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GENERALIZED VISCOSITY APPROXIMATION METHOD FOR NONEXPANSIVE MAPPINGS IN HADAMARD MANIFOLDS

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Abstract. In 2014, Duan et al. proposed the generalized viscosity approximation method to obtain the strong convergence theorem in the framework of Hilbert space. The purpose of this paper is to establish the convergence of generalized Viscosity approximation method for nonexpansive mappings in Hadamard manifolds. Our theorem improves and extends the results that have been proved in this direction for this important class of nonlinear mappings.

Keywords: nonexpansive mappings; generalized viscosity approximation method; fixed points; Hadamard manifolds.

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1. Introduction

Let M be Hadamard manifold and C be a nonempty subset of M. A mapping $T : C \to M$ is called nonexpansive if for any $x, y \in C$,

$$d(T(x), T(y)) \leq d(x, y)$$

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Let us denote the fixed point set of T by

$$\text{Fix}(T) = \{x \in C : T(x) = x\}$$

If $C \subseteq M$ is bounded, closed and convex and T is a nonexpansive mapping of C into itself, then $\text{Fix}(T)$ is nonempty [11]. For approximation of fixed points of a nonexpansive mapping, researchers use some methods. In fixed point theory, approximation methods have attracted so much attention, since they are very powerful and important tools in the study of nonlinear sciences.

Viscosity approximation method for nonexpansive mapping was introduced by Moudafi [22] in 2000. He established strong convergence of both implicit and explicit schemes in a Hilbert space. In 2004, Xu [33] extended Moudafi's results [22] to the framework of uniformly smooth Banach spaces and proved the strong convergence of iterative schemes. Many authors studied the fixed point problems for nonexpansive mappings by the viscosity approximation methods and obtained a series of good results [5, 14, 16, 25, 28, 34]. Over the last decades, Viscosity approximation methods have been applied to convex optimization, linear programming, monotone inclusions, elliptic differential equations and so on.

In the last few years, several important concepts of nonlinear analysis and optimization problems have been extended from Euclidean space to a Riemannian manifold setting. In fact, a manifold is not a linear space. In this setting the linear space is replaced by a Hadamard manifold and the line segment by a geodesic (see [1, 2, 3, 4, 10, 12, 17, 19, 24, 26, 30, 31]). There are lots of optimization problems arising in various applications which cannot be posed in linear spaces and requires a Hadamard manifold structure for their formalization and study. Some algorithms for solving variational inequalities and minimization problems have been extended from the Hilbert space framework to the more general setting of Riemannian manifolds [6, 9, 18, 23, 29, 35]. Most of the extended methods require the Riemannian manifold to have non-positive sectional curvature, i.e., a Hadamard manifold. Since the exponential map cannot be defined in the whole tangent bundle, it is not invertible. Then we will focus in the case of Hadamard manifolds, remarking the statements which remain true in Riemannian manifolds in general.

In 2014, Duan and He [8] proposed a generalized viscosity approximation method for nonexpansive mappings and obtained the following strong convergence theorem in the framework of Hilbert space.

Theorem 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{f_n\}$ be a sequence of ρ_n-contractive self-maps of C with $0 \leq \rho_l = \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n \leq \rho_u < 1$. Let $S : C \to C$ be a nonexpansive mapping. Assume the set $\mathrm{Fix}(S) \neq \phi$ and $\{f_n(x)\}$ is uniformly convergent for any $x \in D$, where D is any bounded subset of C. Given $x_1 \in C$, let $\{x_n\}$ be generated by the following algorithm:*

$$x_{n+1} = \alpha_n f_n(x_n) + (1-\alpha_n)Sx_n \tag{1.1}$$

where the sequence $\{a_n\} \subset (0,1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ defined by (1.1) converges strongly to a point $x^* \in \mathrm{Fix}(S)$, which is also the unique solution of the variational inequality

$$\langle f(x^*) - x^*, p - x^* \rangle \leq 0, \quad \forall\, p \in \mathrm{Fix}(S) \tag{1.2}$$

Recently, Jeong J.U. [13] proved some results using generalized viscosity approximation methods for mixed equilibrium problems and fixed point problems. Motivated and inspired by the works mentioned above, we study the generalized viscosity approximation method (1.1) for nonexpansive mappings in the setting of Hadamard manifolds, i.e., complete simply connected Riemannian manifolds of nonpositive sectional curvature. Also we show that the result proved in this paper extends the corresponding results of Duan et al. [8] and Marquez [21].

