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BEST LINEAR UNBIASED ESTIMATE OF LINEAR TREND-CYCLE COMPONENT BASED ON FBE-DERIVED VARIABLES

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Abstract: The Best linear unbiased estimates (BLUE) of the parameters (slope and intercept) of a linear trend-cycle component based on Fixed Base Estimation (FBE) derived variables are discussed in this paper. The FBE-derived variables were found to have constant mean, non-constant variance but with constant autocorrelation coefficient at all lags. The variance of the variables decreased with recent time points, indicating that estimates of the slope from recent periods are more precise. Best Linear unbiased estimates of the slope from FBE-derived variables also attach greater weights to the more recent observations but have the same minimum variance as those from Chain Base Estimation (CBE) derived variables. Simulated numerical examples were used to illustrate the methods. The simulation results show that BLUE from the FBE and CBE-derived variables outperform the Simple Average and Least Squares Methods in terms of Mean Percentage Error (MPE), Mean Square Error (MSE) and Mean Absolute Percentage Error (MAPE) of forecasts.

Keywords: Best linear unbiased Estimates, minimum variance, Moving Average
Process of order one, Derived variables, partial autocorrelation structure

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1. Introduction

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For short series in which the trend and cyclical components are jointly estimated, the two contending models are the additive and multiplicative models (Chatfield (2004), Kendall and Ord (1990)).

Additive model:

$$X_t = M_t + S_t + e_t \tag{1.1}$$

Multiplicative model:

$$X_t = M_t \; S_t \; e_t \tag{1.2}$$

where M_t is the trend-cycle component ; S_t is the seasonal component with the property that $S_{(i-1)s+j} = S_j$, $i = 1, 2, ..., m$, and e_t is the irregular or random component.

Appropriate assumptions on S_t and e_t for the additive and multiplicative models can be found in Chatfield (2004), Kendall and Ord (1990) and Iwueze and Nwogu (2004). The linear trend-cycle
component $\left(M_t\right)$ can be written as

$$M_t = a + bt \tag{1.3}$$

For a series which do contain a substantial trend, the ttraditional practice is to (i) fit a trend curve by some method and de-trend the series (ii) use the de-trended series to estimate the seasonal indices.

From the periodic averages ($\overline{X}_{i.}$) of the Buys-Ballot Table, Iwueze and Nwogu (2004) derived two sets of variables. On bases of these variables they developed two methods of estimating the parameters of a linear trend-cycle component for short period series. And the results obtained by Iwueze and Nwogu (2004) for the additive and multiplicative models are summarized in Iwueze, Nwogu and Ajaraogu (2010).

The two sets of derived-variables are: (i) the Chain Base Estimation (CBE) derived variables defined as

$$\hat{b}_i^{(c)} = \frac{\overline{X}_{(i+1).} - \overline{X}_{i.}}{s}, \; i = 1, 2, ..., m-1 \tag{1.4}$$

and (ii) the Fixed Base Estimation (FBE) derived variables defined as

$$\hat{b}_i^{(f)} = \frac{\overline{X}_{(i+1).} - \overline{X}_{1.}}{i\,s}, \; i = 1, 2, ..., m-1 \tag{1.5}$$

It is important to note that each set of the derived variables gives (m-1) estimates of the slope of the trend-cycle component. Hence, Iwueze and Nwogu (2004) gave the estimate of the slope of the linear trend-cycle component as a simple average of the derived variables. That is,

$$\hat{b}^{(c)} = \frac{1}{m-1}\sum_{i=1}^{m-1}\hat{b}_i^{(c)} \tag{1.6}$$

for the CBE and

$$\hat{b}^{(f)} = \frac{1}{m-1}\sum_{i=1}^{m-1}\hat{b}_i^{(f)} \tag{1.7}$$

for the FBE.

Iwueze, Nwogu and Ajaraogu (2010) derived the means and variances of $\hat{b}^{(c)}$ and $\hat{b}^{(f)}$ to be

$$E(\hat{b}^{(c)}) = b \tag{1.8}$$

$$\text{var}(\hat{b}^{(c)}) = \frac{2\sigma^2}{s^3} \tag{1.9}$$

$$E(\hat{b}^{(f)}) = b \tag{1.10}$$

$$\text{var}(\hat{b}^{(f)}) = \left(\frac{2\sigma^2}{s^3}\right)\lambda \tag{1.11}$$

where

$$\lambda = \sum_{i=2}^{m}\frac{1}{(i-1)^2} + \sum_{i}^{m}\sum_{<\ j}^{m}\frac{1}{(i-1)(j-1)} \tag{1.12}$$

Iwueze, Nwogu and Ajaraogu (2011) investigated the covariance structures of the CBE and FBE derived variables. For the CBE derived variables, if we let $R^{(c)}(k) = \text{cov}\left(\hat{b}_{i.}^{(c)}, \hat{b}_{(i+k).}^{(c)}\right)$ and $\rho_k^{(c)} = R^{(c)}(k)/R^{(c)}(0)$, then

$$R^{(c)}(k) = \begin{cases} 2\sigma^2/s^3, & k = 0 \\ -\sigma^2/s^3, & k = \pm 1 \\ 0, & k > 1 \end{cases} \tag{1.13}$$

$$\rho_k^{(c)} = \begin{cases} 1, & k = 0 \\ -1/2, & k = \pm 1 \\ 0, & k > 1 \end{cases} \tag{1.14}$$

