. Then for all $A, B \in \mathscr{F}_L(Q)$,*

i) if $A \subseteq B$, then $Ico(A) \subseteq Ico(B)$ and $Fco(A) \subseteq Fco(B)$

ii) $Ico(A) \subseteq A$ and $Fco(A) \subseteq A$

iii) $Ico(A \cap B) = Ico(A) \cap Ico(B)$ and $Fco(A \cap B) = Fco(A) \cap Fco(B)$

iv) $Ico(Ico(A)) = Ico(A)$ and $Fco(Fco(A)) = Fco(A)$.

Proof. We only prove the properties of Ico, because the properties of Fco are similar.

(i) Let $A \subseteq B$.

Then $A(p) \leq B(p), \forall p \in Q$.

Now for any $q \in Q$, we have

$Ico(A)(q) = \bigwedge_{\substack{x \in \Sigma^* \\ p \in Q}} \{\delta^*(p,x,q) \to A(p)\} \leq \bigwedge_{\substack{x \in \Sigma^* \\ p \in Q}} \{\delta^*(p,x,q) \to B(p)\} = Ico(B)(q)$.

Thus, $Ico(A) \subseteq Ico(B)$.

(ii) Let $q \in Q$.

Then $Ico(A)(q) = \bigwedge_{\substack{x \in \Sigma^* \\ p \in Q}} \{\delta^*(p,x,q) \to A(p)\} \leq \delta^*(q, \varepsilon, q) \to A(q) = 1 \to A(q) = A(q)$.

Thus, $Ico(A) \subseteq A$.

(iii) Let $q \in Q$. Then

$$\begin{aligned}
Ico(A \cap B)(q) &= \bigwedge_{\substack{x \in \Sigma^* \\ p \in Q}} \{\delta^*(p,x,q) \to (A \cap B)(p)\} \\
&= \bigwedge_{\substack{x \in \Sigma^* \\ p \in Q}} \{\delta^*(p,x,q) \to (A(p) \wedge B(p))\} \\
&= \bigwedge_{\substack{x \in \Sigma^* \\ p \in Q}} \{(\delta^*(p,x,q) \to A(p)) \wedge (\delta^*(p,x,q) \to B(p))\} \\
&= \bigwedge_{\substack{x \in \Sigma^* \\ p \in Q}} \{\delta^*(p,x,q) \to A(p)\} \wedge \bigwedge_{\substack{x \in \Sigma^* \\ p \in Q}} \{\delta^*(p,x,q) \to B(p)\} \\
&= Ico(A)(q) \wedge Ico(B)(q) \\
&= (Ico(A) \cap Ico(B))(q).
\end{aligned}$$

Thus, $Ico(A \cap B) = Ico(A) \cap Ico(B)$.

(iv) Let $q \in Q$. Then

$$\begin{aligned}
Ico(Ico(A))(q) &= \bigwedge_{\substack{x \in \Sigma^* \\ p \in Q}} \{\delta^*(p,x,q) \to Ico(A)(p)\} \\
&= \bigwedge_{\substack{x \in \Sigma^* \\ p \in Q}} \{\delta^*(p,x,q) \to \bigwedge_{\substack{y \in \Sigma^* \\ r \in Q}} \{\delta^*(r,y,p) \to A(r)\}\} \\
&= \bigwedge_{\substack{x \in \Sigma^* \\ p \in Q}} \bigwedge_{\substack{y \in \Sigma^* \\ r \in Q}} \{\delta^*(p,x,q) \to (\delta^*(r,y,p) \to A(r))\} \\
&= \bigwedge_{\substack{x \in \Sigma^* \\ p \in Q}} \bigwedge_{\substack{y \in \Sigma^* \\ r \in Q}} \{(\delta^*(p,x,q) * \delta^*(r,y,p)) \to A(r)\} \\
&= \bigwedge_{\substack{x,y \in \Sigma^* \\ r \in Q}} \{\bigvee_{p \in Q} (\delta^*(p,x,q) * \delta^*(r,y,p)) \to A(r)\} \\
&= \bigwedge_{\substack{x,y \in \Sigma^* \\ r \in Q}} \{\delta^*(r,yx,q) \to A(r)\} = Ico(A)(q).
\end{aligned}$$

Thus, $Ico(Ico(A)) = Ico(A)$.