2. Preliminaries

Let $q \in M$, where M is a connected n-dimensional Riemannian manifold. A Riemannian manifold is a Riemannian metric $\langle \cdot, \cdot \rangle$, with the corresponding norm denoted by $\|\cdot\|$. We denote the tangent space of M at q by T_qM. We define the length of a piecewise smooth curve, $c : [x,y] \to M$ joining q to r (i.e. $c(x) = q$ and $c(y) = r$), by using the metric as $L(c) = \int_x^y \|c'(t)\| dt$. Then the Riemannian distance $d(q,r)$ is defined to be the minimal length over the set of all such curves joining q to r, which induces the original topology on M.

Let c be a smooth curve and ∇ be the Levi-Civita connection associated to $(M, \langle\,,\rangle)$. A smooth vector field X along c is said to be parallel if $\nabla_{c'}X = 0$. If c' is parallel, then c is a geodesic and here $\|c'\|$ is a constant. A geodesic joining q to r in M is said to be minimal geodesic if its length equals $d(q,r)$. A geodesic triangle $\Delta(q_1,q_2,q_3)$ of a Riemannian manifold is a set consisting of three points q_1, q_2 and q_3 and three minimal geodesic γ_i joining q_i to q_{i+1}, with $i = 1,2,3 \,(\mathrm{mod}\, 3)$.

A Riemannian manifold is complete if for any $q \in M$, all geodesics emanating from q are defined for all $-\infty < t < \infty$. By the Hopf-Rinow theorem we know that if M is complete then any pair of points in M can be joined by a minimizing geodesic. Thus (M,d) is a complete metric space, and bounded closed subsets are compact.

Now, the exponential map $\exp_q : T_qM \to M$ at $q \in M$ is such that $\exp_q v = \gamma_v(1,q)$ for each $v \in T_qM$, where $\gamma(\cdot) = \gamma_v(\cdot,q)$ is the geodesic starting at q with velocity v. Then $\exp_q tv = \gamma_v(t,q)$ for each real number t. The above mentioned definitions and notations can be easily found in [3, 10].

Defintion 2.1 ([27]). A complete simply connected Riemannian manifold of non-positive sectional curvature is called a Hadamard Manifold.

Now, we present some basic results. We assume that M is a n-dimensional Hadamard manifold.

Proposition 2.1 ([27]). *Let $q \in M$. Then $\exp_q : T_qM \to M$ is a diffeomorphism. For any two points $q,r \in M$ there exists a unique normalized geodesic joining q to r, which is in fact a minimal geodesic. This result shows that M has the topology and differential structure similar to $\mathbb{R}^n$. Thus Hadamard manifolds and Euclidean spaces have some similar geometrical properties.*

Proposition 2.2 (Comparison theorem for triangles, [27]). *Let $\Delta(q_1,q_2,q_3)$ be a geodesic triangle. For each $i = 1,2,3 \,(\mathrm{mod}\ 3)$, by $\gamma_i : [0,l_i] \to M$ the geodesic joining q_i to q_{i+1} and set*

$l_i = L(\gamma_i), \alpha_i = \angle(\gamma_i'(0) - \gamma_{i-1}'(l_{i-1}))$. *Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi,$$

$$l_i^2 + l_{i+1}^2 - 2 l_i l_{i+1} \cos\alpha_{i+1} \leq l_{i-1}^2 \,. \tag{2.1}$$

In terms of the distance and the exponential map, the inequality (2.1) *can be rewritten as*

$$d^2(q_i, q_{i+1}) + d^2(q_{i+1}, q_{i+2}) - 2\langle \exp_{q_{i+1}}^{-1} q_i, \exp_{q_{i+1}}^{-1} q_{i+2}\rangle \leq d^2(q_{i-1}, q_i), \tag{2.2}$$

since

$$\langle \exp_{q_{i+1}}^{-1} q_i, \exp_{q_{i+1}}^{-1} q_{i+2}\rangle = d(q_i, q_{i+1}) d(q_{i+1}, q_{i+2}) \cos\alpha_{i+1} \,.$$