Similarly, for the FBE derived variables, if we let $R^{(f)}(k) = \mathrm{cov}\left(\hat{b}_{i.}^{(f)}, \hat{b}_{(i-k).}^{(f)}\right)$ and $\rho_k^{(f)} = R^{(f)}(k)/R^{(f)}(0)$, then

$$R^{(f)}(k) = \begin{cases} 2\sigma^2 / i^2 s^3, & k = 0 \\ \sigma^2 / (i(i-k))\, s^3, & k \neq 0 \end{cases} \tag{1.15}$$

$$\rho_k^{(f)} = \begin{cases} 1, & k = 0 \\ 1/2, & k \neq 0 \end{cases} \tag{1.16}$$

Using the covariance structures (1.13) and (1.14), Iwueze, Nwogu and Ajaraogu (2011) derived the BLUE for the slope based on the CBE derived variables because they exhibit stationarity while those of the FBE do not. The BLUE of the slope, based on the CBE derived variables, was defined as

$$T^{(c)} = \sum_{t=1}^{m-1} \alpha_i \hat{b}_i^{(c)} \tag{1.17}$$

where

$$\sum_{t=1}^{m-1} \alpha_i = 1 \tag{1.18}$$

The mean and variance of $T^{(c)}$ were shown to be

$$E(T^{(c)}) = b \tag{1.19}$$

$$\mathrm{var}(T^{(c)}) = \left(\frac{2\sigma^2}{s^3}\right) S(\boldsymbol{\alpha}) \tag{1.20}$$

where

$$S(\boldsymbol{\alpha}) = \sum_{i=1}^{m-1} \alpha_i^2 - \sum_{i=1}^{m-2} \alpha_i \alpha_{i+1}, \quad \boldsymbol{\alpha}^T = (\alpha_1, \alpha_2, \ldots, \alpha_{m-1}) \tag{1.21}$$

The main objective of this paper is to obtain the BLUE of the slope parameter for the additive model using the non-stationary FBE derived variables. Section 2 presents the preliminaries (the partial autocorrelation structure of FBE derived variables), while Section 3 discusses the main results (including the BLUE of the slope parameter based on the FBE derived variables, the BLUE for the intercept (a), the corresponding standard error and some numerical examples to illustrate the results).

2 Preliminaries

Partial Autocorrelation of the FBE-derived Variables for the Additive Model

The partial autocorrelation function (pacf) of the process $Y_t, t \in Z$, whose autocorrelation function is ρ_k, is given by (Box et al.1994)

$$\phi_{kk} = \begin{cases} \rho_1, & \text{for } k = 1 \\ \left|\mathbf{A}_k^*\right| / \left|\mathbf{A}_k\right| & \text{for } k > 1 \end{cases} \tag{2.1}$$

where

$$\mathbf{A}_k = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{k-4} & \rho_{k-3} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-2} & \cdots & \rho_1 & 1 \end{bmatrix} \tag{2.2}$$

$\left|\mathbf{A}_k\right|$ is the determinant of $\mathbf{A}_k$ and $\mathbf{A}_k^*$ is composed of the first k-1 columns of $\mathbf{A}_k$ with the kth column replaced by $\boldsymbol{\rho} = \left(\rho_1 \;\; \rho_2 \ldots \rho_{k-1} \;\; \rho_k \right)^T$. Theorem 1 below is needed to define the partial autocorrelation function for FBE derived variables.

Theorem 1: *If* $\rho_k = \rho$, for all k, *then the partial autocorrelation function is given by*

$$\phi_{kk} = \frac{\rho}{1 + (k-1)\rho}, \quad \text{for } k \geq 1 \tag{2.3}$$

Proof

By definition in (2.1), if $\rho_k = \rho$ for all k then

$$\phi_{kk} = \rho, \quad \text{for } k = 1$$

$$\phi_{22} = \frac{\begin{vmatrix} 1 & \rho \\ \rho & \rho \end{vmatrix}}{\begin{vmatrix} 1 & \rho \\ \rho & 1 \end{vmatrix}} = \frac{\rho - \rho^2}{1 - \rho^2} = \frac{\rho(1-\rho)}{(1-\rho)(1+\rho)} = \frac{\rho}{(1+\rho)} = \frac{\rho}{(1+(2-1)\rho)}, \quad \text{for } k = 2$$

$$\phi_{33} = \frac{\begin{vmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & \rho \end{vmatrix}}{\begin{vmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{vmatrix}} = \frac{\rho(-1+\rho)^2}{(1+2\rho)(-1+\rho)^2} = \frac{\rho}{(1+2\rho)} = \frac{\rho}{1+(3-1)\rho}, \text{ for } k=3$$

In general,

$$\phi_{kk} = \frac{\begin{vmatrix} 1 & \rho \ldots & \rho\,\rho \\ \rho & 1 \ldots & \rho\,\rho \\ \ldots & \ldots & \ldots \\ \rho & \rho \ldots & \rho\,\rho \end{vmatrix}}{\begin{vmatrix} 1 & \rho \ldots & \rho \\ \rho & 1 \ldots & \rho \\ \ldots & \ldots & \rho \\ \rho & \rho \ldots & 1 \end{vmatrix}} = \frac{\rho(-1+\rho)^{k-1}}{(1+(k-1)\rho)(-1+\rho)^{k-1}} = \frac{\rho}{1+(k-1)\rho}, \text{ for } k>1$$

