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CHROMATIC NUMBER OF GRAPHS WITH SPECIAL DISTANCE SETS-III

V.YEGNANARAYANAN* AND A.PARTHIBAN

Department of Mathematics, Velammal Engineering College, Chennai, India

Abstract. An integer distance graph is a graph $G(Z, D)$ with the set of integers as vertex set and an edge joining two vertices u and v if and only if $| u - v | \in D$ where D is a subset of the positive integers. It is known that $\chi(G(Z, P)) = 4$ where P is a set of prime numbers. In this paper we have identified the class three sets when the distance set D is not only a subset of primes P but also a special class of primes like Markov primes, Bell primes, Dihedral primes, Mills primes, Ramanujan primes, Quartan primes, Isolated primes and Thabit Number primes. We also indicate the existence of an alternative formulation for a prime distance graph and raise certain open problems.

Keywords: Primes, Chromatic Number, Distance Graphs.

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1. Introduction

The graphs considered in this paper are simple and undirected. A k-coloring of a graph G is an assignment of k different colors to the vertices of G such that adjacent vertices receive different colors. The minimum cardinality k for which G has a k-coloring is called the chromatic number of G and is denoted by $\chi(G)$.

*Corresponding author

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We begin with the plane coloring problem. What is the least number of colors needed to color all the points of the Euclidean plane such that no two points of distance one have the same color? The corresponding problem for the real line is easy. $\chi(G(R, D = \{1\})) = 2$. To see this, partition the vertex set of G into two non empty disjoint sets such that their union is R. That is, $V(R) = V_1 \cup V_2$, where $V_1 = \bigcup_{n=-\infty}^{\infty}[2n, 2n+1)$ and $V_2 = \bigcup_{n=-\infty}^{\infty}[2n+1, 2n+2)$. Now color all the vertices of V_1 with color 1 and the vertices of V_2 with color 2. As $V_i, i = 1, 2$ are independent and $G(R, D = \{1\})$ is bipartite the result follows. Clearly $G(R^2, \{1\})$ is an infinite graph. The problem of finding the chromatic number of $G(R^2, \{1\})$ is enormously difficult. Paul Erdos has mentioned this problem as one of his favorite problems. Although he could not solve this problem he has made a significant progress towards the solution of this problem with a vital result given as follows: First let us state the famous Rado's Lemma[13].

Lemma 1.1. Let M and M_1 be arbitrary sets. Assume that to any $v \in M_1$ there corresponds a finite subset A_v of M. Assume that to any finite subset $N \subset M_1$, a choice function $x_N(v)$ is given, which attaches an element of A_v to each element v of N:$x_N(v) \in A_v$. Then there exists a choice function $x(v)$ defined for all $v \in M_1$ ($x(v) \in A_v$ if $v \in M_1$) with the following property. If K is any finite subset of M_1, then there exists a finite subset $N(K \subset N \subset M_1)$, such that, as far as K is concerned, the function $x(v)$ coincides with $x_N(v) : x(v) = x_N(v)(v \in K)$.

Theorem 1.1. Let k be a positive integer, and let the graph G have the property that any finite subgraph is k-colorable. Then G is k-colorable itself[5].

2. Preliminaries

A prime distance graph is a graph $G(Z, P)$ with the set of integers as vertex set and with an edge joining two vertices u and v if and only if $| u - v | \in P$ where P is the set of all prime numbers.

By a chromatic subgraph of a graph G we mean a minimal subgraph of G with the same chromatic number as G.What class of graphs will include a chromatic subgraph of $G(Z, D)$ for $D \subseteq Z$?.

A graph G is color critical if its only chromatic subgraph is G itself. For any positive integer m, n, let $G(m, n)$ be the graph comprising $(m+1)$ distinct vertices $u_0, u_1, ..., u_m$ and m distinct subgraphs $H_1, ..., H_m$ each of which is a copy of K_n, (Where K_n is the complete graph on n vertices)such that u_0 is adjacent to u_m and each vertex of H_i is adjacent to u_{i-1} and u_i, for $1 \leq i \leq m$. $G(m, 1) \cong C_{2m+1}, G(1, n) \cong K_{n+2}$, where C_{2m+1} is the cycle on 2m+1 vertices.

