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STRONG COMMUTATIVITY PRESERVING BIDERIVATIONS ON PRIME RINGS

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Abstract. In the present paper we investigate the commutativity in a prime ring R which admits biderivation $D : R \times R \longrightarrow R$ satisfying $[D(x,x), D(y,y)] = [x,y]$ for all $x,y \in R$. More precisely, we generalize the result of Bell et.al.[3] on strong commutativity preserving biderivations. Moreover, we obtain that generalized biderivation acts as left bimultiplier, whenever it behaves as right R-homomorphisms.

Keywords: prime (semiprime) ring; symmetric biderivation; generalized biderivation; bimultiplier.

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1. Introduction

Throughout the paper R will denote a ring with centre $Z(R)$. A ring R is said to be prime (resp. semiprime) if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = \{0\}$ implies that $a = 0$). We shall write $[x,y]$ the commutator $xy - yx$. An additive mapping $d : R \longrightarrow R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$, for all $x,y \in R$. A derivation d_a is inner if there exists

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$a \in R$ such that $d_a(x) = [a,x]$, for all $x \in R$. Maksa [6] introduced the concept of symmetric biderivations. A mapping $D : R \times R \longrightarrow R$ is said to be symmetric if $D(x,y) = D(y,x)$, for all $x,y \in R$. A mapping $f : R \longrightarrow R$ defined by $f(x) = D(x,x)$, where $D : R \times R \longrightarrow R$ is a symmetric and biadditive mapping, is called the trace of D. The trace f of D satisfies the relation $f(x+y) = f(x) + f(y) + 2D(x,y)$, for all $x,y \in R$. A biadditive mapping $D : R \times R \longrightarrow R$ is called a biderivation if for every $x \in R$, the map $y \mapsto D(x,y)$ as well as for every $y \in R$, the map $x \mapsto D(x,y)$ is a derivation of R, i.e., $D(xy,z) = D(x,z)y + xD(y,z)$ for all $x,y,z \in R$ and $D(x,yz) = D(x,y)z + yD(x,z)$ for all $x,y,z \in R$.

It was shown in [6] that symmetric biderivations are related to general solution of some functional equations. The notion of additive commuting mappings is closely connected with the notion of biderivations. Every commuting additive mapping $f : R \longrightarrow R$ gives rise to a biderivation on R. Namely linearizing $[x, f(x)] = 0$ for all $x,y \in R$, $(x,y) \mapsto [f(x),y]$ is a biderivation. Typical examples are mapping of the form $(x,y) \longmapsto \lambda[x,y]$ for all $x,y \in R$, where $\lambda \in C$, the extended centroid of R. We shall call such maps inner biderivations. There has been ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations. Some results on symmetric biderivation in prime and semiprime rings can be found in [1, 5, 8].

For any semiprime (prime) ring R, one can construct Martindale ring of quotients Q of R (see [2]). As R can be embedded isomorphically in Q, we consider R as a subring of Q. If the element $q \in Q$ commutes with every element in R, then $q \in C$, the centre of Q. C contains the centroid of R and it is called the extended centroid of R. In general, C is a von Neumann regular ring and C is a field if and only if R is prime [2, Theorem 5]. For more details on Martindale ring of quotients, one can see [4].

A map $f : R \longrightarrow R$ is centralizing on R if $[f(x),x] \in Z(R)$ for all $x \in R$; in particular if $[f(x),x] = 0$ for all $x \in R$, then f is called commuting on R. A map $f : R \longrightarrow R$ is called commutativity preserving on R if $[f(x), f(y)] = 0$ whenever $[x,y] = 0$, for all $x,y \in R$. In particular, if $[f(x), f(y)] = [x,y]$ for all $x,y \in R$, then f is called strong commutativity preserving on R. In the sequel several results has been proved for strong commutativity preserving condition for example [3] and the references there in.

2. Strong Commutativity Preserving Biderivations

To prove our main results, we require the following lemmas which play key role in the proof of theorems.

Lemma 2.1. [5] *Let S be a set and R be a semiprime ring. If functions f and g of S into R satisfy $f(s)xg(t) = g(s)xf(t)$ for all $s,t \in S$, $x \in R$, then there exist idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ and an invertible element $\lambda \in C$ such that $\varepsilon_i \varepsilon_j = 0$ for $i \neq j$, $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$ and $\varepsilon_1 f(s) = \lambda \varepsilon_1 g(s)$, $\varepsilon_2 g(s) = 0$, $\varepsilon_3 f(s) = 0$ for all $s \in S$.*