In particular, if $\rho = \frac{1}{2}$, as in FBE (Equation (1.16) then Equation (2.3) reduces to

$$\phi_{kk} = \frac{\rho}{1+(k-1)\rho} = \frac{1/2}{1+(k-1)\,1/2} = \frac{1}{(k+1)} \quad (2.4)$$

3. Main results

3.1 Best Linear Unbiased Estimates of Slope Using the FBE Derived Variables.

The sequence of FBE derived random variables, $\hat{b}_i^{(f)}$, i =1, 2, 3, ..., $m-1$, have been found to have the covariance structure of a non-stationary series with the autocorrelation function and partial autocorrelation function given by (1.16) and (2.4). Thus, the use of their simple average given in Equation (1.7) as an estimate of the slope (b), as recommended by Iwueze and Nwogu (2004), may

not give a reliable estimate. This calls for an alternative method which takes cognisance of the correlation among the variables to produce more reliable estimate of the slope (b) of a linear trend-cycle component.

A linear combination of variables which is an unbiased estimate of a parameter that has minimum variance (among all linear unbiased estimates) is called "best linear unbiased estimate (BLUE)". If let $\beta_1, \beta_2, , \beta_{m-1}$ be any set of real numbers. A linear estimate of the slope b $\left(E\left(\hat{b}_i^{(f)} = b\right)\right)$ is given by

$$T^{(f)} = \sum_{i=1}^{m-1} \beta_i \hat{b}_i^{(f)} \tag{3.1}$$

The expected value of $T^{(f)}$ is

$$E\left(T^{(f)}\right) = \sum_{i=1}^{m-1} \beta_i E\left(\hat{b}_i^{(f)}\right) = \sum_{i=1}^{m-1} \beta_i b = = b\sum_{i=1}^{m-1} \beta_i \tag{3.2}$$

$T^{(f)}$ is unbiased if and only if

$$\sum_{i=1}^{m-1} \beta_i = 1 \tag{3.3}$$

The variance of $T^{(f)}$ is given by

$$\mathrm{var}\left(T^{(f)}\right) = \sum_{i=1}^{m-1} \beta_i^2 \, \mathrm{var}\left(\hat{b}_i^{(f)}\right) + 2\sum_{i}\sum_{<j} \beta_i \beta_j \, \mathrm{cov}\left(\hat{b}_i^{(f)}, \hat{b}_j^{(f)}\right) \tag{3.4}$$

For the sequence of random variables, $\hat{b}_i^{(f)}$, $i = 1, 2, 3, ..., m-1$, with autocovariance structure given in (1.15) and autocorrelation structure given in (1.16), $\mathrm{var}\left(T^{(f)}\right)$ can be written as

$$\mathrm{var}\left(T^{(f)}\right) = R(0)\sum_{i=1}^{m-1} \frac{\beta_i^2}{(i)^2} + 2R(k)\sum_{i<}^{m-1}\sum_{j}^{m-1} \frac{\beta_i \beta_j}{(i)(j)} \tag{3.5}$$

$$= R(0)\left\{\sum_{i=1}^{m-1} \frac{\beta_i^2}{(i)^2} + \sum_{i<}^{m-1}\sum_{j}^{m-1} \frac{\beta_i \beta_j}{(i)(j)}\right\}$$

$$= \frac{2\sigma^2}{s^3}\left\{\sum_{i=1}^{m-1} \frac{\beta_i^2}{(i)^2} + \sum_{i<}^{m-1}\sum_{j}^{m-1} \frac{\beta_i \beta_j}{(i)(j)}\right\} = \frac{2\sigma^2}{s^3} S(\boldsymbol{\beta}) \tag{3.6}$$

where

$$S(\boldsymbol{\beta}) = \sum_{i=1}^{m-1} \frac{\beta_i^2}{(i)^2} + \sum_{i<}^{m-1}\sum_{j}^{m-1} \frac{\beta_i \beta_j}{(i)(j)} \tag{3.7}$$

Then a linear unbiased estimate of b will have minimum variance (among all linear unbiased estimates) if $S(\boldsymbol{\beta})$ is minimum. Hence, the BLUE of b is obtained if we choose $\beta_1, \beta_2, ..., \beta_{m-1}$ that minimize $S(\boldsymbol{\beta})$ with respect to the constraint $\sum_{i=1}^{m-1} \beta_i = 1$.

Note that the expression in (3.7) can be rewritten as

$$S(\boldsymbol{\beta}) = \sum_{i=1}^{m-1} \frac{\beta_i^2}{(i)^2} + \frac{1}{2}\sum_{i\neq}^{m-1}\sum_{j}^{m-1}\frac{\beta_i\beta_j}{(i)(j)}$$

$$= \frac{1}{2}\left\{\sum_{i=1}^{m-1}\frac{2\beta_i^2}{(i)^2} + \sum_{i\neq}^{m-1}\sum_{j}^{m-1}\frac{\beta_i\beta_j}{(i)(j)}\right\} \quad (3.8)$$

In matrix form the constraint in Equation (3.3) is equivalent to

$$\boldsymbol{\beta}'\,\mathbf{1} = 1. \quad (3.9)$$

where $\boldsymbol{\beta}' = (\beta_1, \beta_2, \ldots \beta_{m-1})$ and $\mathbf{1}' = (1, 1, \ldots, 1)$ is a m-1 component vector.