The following results are known.[6]

Lemma 2.1. For any positive integers m, n the graph $G(m, n)$ is color critical with $\chi(G(m, n)) = n + 2$.

Theorem 2.1. $\chi(G(Z, \{2, 3, 5\})) = 4$ and hence $\chi(G(Z, P)) = 4$.

In view of Theorem 2.1,we can allocate the subsets D of P to four classes, according as $G(Z, D)$ has chromatic number 1,2,3 or 4. Obviously empty set is the only member of class 1 and every singleton subset is in class 2.

Theorem 2.2. $\chi(G(Z, D = \{r, s\})) = 2$ if r, s are both odd and are coprime.

Theorem 2.3. Let $D \subseteq P$, with $| D | \geq 2$. Then D may be classified as follows:

(a) D is in class 2 if $2 \notin D$; otherwise D is in class 3 or 4.

(b) If $2 \in D$ and $3 \notin D$, then D is in class 3.

(c) If $\{2, 3\} \subseteq D \subseteq \{p \in P : p \equiv \pm 2 (\text{mod } 5)\}$, then D is in class 3.

(d) If $\{2, 3\} \subseteq D \subseteq \{p \in P : p \equiv \pm 2, \pm 3, 7 (\text{mod } 14)\}$, then D is in class 3.

Proof of (a). By Theorem 2.2, If D is a subset of the odd primes then it is in class 2. If D contains 2 and any odd prime p, then $G(Z, D)$ contains a cycle $v_0 v_1 ... v_p w v_0$ where $v_i = 2i$ for $0 \leq i \leq p$, and $w = p$. This cycle has order $p + 2$, which is odd, so has chromatic number 3. Hence with Theorem 2.1, D is in class 3 or 4.

Proof of (b). Here a proper coloring of $G(Z, D)$ is obtained by assigning the integer x to color class i precisely when $x \equiv i (\bmod 3)$, for $0 \leq i < 3$. In view of (a), D is in class 3.

Proof of (c). Assign an integer x to color class i precisely when $x \equiv 2i$ or $2i + 1 (\bmod 5)$, for $0 \leq i \leq 1$ and assign x to color class 2 if $x \equiv 4 (\bmod 5)$. The difference between any two integers in the same color class is congruent to 0 or $\pm 1 (\bmod 5)$, no such pair is adjacent in $G(Z, D)$, if $D = \{p \in P : p \equiv \pm 2 (\bmod 5)\}$. So with (a), if $2 \in D$ it follows that D is in class 3.

Proof of (d). Assign each integer x to color class i, where $i = 0$ when $x \equiv$0,1,5,6 or 10(mod 14); $i = 1$ when $x =$2,3,7,11 or 12(mod 14); $i = 2$ when $x =$4,8,9 or 13(mod 14). The difference between any two integers in the same color class is congruent to 0,$\pm 1, \pm 4, \pm 5, \pm 6 (\bmod 14)$, so no such pair is adjacent in $G(Z, D)$ if $D \subseteq \{p \in P : p \equiv \pm 2, \pm 3, 7 (\bmod 14)\}$. With (a), if $2 \in D$, then D is in class 3.

Theorem 2.4. For any prime $p > 5, \{2, 3, p\}$ is in class 3.

Proof. In view of Theorem 2.3, it is enough to demonstrate a proper 3-coloring for $G(Z, \{2, 3, p\})$. First suppose $p = 6k + 1$, for some $k > 0$. Assign each integer x to color class i, where $0 \leq i \leq 2$ as follows: $x \equiv a (\bmod 6k + 4)$ with $-4 \leq a <\neq 6k$. If $a \geq 0$ then $x \to i$ when $a = 2i$ or $2i + 1 (\bmod 6)$. Also $x \to 0$ if $a = -4$; $x \to 1$ if $a = -3$ or -2, $x \to 2$ if $a = -1$. No two integers in the same color class differ by ± 2 or ± 3 (mod $6k+4$). So this is a proper coloring. If $p = 6k - 1$, for some $k \geq 2$. This time, for any integer x let $x \equiv a (\bmod 6k + 2)$, with $-14 \leq a < 6k - 12$. If $a \geq 0$ then $x \to i$ when $a = 2i$ or $2i + 1 (\bmod 6)$ with $0 \leq i \leq 2$. As in the proof of Theorem 2.3(d), when $a < 0$ we assign $x \to 0$ if $a \in \{-14, -13, -9, -8, -4\}, x \to 1$ if $a \in \{-12, -11, -7, -3, -2\}$ and $x \to 2$ if $a \in \{-11, -6, -5, -1\}$. No two integers in the same color class differ by ± 2 or ± 3(mod $6k + 2$). So this is a proper coloring and covers all the cases.

