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J. Math. Comput. Sci. 6 (2016), No. 6, 1177-1186

ISSN: 1927-5307

NUMERICAL INTEGRATION OF ARBITRARY FUNCTIONS OVER A CONVEX AND NON-CONVEX POLYGONAL DOMAIN BY QUADRATURE METHOD

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Abstract: The objective of this paper is to present some of our recent development in numerical integration of arbitrary function over a convex and non-convex polygonal domain are approximated by quadrature method based on Generalized Gaussian quadrature rule, to evaluate the typical integrals governed by the proposed method.

Keywords: finite element method; numerical integration; generalised Gaussian quadrature; convex and non-convex region.

2010 AMS Subject Classification: 65L60.

1. INTRODUCTION

Finite element method is one of the most powerful computational technique for approximate solution of a variety of "real world" applied science and engineering problems for over half a century since its inception in the mid 1960. Today, finite element analysis (FEA) has become an integral and major component in the design or modeling of a physical phenomenon in various disciplines. The finite element method has proved superior to other numerical methods due to its better adaptability to any complex geometry. The finite element method (FEM) is a numerical procedure that can be used to obtain solutions to a large class of engineering problems in stress analysis, heat transfer, electromagnetism and fluid flow etc The calculation of integrals of arbitrary functions over convex and Non convex polygonal domain is one of the most difficult

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Received June 26, 2016

part in solving applied problems in CFD, electrodynamics, quantum mechanics, heat flow across a boundary between materials with different conductivity etc. especially to solving two dimensional partial differential in convex and Non convex region by explicit integration method to extract the stiffness matrix, The integrals arising in practical problems are not always simple and other quadrature scheme cannot evaluate exactly. Numerical Integration over triangle and square region are simple but convex and non convex polygonal domain is challenging task to integration over a arbitrary function. The integration points have to be increased in order to improve the integration accuracy and is desirable to make these evaluations by few gauss points as possible. The method proposed here is termed as Generalized Gaussian quadrature rule

From the literature of review we may realize that several works in numerical integration using Gaussian quadrature over various regions have been carried out [1-5]. Gaussian Legendre quadrature over two- dimensional triangle region given in [6-9], to construct the numerical algorithm based on optimization and group theory to compute quadrature rule for numerical integration of bivariate polygonal over polygonal domain are presented in [10]. Numerical integration over polygonal region by using, spline finite element method and cubature formula over polygons are discussed in [11]. In this paper, convex and Non convex polygonal domain is discretized into sub polygons by 4 noded quadrilateral elements and then we apply Generalised Gaussian quadrature method to evaluate the typical integral of arbitrary functions over the convex and Non convex polygonal domain.

The paper is organized as follows. In section 2. we present the Generalized Gaussian Quadrature formula. In section 3. presents the mathematical preliminaries required for understanding the derivation and discretised the convex and Non convex polygonal domain into sub 4- noded quadrilateral elements and then derive a new Gaussian quadrature formula over a quadrilateral region to calculate sampling points and weight coefficients of various order and also plotted the extracted sampling points in both convex and non convex region. In section 4. we compare the numerical results with some illustrative examples.

2. GENERALISED GAUSSIAN QUADRATURE RULE

Suppose the integral of the form

$$\int_a^b k(x)\Omega(x)dx = \sum_{i=1}^{n} w_i^n \Omega(x_i^n) \tag{1}$$

$x_i^n \in [a,b]$ and $w_i^n \in \mathrm{R}$, $\forall \; i = 1,2,3,4,----,n$

Where x_i^n and w_i^n are sampling points and corresponding weights respectively. The quadrature formula are given in Eq.(1) is said to be Generalized Gaussian quadrature rule with respect to the set of functions

$\{1, \ln x, x, x\ln x, x^2, x^2 \ln x, x^2, x^3, x^3 \ln x, -- x^{n-1} \ln x\}$ on [0, 1] of order N = 5, 10, 15, 20, 40 are given in Table 1. (J. Ma.et al. [12]) we using these sampling points and its weights in the product of polynomial and logarithmic function.