The $S(\boldsymbol{\beta})$ in (3.8) can be re-written as

$$S(\boldsymbol{\beta}) = \boldsymbol{\beta}'\,\mathbf{V}\,\boldsymbol{\beta} \quad (3.10)$$

where

$$\mathbf{V}_{m-1} = \frac{1}{2}\begin{bmatrix} 2.1 & \left(1.\frac{1}{2}\right) & \left(1.\frac{1}{3}\right) & \cdots & \left(1.\frac{1}{m-2}\right) & \left(1.\frac{1}{m-1}\right) \\ \left(\frac{1}{2}.1\right)_1 & 2\left(\frac{1}{2}\right)^2 & \left(\frac{1}{2}.\frac{1}{3}\right) & \cdots & \left(\frac{1}{2}.\frac{1}{m-2}\right) & \left(\frac{1}{2}.\frac{1}{m-1}\right) \\ \left(\frac{1}{3}.1\right) & \left(\frac{1}{3}.\frac{1}{2}\right) & 2\left(\frac{1}{3}\right)^2 & \cdots & \left(\frac{1}{3}.\frac{1}{m-2}\right) & \left(\frac{1}{3}.\frac{1}{m-1}\right) \\ . & & & & & \\ . & & & & & \\ \left(\frac{1}{m-1}.1\right) & \left(\frac{1}{m-1}.\frac{1}{2}\right) & \left(\frac{1}{m-1}.\frac{1}{3}\right) & \cdots & \left(\frac{1}{m-1}.\frac{1}{m-2}\right) & 2\left(\frac{1}{m-1}\right)^2 \end{bmatrix} =$$

$$\frac{1}{2}\left\{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \left(\frac{1}{2}\right)^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \left(\frac{1}{3}\right)^2 & \cdots & 0 & 0 \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & \cdots & 0 & \left(\frac{1}{n}\right)^2 \end{bmatrix} + \begin{bmatrix} 1 & \left(1.\frac{1}{2}\right) & \left(1.\frac{1}{3}\right) & \cdots & \left(1.\frac{1}{n}\right) \\ \left(\frac{1}{2}.1\right)_1 & \left(\frac{1}{2}\right)^2 & \left(\frac{1}{2}.\frac{1}{3}\right) & \cdots & \left(\frac{1}{2}.\frac{1}{n}\right) \\ \left(\frac{1}{3}.1\right) & \left(\frac{1}{3}.\frac{1}{2}\right) & \left(\frac{1}{3}\right)^2 & \cdots & \left(\frac{1}{3}.\frac{1}{n}\right) \\ . & & & & \\ . & & & & \\ \left(\frac{1}{n}.1\right) & \left(\frac{1}{n}.\frac{1}{2}\right) & \left(\frac{1}{n}.\frac{1}{3}\right) & \cdots & \left(\frac{1}{n}\right)^2 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\left[\mathbf{D}_n\,\mathbf{D}'_n + \mathbf{e}_n\,\mathbf{e}'_n\right] \quad (3.11)$$

where $n = m-1$

$$\mathbf{D}_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \left(\frac{1}{2}\right) & 0 & \dots & 0 & 0 \\ 0 & 0 & \left(\frac{1}{3}\right) & \dots & 0 & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & \dots & 0 & \left(\frac{1}{n}\right) \end{bmatrix} \text{ and } \mathbf{e}_n = \begin{bmatrix} 1 \\ \left(\frac{1}{2}\right) \\ \left(\frac{1}{3}\right) \\ . \\ . \\ . \\ \left(\frac{1}{n}\right) \end{bmatrix}$$

Therefore, the problem of finding the minimum variance is equivalent to minimizing $S(\boldsymbol{\beta}) = \boldsymbol{\beta}' \mathbf{V} \boldsymbol{\beta}$, subject to $\boldsymbol{\beta}'\mathbf{1} = 1$. Using Lagrange's multiplier, this is equivalent to

$$S(\boldsymbol{\beta}, \lambda) = \boldsymbol{\beta}' \mathbf{V} \boldsymbol{\beta} - \lambda(\boldsymbol{\beta}'\mathbf{1} - 1) \tag{3.12}$$

Taking partial derivatives of (3.12) with respect to $\boldsymbol{\beta}$ and λ and equating each to zero give the following Equations

$$\frac{\partial S(\boldsymbol{\beta}, \lambda)}{\partial \boldsymbol{\beta}} = 2\mathbf{V}\boldsymbol{\beta} - \lambda\mathbf{1} = 0 \tag{3.13}$$

$$\frac{\partial S(\boldsymbol{\beta}, \lambda)}{\partial \lambda} = -(\boldsymbol{\beta}'\mathbf{1} - 1) = 0 \tag{3.14}$$