3. Some Special Class of Primes

In this section we introduce certain interesting set of prime numbers. Each such set of primes have certain unique properties that make them special and are found to be useful from both theoretical and practical point of view.

3.1.Markov Primes

Markov numbers are defined to be the integers occurring as solutions of the Diophantine equation: $x^2 + y^2 + z^2 - 3xyz = 0...(1)$. When talking about a triplet satisfying the equation (1), say, (x_0, y_0, z_0) we will always assume that $x_0 \leq y_0 \leq z_0...(2)$. The only triplets satisfying (1) in which an equality occurs in (2) are (1, 1, 1) and (1, 1, 2). To see this, assume that $x_0 = y_0$. This leads to the equation in $z : 2x^2+z^2-3x^2z = 0$. Solving for z, we get $z = \frac{1}{2}(3x^2 \pm x\sqrt{9x^2 - 8})$. This requires that $9x^2 - 8$ to be a perfect square, say, t^2. Hence $9x^2-8 = t^2$ or $9x^2-t^2 = 8 = 1.8 = 2.4$. That is $(3x-t)(3x+t) = 8 = 1.8 = 2.4$. This leads to two possibilities: $3x - t = 1$ and $3x + t = 8$ which implies $6x = 9$ so no integer solution and $3x - t = 2$ and $3x + t = 4$ implies $x = 1$. To solve for z, we plug $x = y = 1$. Then (1) becomes $z^2 - 3z + 2 = 0$ giving $z = 1$ or $z = 2$. Next assume that $y_0 = z_0$. This leads to the equation in $z : 2y^2 + z^2 - 3y^2z = 0$ which is identical with with the above case except that y replaces x. So the only solution in this case are also (1,1,1) and (1,1,2).

Assume that (x, y, z) in a solution of (1) with $x \leq y \leq z$. If we keep two of the variables fixed; this produces a quadratic equation in the third variable, with two solutions. Since the equation is monic with integer coefficients , the other solution is also an integer, call it x', y', z', whichever is the case. If $t^2 + bt + c$=0 is a quadratic equation in t with roots r_1, r_2, then $r_{1,2} = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c})$, $r_1 + r_2 = -b$, $r_1r_2 = c$. Thus we have the following formulas: $x' = 3yz - x = \frac{(y^2+z^2)}{x} = \frac{1}{2}(3yz \pm \sqrt{9y^2z^2 - 4(y^2 + z^2)})...(3)$. $y' = 3xz - y = \frac{x^2+z^2}{y} = \frac{1}{2}(3xz \pm \sqrt{9x^2z^2 - 4(x^2 + z^2)})...(4)$, $z' = 3xy - z = \frac{x^2+y^2}{z} = \frac{1}{2}(3xy \pm \sqrt{9x^2y^2 - 4(x^2 + y^2)})...(5)$. For example, the triplet of solutions (1,2,5) leads to three new solutions: (2,5,29) where $x' = 29$, (1,5,13) where $y' = 13$ and (1,1,2) where $z' = 1$. We can actually figure out the signs in the formulas (3),(4) and (5).

Suppose $1 \leq t < z$ and $1 \leq x < y$ are integers. Then clearly $\frac{1}{2}(3tz+\sqrt{9t^2z^2 - 4(t^2 + z^2)}) > \frac{3tz}{2} > z$ and $\frac{1}{2}(3xy-\sqrt{9x^2y^2 - 4(x^2 + y^2)}) < y$. To see the second inequality, let $\psi(x, y) =$