Proposition 2.3 ([27])**.** *A subset $N \subseteq M$ is said to be convex if for any two points q and r in N, the geodesic joining q to r is contained in N, i.e., if $\gamma : [a,b] \to M$ is a geodesic such that $q = \gamma(a)$ and $r = \gamma(b)$ then $\gamma((1-t)a + tb) \in N$ for all $t \in [0,1]$. From now N will denote a nonempty, closed and convex set in M.*

A real valued function f defined on M is said to be convex if for any geodesic γ of M, the composition function $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is convex, that is,

$$(f \circ \gamma)(ta + (1-t)b) \leq t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b) \quad \text{for any } a, b \in \mathbb{R} \text{ and } 0 \leq t \leq 1.$$

Proposition 2.4 ([27])**.** *Let $d : M \times M \to \mathbb{R}$ be a distance function. Then d is a convex function with respect to the product Riemannian metric, i. e., given any pair of geodesics $\gamma_1 : [0,1] \to M$ and $\gamma_2 : [0,1] \to M$ the following inequality holds for all $t \in [0,1]$:*

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t) d(\gamma_1(0), \gamma_2(0)) + t d(\gamma_1(1), \gamma_2(1))$$

In particular, for each $q \in M$, the function $d(\cdot, q) : M \to \mathbb{R}$ is a convex function. Let P_N denotes the projection onto N defined by

$$P_N(q) = \{q_0 \in N : d(q, q_0) \leq d(q, r), \textit{ for all } r \in N\} \textit{ for all } q \in M \,.$$

Proposition 2.5 ([31])**.** *For any point $q \in M$, $P_N(q)$ is a singleton and the following inequality holds for all $r \in N$*

$$\langle \exp_{P_N(q)}^{-1} q, \exp_{P_N(q)}^{-1} r\rangle \leq 0.$$

Lemma 2.1 ([32]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n, \quad n \geq 0,$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{b_n\}_{n=0}^{\infty}$ be a sequence such that

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$

(ii) *either* $\limsup_{n\to\infty} b_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\alpha_n b_n| < \infty$.

Then the sequence $\{a_n\}_{n=0}^{\infty}$ converges to zero.

3. Main Result

Theorem 3.1. *Let C be a closed convex subset of Hadmard manifold M and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \phi$. Let $\psi_n : C \to C$ be ρ_n-contraction with $0 \leq \rho_l = \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n = \rho_u < 1$. Suppose that $\{\psi_n(x)\}$ is uniformly convergent for any $x \in A$, where A is any bounded subset of C. If the sequence $\lambda_n \subset (0,1)$ satisfies the following conditions:*

(i) $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$,

(ii) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *and*

(iii) $\lim_{n\to\infty} \frac{(\lambda_n - \lambda_{n-1})}{\lambda_n} = 0$.

Then the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = \exp_{\psi_n(x_n)}((1-\lambda_n)\exp^{-1}_{\psi_n(x_n)} T(x_n)) \tag{3.1}$$

converges strongly to $\bar{x} \in C$, which is also the unique solution of the variational inequality

$$\langle \exp^{-1}_{\bar{x}} \psi(\bar{x}), \exp^{-1}_{\bar{x}} x\rangle \leq 0, \quad \forall\, x \in \mathrm{Fix}(T). \tag{3.2}$$

The algorithm (3.1) *is equivalent to $x_{n+1} = \gamma_n(1-\lambda_n)$, $\forall\ n \geq 0$ where $\gamma_n : [0,1] \to M$ is the geodesic joining $\psi_n(x_n)$ to $T(x_n)$.*

Proof. We first prove the boundedness of $\{x_n\}$.

For this, take $x \in F(T)$. Then, by the convexity of the distance function and nonexpansivity of T, we have

$$\begin{aligned} d(x_{n+1},x) &\leq d(\gamma_n(1-\lambda_n),x) \\ &\leq \lambda_n d(\psi_n(x_n),x)+(1-\lambda_n)d(x_n,x) \\ &\leq \lambda_n\rho_n d(x_n,x)+\lambda_n d(\psi_n(x),x)+(1-\lambda_n)d(x_n,x) \\ &\leq (1-\lambda_n(1-\rho_n))d(x_n,x)+\lambda_n(1-\rho_n)\frac{d(\psi_n(x),x)}{(1-\rho_n)} \\ &\leq \max\left\{d(x_n,x),\frac{1}{1-\rho_n}d(\psi_n(x),x)\right\} \end{aligned} \tag{3.3}$$