From (3.13)

$$\boldsymbol{\beta} = \frac{\lambda}{2}\mathbf{V}^{-1}\mathbf{1} \tag{3.15}$$

Substituting (3.15) into (3.14) gives

$$\frac{\lambda}{2} = \frac{1}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} \tag{3.16}$$

Hence,

$$\boldsymbol{\beta} = \frac{\lambda}{2}\mathbf{V}^{-1}\mathbf{1} = \frac{\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} \tag{3.17}$$

and

$$S(\boldsymbol{\beta}) = \boldsymbol{\beta}'\mathbf{V}\boldsymbol{\beta} = \left(\frac{\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}\right)'\mathbf{V}\left(\frac{\mathbf{V}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}\right) = \frac{1}{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}} \tag{3.18}$$

As an example of the minimization of (3.10) subject to the constraint in (3.9), we let $m - 1 = 4 \Rightarrow m = 5$.

$$\mathbf{D}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0 & 0.5000 & 0 & 0 \\ 0 & 0 & 0.3333 & 0 \\ 0 & 0 & 0 & 0.2500 \end{bmatrix}$$

$$, \quad e_4 = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1.0000 \\ 0.5000 \\ 0.3333 \\ 0.2500 \end{bmatrix}$$

$$\mathbf{V}_4 = \frac{1}{2}\left[\mathbf{D}_n \mathbf{D'}_n + \mathbf{e}_n \mathbf{e'}_n\right] =$$

$$\begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{6} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{12} & \frac{1}{16} \\ \frac{1}{6} & \frac{1}{12} & \frac{1}{9} & \frac{1}{24} \\ \frac{1}{8} & \frac{1}{16} & \frac{1}{24} & \frac{1}{16} \end{bmatrix} = \begin{bmatrix} 1.0000 & 0.2500 & 0.1666 & 0.1250 \\ 0.2500 & 0.2500 & 0.0833 & 0.0625 \\ 0.1666 & 0.0833 & 0.1111 & 0.0417 \\ 0.1250 & 0.0625 & 0.0417 & 0.0625 \end{bmatrix}$$

Hence, Equation (3.17) reduces to

$$\boldsymbol{\beta}_4 = \frac{\mathbf{V}_4^{-1}\mathbf{1}}{\mathbf{1'V}_4^{-1}\mathbf{1}} = \begin{bmatrix} -0.100 \\ 0.000 \\ 0.300 \\ 0.800 \end{bmatrix} \quad \text{and} \quad S_4(\boldsymbol{\beta}) = \boldsymbol{\beta}'_4 \mathbf{V}_4 \boldsymbol{\beta}_4 = 0.05$$

The weights for $m = 3, 4, ..., 21$ $(m - 1 = 2, 3, ..., 20)$ are given in Table 1. As Table 1 shows, the weights increased as period (i) increased. This indicates that the FBE attaches greater weights to the derived variables ($b_i^{(f)}$)'s of the more recent years/periods in computing the BLUE for the slope. This is, in agreement with the views expressed by Equation (1.15), which indicates that the variance of $b_i^{(f)}$ decreases (i.e. $b_i^{(f)}$ becomes more precise) as the period (i) in creases.

Table 1: Sample sizes (m) and their corresponding FBE weights

(β_i, i = 1, 2, ..., m − 1)

i	Sample size m									
	3	4	5	6	7	8	9	10	11	12
1	0.000	-0.100	-0.100	-0.086	-0.071	-0.060	-0.050	-0.042	-0.036	-0.032
2	1.000	0.200	0.000	-0.057	-0.071	-0.071	-0.067	-0.061	-0.055	-0.049
3		0.900	0.300	0.086	0.000	-0.036	-0.050	-0.055	-0.055	-0.052
4			0.800	0.343	0.143	0.048	0.000	-0.024	-0.036	-0.042
5				0.714	0.357	0.179	0.083	0.030	0.000	-0.018
6					0.643	0.357	0.200	0.109	0.054	0.021
7						0.583	0.350	0.212	0.127	0.073
8							0.533	0.339	0.218	0.140
9								0.492	0.328	0.221
10									0.454	0.315
11										0.423
$S(\beta)$	0.25	0.100	0.05	0.029	0.018	0.012	0.008	0.006	0.005	0.004

Table 1 continued

i	Sample size m								
	13	14	15	16	17	18	19	20	21
1	-0.028	-0.024	-0.021	-0.019	-0.017	-0.016	-0.014	-0.013	-0.012
2	-0.044	-0.04	-0.036	-0.032	-0.029	-0.027	-0.025	-0.023	-0.021
3	-0.049	-0.046	-0.043	-0.040	-0.037	-0.034	-0.032	-0.029	-0.027
4	-0.044	-0.044	-0.043	-0.041	-0.039	-0.037	-0.035	-0.033	-0.031
5	-0.028	-0.033	-0.036	-0.037	-0.037	-0.036	-0.035	-0.034	-0.033
6	0.000	-0.013	-0.022	-0.027	-0.029	-0.031	-0.032	-0.032	-0.031
7	0.038	0.015	0.000	-0.010	-0.017	-0.022	-0.025	-0.026	-0.027
8	0.088	0.053	0.029	0.012	0.000	-0.008	-0.014	-0.018	-0.021
9	0.149	0.099	0.064	0.040	0.022	0.009	0.000	-0.007	-0.012
10	0.220	0.154	0.107	0.074	0.049	0.031	0.018	0.008	0.000
11	0.302	0.218	0.157	0.113	0.081	0.057	0.039	0.025	0.014
12	0.396	0.29	0.214	0.159	0.118	0.087	0.064	0.045	0.031
13		0.372	0.279	0.210	0.159	0.121	0.092	0.069	0.051
14			0.350	0.268	0.206	0.159	0.124	0.095	0.073
15				0.330	0.257	0.201	0.159	0.124	0.097
16					0.314	0.248	0.198	0.156	0.125
17						0.299	0.240	0.192	0.155
18							0.280	0.230	0.187
19								0.272	0.223
20									0.260
$S(\beta)$	0.003	0.002	0.002	0.002	0.001	0.001	0.001	0.001	0.001