$\frac{1}{2}(3xy - \sqrt{9x^2y^2 - 4(x^2+y^2)})$. Then $\psi(x,y) = \frac{1}{2}3xy(1 - \sqrt{1 - \frac{4}{9}(\frac{1}{x^2} + \frac{1}{y^2})})$ and a little more computation yields $\psi(x,y) = \frac{1}{1+\sqrt{1-\frac{4}{9}(\frac{1}{x^2}+\frac{1}{y^2})}}.\frac{1}{2}.3xy.\frac{4}{9}(\frac{1}{x^2} + \frac{1}{y^2})$. Now , the first fraction, the one with the square root in the denominator, assumes maximum when $x = 1$ and $y = 1$, because of the hypothesis on x and y. Hence $\psi(x,y) \leq \frac{3}{5}.\frac{1}{2}.3.\frac{4}{9}.(\frac{y}{x} + \frac{x}{y}) \leq \frac{2}{5}(y+1) < y$. So in view of this observation, if $x < y < z$ then in the formula (3),(4) the sign is positive and in the formulas (5) the sign is negative. To note this if negative sign holds in (3) or (4) then x or $y = \frac{1}{2}(3tz + \sqrt{9t^2z^2 - 4(t^2+z^2)})$ where $t = y$ or x, which would imply that $x > z$ or $y > z$, contrary to hypothesis. If positive sign holds in (5) then $z = \frac{1}{2}(3xy - \sqrt{9x^2y^2 - 4(x^2+y^2)}) < y$ again contradicting the hypothesis.

Let (x,y,z) be a solution of (1) satisfying (2). Define $X(x,y,z) = (y,z,x')$, $Y(x,y,z) = (x,z,y'), Z(x,y,z) = (x,z',y)$ or (z',x,y) where x', y', z' are given by the formulas (3), (4) and (5). If $x < y < z$ then $z < y' < x'$ and $z' < y$. From the above observation $z < y'$ and $z' < y$ are immediate. Now $x' - y' = (3yz - x) - (3xz - y) = (3z+1)(y-x) > 0$. So $y' < x'$. Every solution of (1) can be obtained by starting with the triple (1,1,1) and repeatedly obtaining new solutions by one of the transformations X and/or Y[12]. There is a conjecture due to Frobenius[10]. Given a Markov number z, the sequence of transformations of X's and Y's leading from (1,1,1) to (x,y,z) is unique. It is known that this is true when z is a power of a prime[1, 2, 3, 4].

Markov primes are those primes p for which there exist integers x and y such that $x^2 + y^2 + p^2 = 3xyp$. The first few such prime numbers are 2, 5, 13, 29, 89, 233, 433, 1597, 2897, 5741, 7561, 28657, 33461, 43261, 96557, 426389, 514229, 1686049, 2922509, 3276509, 94418953,321534781, 433494437, 780291637, 1405695061, 2971215073, 19577194573, 25209506681...

3.2.Bell Primes

The nth Bell number, named after Eric Temple Bell, is the number of partitions of a set with n members, or equivalently, the number of equivalence relations on it. Starting with $B_0 = B_1 = 1$, the first few Bell numbers are:1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975,... In general, B_n is the number of partitions of a set of size n. A partition of a set S is defined as a set of nonempty, pairwise disjoint subsets of S whose union is S. For example, $B_3 = 5$ because the 3-element set $\{a, b, c\}$ can be partitioned in 5 distinct ways:$\{\{a\}, \{b\}, \{c\}\}$, $\{\{a\}, \{b, c\}\}$, $\{\{b\}, \{a, c\}\}$,$\{\{c\}, \{a, b\}\}$, $\{\{a, b, c\}\}$. B_0 is 1 because there is exactly one partition of the empty set. Every member of the empty set is a nonempty set (that is vacuously true), and their union is the empty set. Therefore, the empty set is the only partition of itself. Note that, as suggested by the set notation above, we consider neither the order of the partitions nor the order of elements within each partition. This means the following partitioning are all considered identical:$\{\{b\}, \{a, c\}\}$, $\{\{a, c\}, \{b\}\}$, $\{\{b\}, \{c, a\}\}$, $\{\{c, a\}, \{b\}\}$. The Bell numbers can easily be calculated by creating the so-called Bell triangle, also called Aitken's array or the Peirce triangle. 1.Start with the number one. Put this on a row by itself. 2.Start a new row with the rightmost element from the previous row as the leftmost number, 3. Determine the numbers not on the left column by taking the sum of the number to the left and the number above the number to the left (the number diagonally up and left of the number we are calculating), 4.Repeat step three until there is a new row with one more number than the previous row, 5. The number on the left hand side of a given row is the Bell number for that row. The first few Bell numbers that are primes are: 2, 5, 877, 27644437, 35742549198872617291353508656626642567,...