Theorem 5.3. *Let $\mathscr{M} = (Q, \Sigma, \delta)$ be an L-VFSM. Then for any $A \in \mathscr{F}_L(Q)$,*
(i) $so(A) = A$ iff $Ico(A) = A$,
(ii) $su(A) = A$ iff $Fco(A) = A$.

Proof. (i) Let $so(A) = A$. For any $q \in Q, so(A)(q) = A(q) \Rightarrow$ for any $q \in Q$, $\bigvee_{\substack{x \in \Sigma^* \\ p \in Q}} \{\delta^*(q,x,p) * A(p)\} = A(q) \Rightarrow$ for any $p,q \in Q, x \in \Sigma^*, \delta^*(q,x,p) * A(p) \le A(q) \Rightarrow$ for any $p,q \in Q, x \in \Sigma^*, A(p) \le \delta^*(q,x,p) \to A(q) \Rightarrow$ for any $p \in Q, A(p) \le \bigwedge_{\substack{x \in \Sigma^* \\ p \in Q}} \{\delta^*(q,x,p) \to A(q)\} \Rightarrow$ for any $p \in Q, A(p) \le Ico(A)(p)$ whence,$A \le Ico(A)$. But, by Theorem 5.2 $Ico(A) \le A$ and thus $Ico(A) = A$. Similarly, we can prove converse.

(ii) On the same line we can prove the result.

6. Conclusion

In this paper we have introduced successor core operator in an analogue to the source core operator. Two more new fuzzy operators namely fuzzy initial core and fuzzy final core are introduced and their properties are studied.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] Belohlavek R., Some Properties of residuated lattices, *Czechoslovak Mathematical Journal*, 53(1)(2003), 161-171.

[2] Ciric M., Stamenkovic A., Ignjatovic J. and Petkovic, Fuzzy relation equations and reduction of fuzzy automata, *Journal of Computer and System Sciences*, 76(2010), 609-633.

[3] Das P., A fuzzy topology associated with a fuzzy finite state machine, *Fuzzy Sets and Systems*, 105(1999), 469-479.

[4] Guo X., Grammar theory based on lattice-ordered monoid, *Fuzzy Sets and Systems*, 160(2009), 1152-1161.

[5] Gupta M.M., Saridis G.N. and Gaines B.R., *Fuzzy Automata and Dicision Processes*, North-Holland, New York, 1979.

[6] Hohle U. and Klement E.P., *Non-classical Logics and Their Applications to Fuzzy Subsets*, Kluwer Academic Publishers, Dordrecht, Boston, London 1995.

[7] Ignjatović J., Ćirić M. and Bogdanović S., Determinization of fuzzy automata with membership values in complete residuated lattices, *Information Sciences*, 178(2008), 164180.

[8] Kandel A. and Lee S.C., *Fuzzy Switching and Automata: Theory and Applications*, London: Grane Russak, 1980, 171-262.

[9] Kim Y.H., Kim J.G. and Cho S.J., Products of T-generalized state machine and T-generalized transformation semigroups, *Fuzzy Sets and Systems*, 93(1998), 87-97.

[10] Kumbhojkar H.V. and Chaudhari S.R., Fuzzy recognizers and recognizable sets, *Fuzzy Sets and Systems*, 131(2002), 381-392.

[11] Kumbhojkar H.V.and Chaudhari S.R., On covering of products of fuzzy finite state machines, *Fuzzy Sets and Systems*, 125(2002), 215-222.

[12] Lee E. T. and Zadeh L. A., Note on Fuzzy Languages,*Information Sciences*, 1(1969), 421-434.

[13] Li Y., Finite automata theory with membership values in lattices, *Information Sciences*, 181(2011), 10031017.

[14] Li P. and Li Y., Algebraic properties of *LA*-languages, *Information Sciences*, 176(2006), 3232-3255.

[15] Li Y. and Pedrycz W., Fuzzy finite automata and fuzzy regular expressions with membership values in lattice-ordered monoids, *Fuzzy Sets and Systems*, 156(2005), 68-92.