For the CBE, Iwueze, Nwogu and Ajaraogu (2010) showed that the BLUE for the slope assigns weights to the $\hat{b}_i^{(c)}$s in a symmetrical form such that the $\hat{b}_i^{(c)}$s within the central periods have greater weights than those at the beginning and end of the period. The distribution of weights among the CBE and FBE derived variables for m = 10 is further illustrated in Figure 1.

For the same m, the two sets of weights (CBE and FBE), lead to the same sum of squares, ie $S(\boldsymbol{\alpha}) = S(\boldsymbol{\beta})$. In other words, the BLUE for both the FBE and CBE derived variables have the same minimum variance for any given set of data.

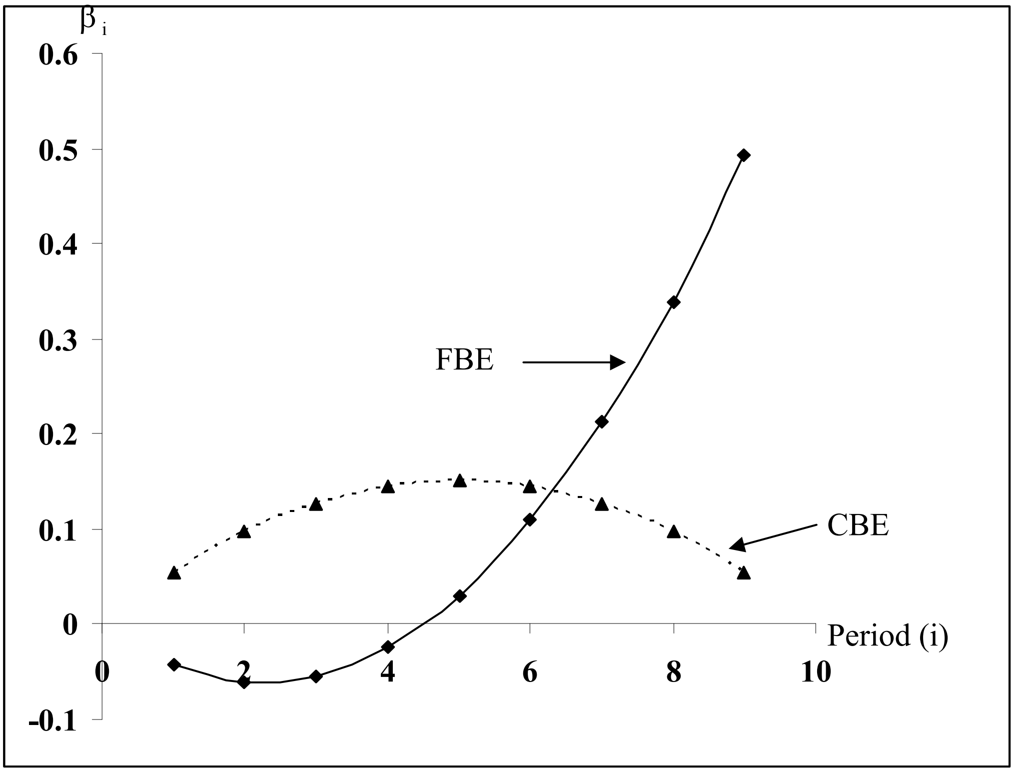

Figure 1: CBE and FBE weights for m = 10.

3.2. Best Linear Unbiased Estimates of the Intercept Using the FBE Derived Variables.

The estimate of the intercept (a) is given by Iwueze and Nwogu (2004) in terms of $\hat{b}^{(f)}$ as

$$\hat{a} = \overline{X}_{..} - \frac{\hat{b}}{2}(n+1) \tag{4.1}$$

where $\hat{b} = \hat{b}^{(f)}$ for the FBE method. Hence, the BLUE for the intercept (a) can be obtained by substituting the BLUE for the slope into (4.1).

The variance of $\hat{a}$ was given by Iwueze, Nwogu and Ajaraogu (2010) as

$$\operatorname{var}(\hat{a}) = \frac{\sigma^2}{n} + \left(\frac{n+1}{2}\right)^2 \operatorname{var}(\hat{b}) \tag{4.2}$$

Hence, the variance of the BLUE for the intercept (a) can be derived by substituting the variance of the BLUE for the slope given in Equation (3.6) into (4.2).