3.3.Dihedral Primes

A dihedral prime or dihedral calculator prime is a prime number that still reads like itself or another prime number when read in a seven-segment display, regardless of orientation (normally or upside down), and surface (actual display or reflection on a mirror). The first few decimal dihedral primes are 2, 5, 11, 101, 181, 1181, 1811, 18181, 108881, 110881, 118081, 120121, 121021, 121151, 150151, 151051, 151121, 180181, 180811, 181081... The smallest dihedral prime that reads differently with each orientation and surface combination is 120121 which becomes 121021 (upside down), 151051 (mirrored), and 150151 (both upside down and mirrored). The digits 0, 1 and 8 remain the same regardless of orientation or surface (the fact that 1 moves from the right to the left of the seven-segment cell when reversed is ignored). 2 and 5 remain the same when viewed upside down, and turn into each other when reflected in a mirror. In the display of a calculator that can handle hexadecimal, 3 would become E reflected, but E being an even digit, the 3 can't be used as the first digit because the reflected number will be even. Though 6 and 9 become each other upside down, they are not valid digits when reflected, at least not in any of the numeral systems pocket calculators usually operate in. Strobogrammatic primes that don't use 6 or 9 are dihedral primes. This includes repunit primes and all other palindromic primes which only contain digits 0, 1 and 8 (in binary, all palindromic primes are dihedral). It appears to be unknown whether there exist infinitely many dihedral primes, but this would follow from the conjecture that there are infinitely many repunit primes. The palindromic prime $10^{180054} + 8 \times \frac{(10^{58567}-1)}{9} \times 10^{60744} + 1$, discovered in 2009 by Darren Bedwell, is 180055 digits long and may be the largest known dihedral prime as of 2009.

3.4.Mills Primes

Mills' constant is defined as the smallest positive real number A such that the floor of the double exponential function $\lfloor A^{3^n} \rfloor$ is a prime number, for all positive integers n. This constant is named after William H. Mills who proved in 1947 the existence of A based on results of Guido Hoheisel and Albert Ingham on the prime gaps. Its value is unknown, but if the Riemann hypothesis is true it is approximately 1.3063778838630806904686144926... The primes generated by Mills' constant are known as Mills primes; if the Riemann hypothesis is true, the sequence begins 2, 11, 1361, 2521008887... If $a(i)$ denotes the ith prime in this sequence, then $a(i)$ can be calculated as the smallest prime number larger than $a(i-1)3$. In order to ensure that rounding A^{3^n}, for n=1,2,3,... produces this sequence of primes, it must be the case that $a(i) < (a(i-1)+1)^3$. The Hoheisel-Ingham results guarantee that there exists a prime between any two sufficiently large cubic numbers, which is sufficient to prove this inequality if we start from a sufficiently large first prime $a(1)$. The Riemann hypothesis implies that there exists a prime between any two consecutive cubes, allowing the sufficiently large condition to be removed, and allowing the sequence of Mills' primes to begin at $a(1) = 2$. Currently, the largest known Mills prime (under the Riemann hypothesis) is $((((((((2^3+3)^3+6)^3+80)^3+12)^3+450)^3+894)^3+3636)^3+70756)^3+97220$, which is 20,562 digits long.

3.5.Ramanujan Primes

Ramanujan prime is a prime number that satisfies a result proven by Srinivasa Ramanujan relating to the prime-counting function. In 1919, Ramanujan published a new proof of Bertrand's postulate which, as he notes, was first proved by Chebyshev [14]. At the end of the two-page published paper, Ramanujan derived a generalized result, and that is: $\pi(x) - \pi(x/2) \geq 1, 2, 3, 4, 5, ...$ for all $x \geq 2, 11, 17, 29, 41,$ ...respectively, where $\pi(x)$ is the prime-counting function, equal to the number of primes less than or equal to x. The converse of this result is the definition of Ramanujan primes: The nth Ramanujan prime is the least integer R_n for which $\pi(x) - \pi(x/2) \geq$n, for all $x \geq R_n$. The first five Ramanujan primes are thus 2, 11, 17, 29, and 41. Equivalently, Ramanujan primes are the least integers R_n for which there are at least n primes between x and $x/2$ for all $x \geq R_n$. Note that the integer R_n is necessarily a prime number: $\pi(x) - \pi(x/2)$ and, hence,$\pi(x)$ must increase by obtaining another prime at $x = R_n$. Since $\pi(x) - \pi(x/2)$ can increase by at most 1, $\pi(R_n) - \pi(R_n/2) = n$.