3. FORMULATION OF INTEGRALS OVER A LINEAR CONVEX QUADRILATERAL ELEMENT

The integral of an arbitrary function $f(x,y)$ over an arbitrary convex quadrilateral region Ω is given by

$$I = \iint_{\Omega} f(x,y)dxdy \tag{2}$$

Let us consider an arbitrary four noded quadrilateral element in the global coordinate is mapped into 2- square in the local coordinate (ξ,η). The isoperimetric coordinate transformation from (x, y) plane to (ξ,η) is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \sum_{k=1}^{4} \begin{pmatrix} x_k \\ y_k \end{pmatrix} N_k(\xi,\eta) \tag{3}$$

Where (x_k, y_k), (k=1,2,3,4) are the vertices of the quadrilateral element in (x, y) plane and $N_k(\xi,\eta)$ denotes the shape function of node k such that

$$N_k(\xi,\eta) = \frac{1}{4}(1+\xi_k\xi)(1+\eta_k\eta) \tag{4}$$

Where $((\xi_k,\eta_k), k = 1,2,3,4)$ = ((-1,-1),(1,-1),(1,1),(-1,1))

From the Eq.(3) and Eq.(4) , we have

$$\frac{\partial x}{\partial \xi} = \sum_{k=1}^{4} x_k \frac{\partial N_k}{\partial \xi} = \frac{1}{4}[(-x_1 + x_2 + x_3 - x_4) + (x_1 - x_2 + x_3 - x_4)\eta]$$

$$\frac{\partial x}{\partial \eta} = \sum_{k=1}^{4} x_k \frac{\partial N_k}{\partial \eta} = \frac{1}{4}[(-x_1 - x_2 + x_3 - x_4) + (x_1 - x_2 + x_3 - x_4)\xi]$$

Similarly

$$\frac{\partial y}{\partial \xi} = \sum_{k=1}^{4} y_k \frac{\partial N_k}{\partial \xi} = \frac{1}{4}[(-y_1 + y_2 + y_3 - y_4) + (y_1 - y_2 + y_3 - y_4)\eta]$$

$$\frac{\partial y}{\partial \eta} = \sum_{k=1}^{4} y_k \frac{\partial N_k}{\partial \eta} = \frac{1}{4}[(-y_1 - y_2 + y_3 + y_4) + (y_1 - y_2 + y_3 - y_4)\xi]$$

and

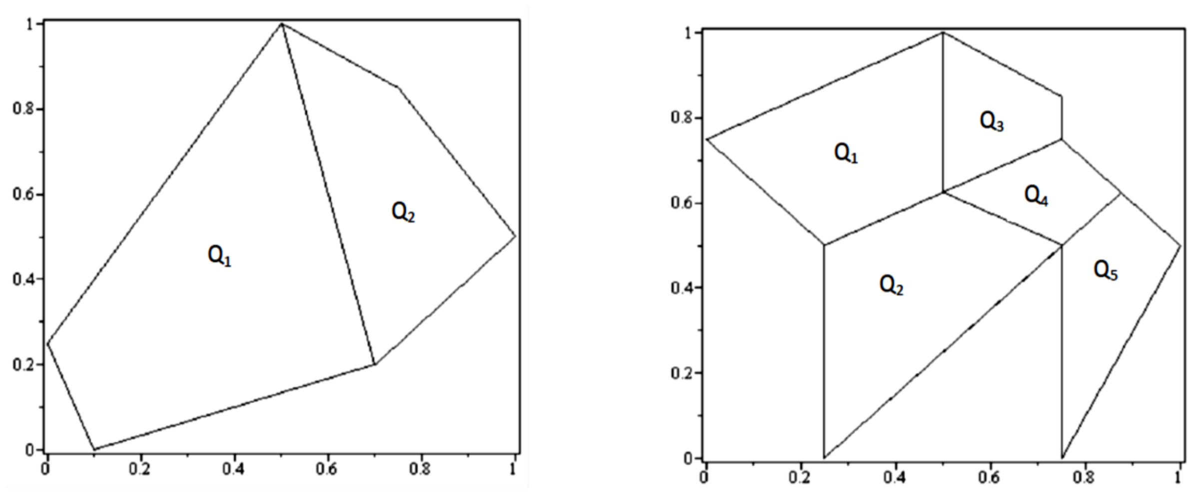
$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} = l + m\xi + n\eta \tag{5}$$