3.3. Empirical Examples

The computation of BLUE from FBE derived variables is illustrated here with simulations of $n = sm$ ($m = 8$ and 9) observations from $X_t = a + bt + S_t + e_t$ with $s = 12$, $a = 1.0$, $b = 0.2$, $S_1 = 5.65$, $S_2 = -4.60$, $S_3 = 1.59$, $S_4 = -2.38$, $S_5 = 0.38$, $S_6 = 1.32$, $S_7 = 1.23$, $S_8 = -1.98$, $S_9 = -1.21$, $S_{10} = -2.05$, $S_{11} = 0.73$, $S_{12} = 1.32$ and $e_t \sim N(0, 1)$. The first $n_0 = (m-1)s$ observations were used to determined and compare the properties of the BLUE for the FBE with those from the Least Squares Estimation (LSE) method, Simple Average Estimation (SAE) method of the FBE and CBE derived variables and BLUE for CBE derived variables as shown in Table 2. The last $s = 12$ observations were used to assess the forecasting performances of five methods of estimation as shown in Table 3.

As Table 2 shows, the BLUE of the slope (b) and the estimate of standard error from the FBE method are the same as those from the CBE method. This is expected, since the multipliers, $S(\boldsymbol{\alpha})$ and $S(\boldsymbol{\beta})$, for FBE and CBE respectively are the same. The BLUE for the slope are more precise than the estimates from simple average method which ignored the serial correlation among derived variables. The BLUE for the intercept is also the same for the FBE and CBE. This is expected, since estimate of the intercept depends on the estimate of the slope. The BLUE for the intercept is also more precise than those from the other estimation methods.

The adequacy of the FBE fitted model and its forecasting performance are examined, while comparing it with those of the other methods of estimation as shown in Table 3. The adequacy of the fitted model was assessed using the JB statistic defined by Bera and Jarque (1980) as

$$JB = \frac{n}{6}\gamma_1^2 + \frac{n}{24}(\gamma_2)^2 \tag{5.1}$$

where γ_1 is a measure of skewness and $\alpha_4 = \gamma_2 + 3$ is a measure of kurtosis. The JB statistic is used in residual analysis to test for normality, homoscedasticity and serial independence of regression residuals. When regression residuals are independent and normally distributed with constant mean and variance the statistic follows the Chi-square distribution with 2 degrees of freedom.

When compared with the tabulated values of Chi-square ($\chi^2_2(0.05)$), the values of the JB statistic in Table 3 indicate that the fitted models are adequate for the series. Table 3 also shows that BLUE from both CBE and FBE methods recover the error mean and standard deviation equally well. In terms of MPE, MSE and MAPE of forecasts, BLUE from CBE and FBE performed equally well and outperformed the Simple Average Estimation (SAE) and Least Squares Estimation (LSE) methods.

Remarks

This study has examined the Best Linear Unbiased Estimator (BLUE) of the slope (b) of a linear trend-cycle component of time series computed from FBE derived variables defined by Iwueze and Nwogu (2004). Since the estimates of the intercept and seasonal indices depend on it, the emphasis in this study is therefore, on the slope. The properties of the BLUE from the FBE derived variables were determined and compared with those from the Least Squares Estimation (LSE) method, Simple Average Estimation (SAE) method from the FBE and CBE derived variables and BLUE for CBE derived variables. The results show that BLUE from both CBE and FBE methods performed equally well in terms of estimating the slope and intercept as well as their corresponding standard errors, recovering the error mean and standard deviation and MPE, MSE and MAPE of forecasts. This is expected since the variances of the BLUE of the slope (the CBE and FBE) are constant multiples of $\frac{2\sigma^2}{s^3}$, with the multipliers, $S(\boldsymbol{\alpha})$ and $S(\boldsymbol{\beta})$ for FBE and CBE respectively, which have been shown to be the same. The BLUE from CBE and FBE outperformed the Simple Average Estimation (SAE) of both CBE and FBE and Least Squares Estimation (LSE) methods. This is because BLUE took serial correlation among the derived variables into consideration which the simple averages ignored.

Therefore, when using Buys-Ballot procedure for time series decomposition of a series with linear trend-cycle component, the BLUE for the slope computed from the FBE or CBE-derived variables should be used. This leads to more precise estimates of time series components. This can also be extended to estimation of the slope in any regression problem. in which variables are serially correlated.

Table 2: Parameter Estimates for $m = 8, s = 12, n = 96, n_0 = 84$ and $m = 9, s = 12, n = 108, n_0 = 96$.