By mathematical induction, we have

$$d(x_{n+1},x) \leq \max\left\{d(x_0,x),\frac{1}{1-\rho_n}d(\psi_n(x),x)\right\} \tag{3.4}$$

which implies that $\{x_n\}$ is bounded, so $\{T(x_n)\}$ and $\{\psi_n(x_n)\}$ is bounded due to uniform convergence of $\{\psi_n\}$ on A. Next, we claim that

$$d(x_{n+1},x_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.5}$$

Due to boundedness of $\{x_n\}$, we can choose a constant K such that

$$d(x_n,x_{n-1}) \leq K \text{ and } d(\psi_n(x_n),x_n) \leq K \quad \text{for all } n \geq 0 \tag{3.6}$$

$$\begin{aligned} d(x_{n+1},x_n) &\leq d(\gamma_n(1-\lambda_n),\gamma_{n-1}(1-\lambda_{n-1})) \\ &\leq d(\gamma_n(1-\lambda_n),\gamma_{n-1}(1-\lambda_n))+d(\gamma_{n-1}(1-\lambda_n),\gamma_{n-1}(1-\lambda_{n-1})) \\ &\leq \lambda_n d(\psi_n(x_n),\psi_n(x_{n-1}))+(1-\lambda_n)d(x_n,x_{n-1})+|\lambda_n-\lambda_{n-1}|d(\psi_n(x_{n-1}),x_{n-1}) \\ &\leq (1-\lambda_n(1-\rho_n))d(x_n,x_{n-1})+|\lambda_n-\lambda_{n-1}|d(\psi_n(x_{n-1}),(x_{n-1}) \end{aligned} \tag{3.7}$$

By putting $\lambda_n(1-\rho_n)=\lambda_n^*$ and combining (3.6) and (3.7), we obtain

$$d(x_{n+1},x_n) \leq (1-\lambda_n^*)d(x_n,x_{n-1})+K(|\lambda_n-\lambda_{n-1}|) \tag{3.8}$$

If assumption (iii) holds, consider $\alpha_n=\lambda_n^*$ and $b_n=\frac{K(|\lambda_n-\lambda_{n-1}|}{\lambda_n}$ and using Lemma 2.1, we find that (3.5) holds.

Now if assumption (ii) holds, for $k \leq n$ we have

$$\begin{aligned} d(x_{n+1},x_n) &\leq \prod_{i=k}^{n}(1-\lambda_i^*)d(x_k,x_{k-1}) + K\prod_{i=k}^{n}(|\lambda_i - \lambda_{i-1}|) \\ &\leq K\prod_{i=k}^{n}(1-\lambda_i^*) + K\prod_{i=k}^{n}(|\lambda_i - \lambda_{i-1}|) \end{aligned}$$

$$\lim_{n\to\infty} d(x_{n+1},x_n) \leq K\prod_{i=k}^{\infty}(1-\lambda_i^*) + K\prod_{i=k}^{\infty}(|\lambda_i - \lambda_{i-1}|) \tag{3.9}$$

By assumptions (i) and (ii), we obtain, $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$.

Next we prove that

$$\limsup_{n\to\infty} \langle \exp_{\bar{x}}^{-1} \psi_n(\bar{x}), \exp_{\bar{x}}^{-1} x_n)\rangle \leq 0 \tag{3.10}$$

where $\bar{x} = P_F(T)\psi_n(\bar{x})$ is a unique solution of the variational inequality (3.2).

It is proved above that $\{x_n\}$ and $\{\psi_n(x_n)\}$ are bounded, thus $\{\langle \exp_{\bar{x}}^{-1} \psi_n(\bar{x}), \exp_{\bar{x}}^{-1} x_n\rangle\}$ is bounded; hence its upper limit exists. Thus we can find a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\limsup_{n\to\infty} \langle \exp_{\bar{x}}^{-1} \psi_n(\bar{x}), \exp_{\bar{x}}^{-1} x_n\rangle = \lim_{k\to\infty} \langle \exp_{\bar{x}}^{-1} \psi_n(\bar{x}), \exp_{\bar{x}}^{-1} x_{n_k}\rangle \tag{3.11}$$

Without loss of generality, we may assume that $x_{n_k} \to x^*$ for some $x^* \in M$, since $\{x_n\}$ is bounded. Using the convexity of distance function, we have

$$d(x_{n_k+1}, T(x_{n_k})) \leq \lambda_n d(\psi_n(x_{n_k}), T(x_{n_k})).$$

Since $\{d(\psi_n(x_{n_k}), T(x_{n_k}))\}$ is bounded as $\{x_n\}$ and $\{\psi_n(x_n)\}$ are bounded.