Lemma 2.2. [5] *Let R be a 2-torsionfree semiprime ring, and let $g : R \longrightarrow R$ be a centralizing additive mapping. Then there exist $\lambda \in C$ and an additive mapping $\xi : R \longrightarrow C$ such that $g(x) = \lambda x + \xi(x)$ for all $x \in R$.*

Theorem 2.1. *Let R be a 2-torsionfree prime ring. If R admits a symmetric biderivation D with trace f satisfying $[D(x,x), D(y,y)] = [x,y]$ for all $x,y \in R$, then there exist an idempotent $\varepsilon \in C$ and an element $\alpha \in C$ such that the algebra $(1-\varepsilon)R$ is commutative and $\varepsilon D(x,y) = \alpha \varepsilon [x,y]$ for all $x,y \in R$.*

Proof Consider $[D(x,x), D(y,y)] = [x,y]$ for all $x,y \in I$. If $D = 0$, then we have $[x,y] = 0$ and hence R is commutative. Thus $D \neq 0$ and we have

$$[D(x,x), D(y,y)] = [x,y] \text{ for all } x,y \in R. \tag{2.1}$$

Linearization of (2.1) in x yields that

$$2[D(x,z), f(y)] + [f(x), f(y)] + [f(z), f(y)] - [x,y] - [z,y] = 0 \text{ for all } x,y,z \in R. \tag{2.2}$$

Comparing (2.1) and (2.2), we get

$$2[D(x,z), f(y)] = 0 \text{ for all } x,y,z \in R. \tag{2.3}$$

Since R is 2-torsionfree, we have

$$[D(x,z), f(y)] = 0 \text{ for all } x,y,z \in R. \tag{2.4}$$

Replacing x by xu in (2.4) and using (2.4), we find

$$D(x,z)[u, f(y)] + [z, f(y)]D(x,u) = 0 \text{ for all } u,x,y,z \in R. \tag{2.5}$$

Substitute $uf(y)$ for u in (2.5) to get

$$D(x,z)[u, f(y)]f(y) + [z, f(y)]uD(x, f(y)) + [z, f(y)]D(x,u)f(y) = 0 \text{ for all } u,x,y,z \in R. \tag{2.6}$$

Application of (2.5) yields that

$$[z, f(y)]uD(x, f(y)) = 0 \text{ for all } u,x,y,z \in R. \tag{2.7}$$

Since R is prime, we have either $D(x, f(y)) = 0$ or $[z, f(y)]$ for all $x,y,z \in R$. If $D(x, f(y)) = 0$ for all $x,y \in R$, then $D = 0$ by [7]. Which leads to a contradiction. Now consider the later case

$$[w, f(y)] = 0 \text{ for all } w,y \in R. \tag{2.8}$$

Linearizing (2.8) and using 2-torsion freeness of R, we find

$$[w, D(y,z)] = 0 \text{ for all } w,y,z \in R. \tag{2.9}$$

Replacing y by yu in (2.9) and using (2.9), we obtain

$$[w,y]D(u,z) + D(y,z)[w,u] = 0 \text{ for all } u,w,y,z \in R. \tag{2.10}$$

Substitute uv for u in (2.10) and use (2.10) to get

$$[w,y]uD(v,z) + D(y,z)u[w,v] = 0 \text{ for all } u,v,w,y,z \in R. \tag{2.11}$$

Define $S : R \times R \longrightarrow R$ by $S(x,y) = [x,y]$. Then (2.11) reduces to the form

$$S(y,w)uD(v,z) = D(y,z)uS(w,v) \text{ for all } u,v,w,y,z \in R. \tag{2.12}$$

Applying Lemma 2.1, there exist mutually orthogonal idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in C$ and an invertible element $\lambda \in C$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$ and $\varepsilon_1 D(x,y) = \lambda \varepsilon_1 [x,y]$, $\varepsilon_2 [x,y] = 0$, $\varepsilon_3 D(x,y) = 0$ for all $x,y \in R$. Setting $\varepsilon_1 + \varepsilon_3 = \varepsilon$ and $\alpha = \lambda \varepsilon_1$, then we have $\varepsilon D(x,y) = \alpha \varepsilon [x,y]$ for all $x,y \in R$.

Theorem 2.2. *Let R be a 2-torsionfree prime ring and $g : R \longrightarrow R$ be any additive map. If R admits a symmetric biderivation D with trace f satisfying $[D(x,x), g(y)] = [x,y]$ for all $x,y \in R$, then one of the conditions hold*

(1) $D = 0$;

(2) there exist $\lambda \in C$ and an additive mapping $\xi : R \longrightarrow C$ such that $g(x) = \lambda x + \xi(x)$ for all $x \in R$.