	True Value	Estimates for $m = 8, s = 12, n = 96, n_0 = 84$					Estimates for $m = 9, s = 12, n = 108, n_0 = 96$				
		LSE	SCBE	BCBE	SFBE	BFBE	LSE	SCBE	BCBE	SFBE	BFBE
$\hat{a}$	1.00	0.9748	1.0578	0.9539	1.0580	0.9488	1.2297	1.2978	1.1830	1.4250	1.1835
$\hat{b}$	0.20	0.2000	0.1981	0.2005	0.1977	0.2006	0.1951	0.1937	0.1961	0.1911	0.1961
$\hat{\sigma}_a$	-	0.6074	0.2598	0.2185	0.4715	0.6684	0.5959	0.2534	0.2046	0.4841	0.2046
$\hat{\sigma}_b$	-	0.0124	0.0056	0.0045	0.0108	0.0155	0.0107	0.0048	0.0037	0.0098	0.0037
S_1	5.65	5.0027	4.9920	5.0054	4.9901	5.0060	5.9638	5.9565	5.9695	5.9422	5.9695
S_2	-4.60	-5.0477	-5.0565	-5.0455	-5.0580	-5.0449	-4.7425	-4.7483	-4.7376	-4.7601	4.7377
S_3	1.59	1.1731	1.1662	1.1748	1.1651	1.1752	1.4763	1.4659	1.4743	1.4567	1.4742
S_4	-2.38	-2.4377	-2.4425	-2.4364	-2.4433	-2.4361	-3.0647	-3.0677	-3.0618	-3.0743	-3.0618
S_5	0.38	0.8622	0.8592	0.8629	0.8588	0.8631	0.4346	0.4331	0.4365	0.4291	0.4365
S_6	1.32	1.3420	1.3410	1.3422	1.3408	1.3423	1.2407	1.2405	1.2416	1.2392	1.2416
S_7	1.23	1.6843	1.6852	1.6840	1.6854	1.6840	1.3911	1.3923	1.3912	1.3936	1.3912
S_8	-1.98	-2.1927	-2.1897	-2.1934	-2.1892	-2.1936	-2.5809	-2.5783	-2.5818	-2.5743	-2.5818
S_9	-1.21	-1.1516	-1.1467	-1.1528	-1.1458	-1.1531	-0.9832	-0.9793	-0.9852	-0.9727	-0.9852
S_{10}	-2.05	-2.0612	-2.0544	-2.0630	-2.0533	-2.0634	-1.7089	-1.7035	-1.7118	-1.6943	-1.7117
S_{11}	0.73	1.1697	1.1785	1.1675	1.1800	1.1670	0.9399	0.9467	0.9361	0.9585	0.9362
S_{12}	1.32	1.6569	1.6677	1.6543	1.6694	1.6535	1.6338	1.6421	1.6290	1.6564	1.6290

Table 4: Residual and Forecasting Analysis for $m = 8$, $s = 12$, $n = 96$, $n_0 = 84$ and $m = 9$, $s = 12$, $n = 108$, $n_0 = 96$.

	$m = 8, s = 12, n = 96$					$m = 9, s = 12, n = 108$				
	LSE	SCBE	BCBE	SFBE	BFBE	LSE	SCBE	BCBE	SFBE	BFBE
Mean	0.0003	-0.0019	0.0000	0.0149	0.0008	0.0008	0.0006	-0.0009	-0.0005	-0.0015
$\hat{\sigma}$	0.9820	0.9836	0.9819	0.9842	0.9819	0.9703	0.9722	0.9700	0.9796	0.9700
γ_1	-0.016	-0.043	-0.008	-0.049	-0.007	-0.18	-0.16	-0.19	-0.13	-0.19
γ_2	-0.670	-0.670	-0.670	-0.670	-0.670	0.193	0.201	0.187	0.203	0.186
JB	1.57	1.59	1.57	1.60	1.57	0.67	0.57	0.72	0.44	0.72
$\chi_2^2(0.05)$	5.99	5.99	5.99	5.99	5.99	5.99	5.99	5.99	5.99	5.99
MPE	0.0105	0.0152	0.0092	0.0171	0.0090	0.0153	0.0188	0.0126	0.0259	0.0126
MSE	0.5420	0.5855	0.5329	0.6074	0.5315	0.0013	0.0014	0.0012	0.0017	0.0012
MAE	0.5096	0.5333	0.5114	0.5513	0.5121	0.7042	0.7412	0.6756	0.8160	0.6753
MAPE	2.62%	2.76%	2.63%	2.86%	2.64%	3.26%	3.44%	3.12%	3.80%	3.12%

REFERENCES

[1] A. R. Bera and O. M. Jarque "Efficient tests of normality, homoscedasticity and serial independence of regression residuals", Econometrics Letters, 6(3) (1980), 255 – 259

[2] G. E. P. Box, G. M. Jenkins and G. C. Reinsel Time Analysis, Forecasting and Control, 3rd ed, Prentice-Hall, Englewood Cliffs N. J, (1994)

[3] C.. Chatfield, The Analysis of Time Series: An Introduction. 6th ed, Chapman and Hall, London, (2004).

[4] I. S. Iwueze and E. C. Nwogu Buys-ballot estimates for time series decomposition, Global Journal of mathematical Sciences 3 (2) (2004), 83-98,.

[5]. I. S Iwueze, E. C. Nwogu and J. C. Ajaraogu. "Properties of the Buys-Ballot estimates when trend-cycle component of a time series is linear: additive Case", International Journal of Mathematics and Computation, 8 (S10) (2010), 59 - 77

[6]. I. S. Iwueze, E. C. Nwogu and J. C. Ajaraogu, "Best Linear Unbiased Estimate using Buys-Ballot Procedure when Trend-Cycle Component is Linear", Pakistan Journal of Statistics and Operation Research (PJSOR), VII (2) (2011), 183 - 198.

[7] M. G. Kendall and J. K, Ord, Time Series, 3rd ed., Charles Griffin, London, (1990).