[16] Lihua W.and Qiu D., Automata theory based on complete residuated lattice-valued logic: Reduction and minimization, *Fuzzy Sets and Systems*, 161(2010), 1635-1656.

[17] Malik D. S., Mordenson J. N. and Sen M. K., On Subsystem of a Fuzzy Finite State Machine, *Fuzzy Sets and Systems*, 68(1994), 83-92.

[18] Malik D. S., Mordeson J. N. and Sen M. K., Product of Fuzzy Finite State Machines, *Fuzzy Sets and Systems*, 92(1997), 95-102.

[19] Malik D. S., Mordenson J. N. and Sen M. K., Semigroups of Fuzzy Finite State Machines, *Advance in Fuzzy Theory and Technology*, 2(1994), 87-98.

[20] Malik D. S., Mordeson J. N. and Sen M. K., Submachines of Fuzzy Finite State Machines, *Journal of Fuzzy Mathematics*, 2(1994), 781-92.

[21] Malik D. S., Mordenson J. N. and Sen M. K., The Cartesian Composition of Fuzzy Finite State Machines, *Kybernetics*, 68(1995), 98-110.

[22] Mordeson J.N. and Malik D.S., *Fuzzy Automata and Languages:Theory and Applications,Chapman & Hall*, CRC, BocaRaton, London, 2002.

[23] Peeva, Ketty, and Yordan Kyosev. *Fuzzy relational calculus: theory, applications and software (with CD-ROM).* World Scientific, 2005.

[24] Qiu D.W., Automata theory based on complete residuated lattice-valued logic (I), *Science in China (Series F)*, 44(6)(2001), 419-429.

[25] Qiu D.W., Automata theory based on complete residuated lattice-valued logic (II), *Science in China (Series F)*, 45(6)(2002), 442-452.

[26] Qiu D.W. and Ying M., Characterizations of quantum automata, *Theoretical Computer Science*, 312(2004), 479-489.

[27] Santos E.S., Fuzzy automata and languages, *Information Sciences*, 10(1976), 193-197.

[28] Santos E.S., Maxmin automata, *Inform. and control*, 13(1968), 363-377.

[29] Santos E.S., Realization of fuzzy languages by probabilistic, max product and maxmin automata, *Information Sciences*, 8(1975), 39-53.

[30] Santos E.S., Regular fuzzy expressions. In Gupta M.M., Saridis G.N. and Gaines B.R. Fuzzy Automata and Dicision Processes. North-Holland, New York, 1977, 169-175.

[31] Srivastava A.K. and Tivari S.P., A topology for fuzzy automata, *Lecture Notes in Artificial Intelligence*, 2275(2002), 485-490.

[32] Thomason M.G. and Marinos P.N., Deterministic acceptors of regular fuzzy languages, *IEEE Trans Syst Man Cybern*, 4(1994), 228-230.

[33] Tiwari S.P. and Sharan S., Fuzzy automata based on Lattice-ordered monoid with algebraic and topological aspects, *Fuzzy.Inf.Eng.*, 2(2012), 155-164.

[34] Tiwari S.P., Singh A.K. and Sharan S., Fuzzy automata based on Lattice-ordered monoid and associated topology, *Journal of Uncertain Systems*, 6(1)(2012), 51-55.

[35] Tiwari S.P., Singh A.K., Sharan S. and Yadav V.K., Bifuzzy core of fuzzy automata, *Iranian Journal of Fuzzy Systems*, 12(2)(2015), 63-73.

[36] Wechler W., *The concept of fuzzyness in automata and languages theory,* Akademie-Verlag, Berlin, 1978.

[37] Wee W.G., *On generalization of the fuzzy sets concept to pattern classification,* Ph.D.Thesis, TR-EE 67/7, Purde Uni. Lafayette, Indiana, 1967.

[38] Wee W.G., Fu K.S., A formation of fuzzy automata and its application as a model of learning system, *IEEE Trans. Systems Sci. Cybernet.*, 5(1969), 215-223.

[39] Xing H., Qiu D.W., Liu F. and Fan Z., Equivalence in automata theory based on complete residuated lattice-valued logic, *Fuzzy Sets and Systems*, 158(2007), 1407-1422.