3.6.Quartan Primes

A quartan prime is a prime number of the form $x^4 + y^4$, where $x > 0, y > 0$. The odd quartan primes are of the form $16n + 1$. For example, 17 is the smallest odd quartan prime: 17 = 14 + 24. The first few quartan primes are 2, 17, 97, 257, 337, 641, 881,...

3.7.Isolated Primes

An isolated prime is a prime number p such that neither $p - 2$ nor $p + 2$ is prime. In other words, p is not part of a twin prime pair. For example, 23 is an isolated prime since 21 and 25 are both composite. The first few isolated primes are 2, 23, 37, 47, 53, 67, 79, 83, 89, 97,...

3.8.Thabit Number Primes

A Thabit number, Th?bit ibn Kurrah number, or 321 number is an integer of the form $3?2^n - 1$ for a non-negative integer n. The first few Thabit numbers are: 2, 5, 11, 23, 47, 95, 191, 383, 767, 1535, 3071, 6143, 12287, 24575, 49151, 98303, 196607, 393215, 786431, 1572863,... The first few Thabit numbers that are prime (also known as 321 primes): 2, 5, 11, 23, 47, 191, 383, 6143, 786431, 51539607551, 824633720831, ... As of April 2008, the known n values which give prime Thabit numbers are: 0, 1, 2, 3, 4, 6, 7, 11, 18, 34, 38, 43, 55, 64, 76, 94, 103, 143, 206, 216, 306, 324, 391, 458, 470, 827, 1274, 3276, 4204, 5134, 7559, 12676, 14898, 18123, 18819, 25690, 26459, 41628, 51387, 71783, 80330, 85687, 88171, 97063, 123630, 155930, 164987, 234760, 414840, 584995, 702038, 727699, 992700, 1201046, 1232255, 2312734, 3136255, 4235414.

4. χ of special Prime Distance Graphs

Here we compute the chromatic number of the distance graph: $G(Z, D)$, when D is a set/subset of any of the above listed primes.

Theorem 4.1. $\chi(G(Z, D))$=3, when D is a finite/infinite set/subset as the case may be of 1)Bell primes, 2)Dihedral primes, 3)Mills primes, 4)Ramanujan primes, 5)Quartan primes, 6)Isolated primes, 7)Markov primes, 8)Thabit Number primes.

Proof. First note that all of the above listed primes have the only even prime integer 2 as one of its element but at the same time the odd prime integer 3 is not appearing in the list of any of the above mentioned special primes. So using Theorem 2.3(b) we deduce that $\chi(G(Z, D))$=3.

5. Alternative Formulations of Prime Distance Graphs

Prime distance graphs were introduced by Eggleton, Erdos and Skilton in 1985[6]. Research in prime distance graphs has since focused on the chromatic number of $G(Z, D)$ where D is a non-empty proper subset of P [8, 9, 15, 16, 17, 18]. Note that all these graphs are infinite(non-induced) subgraphs of $G(Z, D)$.