Where

$$l = \frac{1}{8}\left[(x_4 - x_2)(y_1 - y_3) + (x_3 - x_1)(y_4 - y_2)\right]$$

$$m = \frac{1}{8}\left[(x_4 - x_3)(y_2 - y_1) + (x_1 - x_2)(y_4 - y_3)\right]$$

$$n = \frac{1}{8}\left[(x_4 - x_1)(y_2 - y_3) + (x_3 - x_2)(y_4 - y_1)\right]$$



(a) **(b)**

Fig. 1 (a) Convex polygonal Domain (b) Non convex polygonal Domain

The test the integral domain is shown in Fig. 1(a) with two quadrilateral elements in convex polygonal domain with vertices are (0, 0.25), (0.1, 0), (0.7,0.2), (1, 0.5),(0.75, 0.85) and (0.5, 1) and Fig. 1(b) with five quadrilateral elements in non convex polygonal domain with vertices are (0, 0.75), (0.25, 0.5), (0.25, 0), (0.75, 0), (1, 0.5), (0.75, 0.75), (0.75, 0.85), (0.5, 1), (7/8, 5/8) and (1/2, 5/8)

3.1. Numerical Integration over a convex region

Integral form of Eq. (2) rewritten as

$$I = \iint_{Q_1} f(x,y)dxdy + \iint_{Q_2} f(x,y)dxdy$$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(x(\xi,\eta), y(\xi,\eta))\, J \, d\xi \, d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} f(m1(\xi,\eta), n1(\xi,\eta))J1 d\xi d\eta + \int_{-1}^{1}\int_{-1}^{1} f(m2(\xi,\eta), n2(\xi,\eta))J2 d\xi d\eta$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} w_i w_j [\ f(m1(\xi_i,\eta_j), n1(\xi_i,\eta_j))J1 + \ f(m2(\xi_i,\eta_j), n2(\xi_i,\eta_j))J2]$$

Where

$$m1 = 0.025(1+\xi)(1-\eta) + 0.175(1+\xi)(1+\eta) + 0.125(1-\xi)(1+\eta)$$

$$n1 = 0.0625(1-\xi)(1-\eta) + 0.0500(1+\xi)(1+\eta) + \frac{1}{4}(1-\xi)(1+\eta)$$

$$\mathrm{J1} = 0.09000 - 0.0037500\ \xi + 0.0437500\,\eta$$

$$m2 = 0.125(1-\xi)(1-\eta) + 0.175(1-\xi)(1-\eta) + \frac{1}{4}(1+\xi)(1+\eta) + 0.1875(1-\xi)(1+\eta)$$

$$n2 = \frac{1}{4}(1-\xi)(1-\eta) + 0.050(1+\xi)(1-\eta) + 0.125(1+\xi)(1+\eta) + 0.2125(1-\xi)(1+\eta)$$

$$\mathrm{J2} = 0.0437500 + 0.0162500\ \xi - 0.01500\,\eta$$

Where $\xi_{i\,,}\,\eta_j$ are sampling points and w_i, w_j are corresponding weights

we present the following algorithm to calculate sampling points and its weight coefficients as