Proof consider the given condition

$$[D(x,x), g(y)] = [x,y] \text{ for all } x,y \in R. \tag{2.13}$$

Linearization of (2.13) in x yields that

$$2[D(x,z), g(y)] + [f(x), g(y)] + [f(z), g(y)] - [x,y] - [z,y] = 0 \text{ for all } x,y,z \in R. \tag{2.14}$$

Comparing (2.13) and (2.14), we get

$$2[D(x,z), g(y)] = 0 \text{ for all } x,y,z \in R. \tag{2.15}$$

Since R is 2-torsionfree, we have

$$[D(x,z), g(y)] = 0 \text{ for all } x,y,z \in R. \tag{2.16}$$

Replacing x by xu in (2.16) and using (2.16), we find

$$D(x,z)[u, g(y)] + [z, g(y)]D(x,u) = 0 \text{ for all } u,x,y,z \in R. \tag{2.17}$$

Substitute $f(v)z$ for z in (2.17) and use (2.17) to obtain

$$D(x, f(v))z[u, g(y)] + [f(v), g(y)]zD(x,u) = 0 \text{ for all } u,v,x,y,z \in R. \tag{2.18}$$

Application of (2.16) enable us to have

$$D(x, f(v))z[u, g(y)] = 0 \text{ for all } u,v,x,y,z \in R. \tag{2.19}$$

Primeness of R implies that either $D(x, f(v)) = 0$ or $[g(y), u] = 0$ for all $x,y,v,u \in R$. If $D(x, f(v)) = 0$ for all $x,v \in R$, then $D = 0$ by [7].

Next consider the case when $[g(y), u] = 0$ for all $u,y \in R$. Applying Lemma 2.2, we obtain a

$\lambda \in C$ and an additive mapping $\xi : R \longrightarrow C$ such that $g(y) = \lambda y + \xi(y)$ for all $y \in R$. This completes the proof.

3. Generalized Biderivation Acts as Homomorphism

Let R be a ring and $D : R \times R \longrightarrow R$ be a biadditive map. A biadditive mapping $\Delta : R \times R \longrightarrow R$ is said to be a generalized biderivation if for every $x \in R$, the map $y \mapsto \Delta(x,y)$ is a generalized derivation of R associated with function $y \mapsto D(x,y)$ as well as if for every $y \in R$, the map $x \mapsto \Delta(x,y)$ is a generalized derivation of R associated with function $x \mapsto D(x,y)$ for all $x,y \in R$. It also satisfies $\Delta(x,yz) = \Delta(x,y)z + yD(x,z)$ and $\Delta(xy,z) = \Delta(x,z)y + xD(y,z)$ for all $x,y,z \in R$. For example consider a biderivation θ of R and biadditive a function $\phi : R \times R \longrightarrow R$ such that $\phi(x,yz) = \phi(x,y)z$ and $\phi(xy,z) = \phi(x,z)y$ for all $x,y,z \in R$. Then $\theta + \phi$ is a generalized biderivation of R. The trace g of Δ is defined as $\Delta(x,x) = g(x)$, which satisfies $g(x+y) = g(x) + g(y) + \Delta(x,y) + \Delta(y,x)$ for all $x,y \in R$.

A generalized biderivation Δ is said to be a right R-homomorphism on an ideal I (or left R-homomorphism) if $\Delta(x,yr) = \Delta(x,y)r$ and $\Delta(xr,y) = \Delta(x,y)r$ (or $\Delta(x,ry) = r\Delta(x,y)$ and $\Delta(rx,y) = r\Delta(x,y)$ for all $x,y \in I,\ r \in R$.

Theorem 3.1. *Let R be a prime ring and I be a nonzero right ideal of R. If Δ is a generalized biderivation associated with D such that Δ is a right R-homomorphism, then Δ act as a left bimultiplier.*

Proof Since Δ is a right R-homomorphism, we have

$$\Delta(x,yr) = \Delta(x,y)r \text{ for all } x,y \in I,\ r \in R. \tag{3.1}$$

But we also have

$$\Delta(x,yr) = \Delta(x,y)r + yD(x,r) \text{ for all } x,y \in I,\ r \in R. \tag{3.2}$$

Comparing (3.1) and (3.2), we get

$$yD(x,r) = 0 \text{ for all } x,y \in I,\ r \in R. \tag{3.3}$$

Since the right annihilator of I is zero, we get $D(x,r) = 0$ for all $x \in I,\ r \in R$. This implies that $D(x,z) = 0$ for all $x,z \in I$. We get $\Delta(x,yw) = \Delta(x,y)w + yD(x,w) = \Delta(x,y)w$ for all $x,y,w \in I$. Hence Δ act as left bimultiplier.

Conflict of Interests

The authors declare that there is no conflict of interests.

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