Alternatively a Prime Distance Graph can be defined as one for which there exists a one-to-one labeling $f : V(G) \to Z$ such that for any two adjacent vertices u and v, the integer $\mid f(u) - f(v) \mid$ is prime. We call f a prime distance labeling of G. So G is a Prime Distance Graphs if and only if there exists a prime distance labeling of G. Note that in a prime distance labeling, the labels on the vertices of G must be distinct, but the labels on the edges need not be. For example, the path $P_n = u_1u_2...u_{n+1}$ is a prime distance graph with prime distance labeling $f : V(P_n) \to Z$ defined by $f(u_i) = 3i - 3$ for $1 \leq i \leq n+1$. Similarly every bipartite graph is a prime distance graph. To see this, it is enough to exhibit a prime distance labeling of complete bipartite graph $K_{m,n}$ as every subgraph of a prime distance graph is a prime distance graph. By the famous Green-Tao's Theorem("For every positive integer k, there exists a prime arithmetic progression of length k") there is an arithmetic sequence of $m + n - 1$ primes $p - (m - 1)k, p - (m - 2)k, ..., p - k, p, p + k, ..., p + (n - 2)k, p + (n - 1)k$. Let U and V be the partite sets of $G = K_{m,n}$ with $| U |= m$ and $| V |= n$. Allot to the vertices of V the labels $p, p + k, ..., p + (n - 1)k$, and to the vertices of U the labels $0, k, 2k, ..., (m - 1)k$. Then differences between the labels of vertices of U and the labels of vertices of V are all of the form $p+tk$ with $t \in \{-(m-1), -(m-2), ..., -1, 0, 1, ..., n-2, n-1\}$ and each such $p+tk$ is a prime. In other words, every 2-chromatic graph is a prime distance graph. But note that not every 3-chromatic graph is a prime distance graph as the complete equitripartite graph with three vertices on each part is not a prime distance graph.

5.1. Open Problems

In the alternative definition of prime distance graph we infer that $f(uv)$ may still be prime if uv is not an edge of G. It would be interesting to investigate by having another formulation for prime distance graph to be one where we define $f(uv)$ to be prime if and only if uv is an edge of G. As all prime distance graphs have chromatic number at most 4, are all planar graphs prime distance graphs?

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References

[1] Baragar.A, on the Unicity conjecture for Markoff numbers, Canad. Math.Bull. 39,3-9, 1996.

[2] Button.J.O, The uniqueness of the prime Markoff numbers, Bull.London.math.Soc, 58, 9-17, 1998.

[3] Button.J.O, Markov numbers, Principal Ideals, and continued fraction expansions, Journal of Number Theory, 87, 77-95, 2001.

[4] Cassels.J.W.S, An Introduction to Diophantine Approximation, Chap 2, Cambridge Univ Press, Cambridge, UK, 1957.

[5] Erdos.P and de Bruijn.N.G, A color problem for infinite graphs and a problem in the theory of relations, Proceedings, Sereies A, 54, No.5 and Indag.Math.,13,No.5, 1951.

[6] Eggleton.R.B, Erdos.P and Skilton.D.K, Coloring the real line, J.Combinatorial Theory, Ser B, 39,1985.

[7] Eggleton.R.B, Erdos.P and Skilton.D.K, Erratum: 'Coloring the real line', J.Combinatorial Theory, Ser B,41,(1):139, 1986.

[8] Eggleton.R.B, Erdos.P and Skilton.D.K, Coloring Prime Distance Graphs, Graphs and Combinatorics, 6,(1),17-32, 1990.

[9] Erdos.P, Renyi.A and V.T.Sos, On a problem of Graph Theory, Studia Sci.Math.Hungar., 1, 215-235, 1966.

[10] Frobenius.G, Uberdie Markoffschen Zahlen,S,-B.Press Akad Wiss, 458-487, 1913.

[11] http:primes.utm.edu/primes

[12] Markov.A.A., Sur les formes quadratiques binaires indefinites, I.Mathematische Annalen, 15, 381-409, 1879.

[13] Rado.R, Axiomatic treatment of rank in infinite sets, Canad.J.Math.1, 337-343,(1949).

[14] Ramanujan, S. (1919), "A proof of Bertrand's postulate", Journal of the Indian Mathematical Society 11: 181?182.

[15] Voigt.M and Walther.H, chromatic number of Prime Distance graphs, Discrete Appl.Math.,51(1-2):197-209, 1999. 2nd Twente workshop on Graphs and Combinatorial optimization(Enschede, 1991).

[16] Yegnanarayanan.V, On a question concerning Prime Distance Graphs, Discrete Math, 245(1-3):293-298, 2002.

[17] Yegnanarayanan.V and Parthiban.A., Chromatic Number of Certain Graphs, Proc of International Conference on Mathematics in Engineering and Business Management, Stella Maris College, Chennai, Vol-I, 115-118, March 9-11, 2012.

[18] Yegnanarayanan.V and Parthiban.A., Chromatic Number of Graphs with Special Distance Sets-II, Accepted and to appear in the Proc of International Conference on Mathematical Modeling and Applied Soft Computing, CIT, Coimbatore, India, July, 11-13, 2012.