Case 1

$\boldsymbol{step\ 1}.\ \ k \to 1$

$\boldsymbol{step\ 2}.\ \ i = 1, m$

$\boldsymbol{step\ 3}.\ \ j = 1, n$

$$W_k = J1 * w_i * w_j$$

$$x_k = m1$$

$$y_k = n1$$

$$k = k + 1$$

$\boldsymbol{step\ 4}.\ \ compute\ step\ 3$

$\boldsymbol{step\ 5}.\ \ compute\ step\ 2$

Case 2

$\boldsymbol{step\ 1}.\ \ k \to 1$

$\boldsymbol{step\ 2}.\ \ i = 1, m$

$\boldsymbol{step\ 3}.\ \ j = 1, n$

$$W_k = J2 * w_i * w_j$$

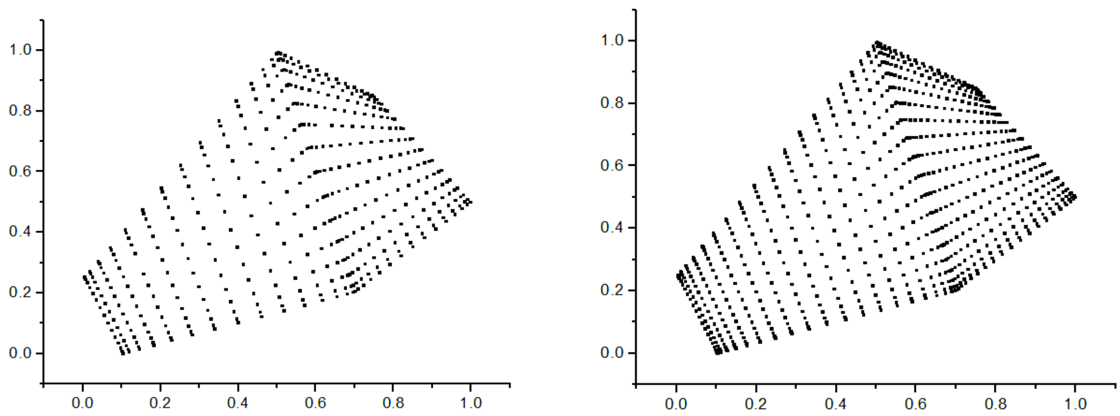
$$x_k = m2$$
$$y_k = n2$$
$$k = k + 1$$

$\boldsymbol{step\ 4}.\ \ compute\ step\ 3$

$\boldsymbol{step\ 5}.\ \ compute\ step\ 2$

result = Case 1 + Case 2

to computed the sampling points and corresponding weights based on the above algorithm for order N = 5, 10, 15, 20 and plotted the distribution of sampling points in convex polygonal domain of various order



(a) N = 15 (b) N = 20

Fig. 2 Distribution of Sampling points in convex polygonal domain

3.2. Numerical Integration over a Non convex region

Integral form of Eq. (2) rewritten as

$$I = \iint_{Q_1} f(x,y)dxdy + \iint_{Q_2} f(x,y)dxdy + \iint_{Q_3} f(x,y)dxdy + \iint_{Q_4} f(x,y)dxdy + \iint_{Q_5} f(x,y)dxdy$$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(x(\xi,\eta), y(\xi,\eta))\, J\, d\xi\, d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} f(m1(\xi,\eta), n1(\xi,\eta))J1 d\xi d\eta + \int_{-1}^{1}\int_{-1}^{1} f(m2(\xi,\eta), n2(\xi,\eta))J2 d\xi d\eta + \int_{-1}^{1}\int_{-1}^{1} f(m3(\xi,\eta), n3(\xi,\eta))J3 d\xi d\eta$$

$$+ \int_{-1}^{1}\int_{-1}^{1} f(m4(\xi,\eta), n4(\xi,\eta))J4 d\xi d\eta + \int_{-1}^{1}\int_{-1}^{1} f(m5(\xi,\eta), n5(\xi,\eta))J5 d\xi d\eta$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} w_i w_j [\ f(m1(\xi_i,\eta_j), n1(\xi_i,\eta_j))J1 + f(m2(\xi_i,\eta_j), n2(\xi_i,\eta_j))J2$$

$$+ f(m3(\xi_i,\eta_j), n3(\xi_i,\eta_j))J3 + f(m4(\xi_i,\eta_j), n4(\xi_i,\eta_j))J4 + f(m5(\xi_i,\eta_j), n5(\xi_i,\eta_j))J5]$$

Where

$$.\,m1=\frac{0.25}{4}(1-\xi)(1-\eta)+\frac{0.25}{4}(1+\xi)(1-\eta)+\frac{0.75}{4}(1+\xi)(1+\eta)+\frac{0.50}{4}(1-\xi)(1+\eta)$$

$$n1=\frac{0.5}{4}(1-\xi)(1-\eta)+\frac{0.5}{4}(1+\xi)(1+\eta)+\frac{0.75}{4}(1-\xi)(1+\eta)$$

$$\mathrm{J1}=0.046875+0.015625\,\xi$$

$$m2=\frac{0.25}{4}(1+\xi)(1-\eta)+\frac{0.5}{4}(1+\xi)(1+\eta)+\frac{0.5}{4}(1-\xi)(1+\eta)$$

$$n2=\frac{0.75}{4}(1-\xi)(1-\eta)+\frac{0.5}{4}(1+\xi)(1-\eta)+\frac{0.75}{4}(1+\xi)(1+\eta)+\frac{1}{4}(1-\xi)(1+\eta)$$

$$\mathrm{J2}=0.03125-0.0781\,\eta-0.0078125\,\xi$$

$$m3=\frac{0.5}{4}(1-\xi)(1-\eta)+\frac{0.5}{4}(1+\xi)(1-\eta)+\frac{0.75}{4}(1+\xi)(1+\eta)+\frac{1}{4}(1-\xi)(1+\eta)$$

$$n3=\frac{1}{4}(1-\xi)(1-\eta)+\frac{0.75}{4}(1+\xi)(1-\eta)+\frac{0.75}{4}(1+\xi)(1+\eta)+\frac{0.9}{4}(1-\xi)(1+\eta)$$

$$\mathrm{J3}=0.02031-0.00781\,\xi-0.003125\,\eta$$

$$m4=\frac{0.5}{4}(1-\xi)(1-\eta)+\frac{0.75}{4}(1+\xi)(1-\eta)+\frac{0.875}{4}(1+\xi)(1+\eta)+\frac{0.75}{4}(1-\xi)(1+\eta)$$

$$n4=\frac{0.75}{4}(1-\xi)(1-\eta)+\frac{0.5}{4}(1+\xi)(1-\eta)+\frac{0.625}{4}(1+\xi)(1+\eta)+\frac{0.75}{4}(1-\xi)(1+\eta)$$

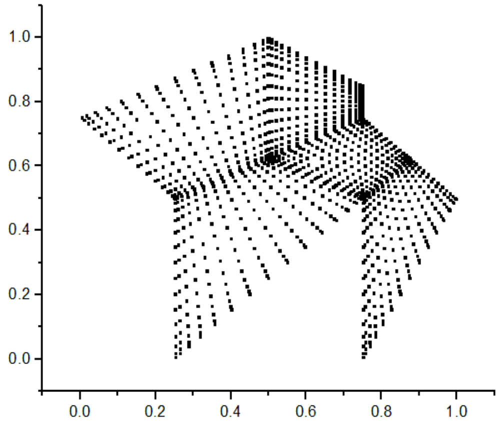
$$\mathrm{J4}=0.0117187-0.00390\,\eta$$

$$m5=\frac{0.75}{4}(1-\xi)(1-\eta)+\frac{0.75}{4}(1+\xi)(1-\eta)+\frac{1}{4}(1+\xi)(1+\eta)+\frac{0.875}{4}(1-\xi)(1+\eta)$$

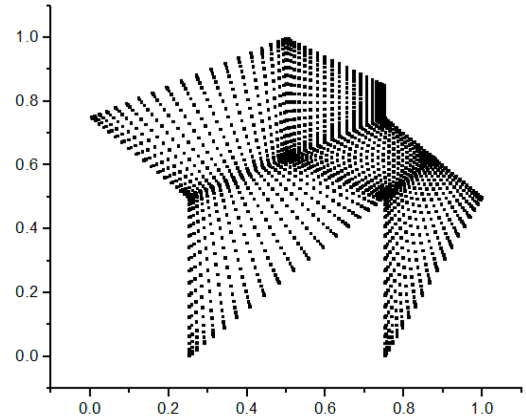
$$n5=\frac{0.5}{4}(1-\xi)(1-\eta)+\frac{0.5}{4}(1-\xi)(1+\eta)+\frac{0.625}{4}(1-\xi)(1+\eta)$$

$$\mathrm{J5}=0.0195312\;+\;0.0078125\,\xi-\;0.00390625\;\eta$$

sampling points and corresponding weights are calculated by the above equations for order N = 5, 10, 15, 20 and plotted the distribution of sampling points in Non convex polygonal domain of various order



(a) N = 15



(b) N = 20

Fig. 3 Distribution of Sampling points in Non convex polygonal domain

4. NUMERICAL RESULTS

We have compared the numerical results obtained using Generalised Gaussian quadrature method with that of numerical results arrived in [11] of various order N = 5, 10, 15, 20 and are tabulated in Table 1 and 2

TABLE. 1 Convex region

Exact value	**Order**	**Computed value**
$\iint_C e^{-100((x-0.5)^2+(y-0.5)^2)}dxdy$	**N=5**	0.03122053
	N=10	0.03141926
0.0314145286323 [Ref. 11]	**N=15**	0.03141457
	N=20	0.03141452
$\iint_C \sqrt{(x-0.5)^2+(y-0.5)^2}\,dxdy$	**N=5**	0.15684370
0.1568251255862 [Ref. 11]	**N=10**	0.15682146
	N=15	0.15682512
	N=20	0.15682512
$\iint_C \lvert x^2+y^2-0.25\rvert dxdy$	**N=5**	0.19978801
	N=10	0.19906357
0.1990625494351 [Ref. 11]	**N=15**	0.19906250
	N=20	0.19906254
$\iint_C \sqrt{\lvert 3-4x-3y\rvert}\,dxdy$	**N=5**	0.54533211
	N=10	0.54538786
0.5453868050054 [Ref. 11]	**N=15**	0.54538680
	N=20	0.54538680

TABLE. 2 Non convex region

Exact value	Order	Computed value
$\iint_N e^{-100((x-0.5)^2+(y-0.5)^2)}dxdy$ 0.0312208389 [Ref. 11]	**N=5** **N=10** **N=15** **N=20**	0.03125429 0.03123803 0.03122035 0.03122083
$\iint_N \sqrt{(x-0.5)^2+(y-0.5)^2}\,dxdy$ 0.1393814567 [Ref. 11]	**N=5** **N=10** **N=15** **N=20**	0.13934531 0.13938197 0.13938145 0.13938145
$\iint_N \left\|x^2+y^2-0.25\right\|dxdy$ 0.2084255960 [Ref. 11]	**N=5** **N=10** **N=15** **N=20**	0.20842330 0.20842555 0.20842550 0.20842559
$\iint_N \sqrt{\left\|3-4x-3y\right\|}\,dxdy$ 0.4545305519 [Ref. 11]	**N=5** **N=10** **N=15** **N=20**	0.45453651 0.45453009 0.45453054 0.45453055

5. CONCLUSIONS

In this paper, a new quadrature method based on Generalised Gaussian quadrature rule is applied for the numerical integration of arbitrary function in convex and non convex polygonal region. The results obtained are in excellent agreement with exact values.

Conflict of Interests

The authors declare that there is no conflict of interests.

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