By assumption (i) it follows that $\lim_{k\to\infty} d(x_{n_k+1}, T(x_{n_k})) = 0$ as $\lambda_{n_k} \to 0$.

Now, by $d(x_{n_k}, T(x_{n_k})) \leq d(x_{n_k+1}, x_{n_k}) + d(x_{n_k+1}, T(x_{n_k}))$, we get $\lim_{n\to\infty} d(x_{n_k}, T(x_{n_k})) = 0$.

Therefore

$$d(x^*, T(x^*)) \leq d(x^*, x_{n_k}) + d(x_{n_k}, T(x_{n_k})) + d(T(x_{n_k}), T(x^*)) \to 0$$

which shows that $x^* \in \mathrm{Fix}(T)$.

Then, since $\langle \exp_{\bar{x}}^{-1} \psi_n(\bar{x}), \exp_{\bar{x}}^{-1} x\rangle \leq 0$ for any $x \in \mathrm{Fix}(T)$, we obtain that

$$\lim_{k\to\infty} \langle \exp_{\bar{x}}^{-1} \psi_n(\bar{x}), \exp_{\bar{x}}^{-1} x_{n_k}\rangle = \langle \exp_{\bar{x}}^{-1} \psi_n(\bar{x}), \exp_{\bar{x}}^{-1} x^*\rangle \leq 0 \tag{3.12}$$

Now combining (3.11) and (3.12), we obtain (3.10).

Finally, we show that

$$\lim_{n\to\infty} d(x_n,\bar{x}) = 0. \tag{3.13}$$

For this, consider the geodesic triangle $\Delta(q_1,q_2,q_3)$ and its comparison triangle $\Delta(q_1,q_2,q_3) \subset \mathbb{R}^2$. Fix $n \geq 0$ and set $q_1 = \psi_n(x_n)$, $q_2 = T(x_n)$, $q_3 = \bar{x}$. So we can write (3.1) as $x_{n+1} = \exp_{q_1}((1-\lambda_n)\exp_{q_1}^{-1} q_2)$. The comparison point of x_{n+1} in $\mathbb{R}^2$ is $x'_{n+1} = \lambda_n q'_1 + (1-\lambda_n)q'_2$. Then

$$d(\psi_n(x_n),\bar{x}) = d(q_1,q_3) = \|q'_1 - q'_3\| \quad \text{and} \quad d(T(x_n),\bar{x}) = d(q_2,q_3) = \|q'_2 - q'_3\|.$$

Let θ and θ' denote the angles at q_3 and q'_3, respectively. Therefore $\theta \leq \theta'$ by Lemma 3.5(1) [18, p. 547] and then $\cos\theta' \leq \cos\theta$. Thus by Lemma 3.5(2) [18, p. 547], we have

$$\begin{aligned}
d^2(x_{n+1},\bar{x}) &\leq \|x'_{n+1} - q'_3\|^2 \\
&= \|\lambda_n(q'_1 - q'_3) + (1-\lambda_n)(q'_2 - q'_3)\|^2 \\
&= \lambda_n^2\|q'_1 - q'_3\| + (1-\lambda_n)^2\|q'_2 - q'_3\|^2 + 2\lambda_n(1-\lambda_n)\|q'_1 - q'_3\|\|q'_2 - q'_3\|\cos\theta' \\
&\leq \lambda_n^2 d^2(\psi_n(x_n),\bar{x}) + (1-\lambda_n)^2 d^2(T(x_n),\bar{x}) + 2\lambda_n(1-\lambda_n)d(\psi_n(x_n),\bar{x})d(T(x_n),\bar{x})\cos\theta \\
&\leq \lambda_n^2 d^2(\psi_n(x_n),\bar{x}) + (1-\lambda_n)^2 d^2(x_n,\bar{x}) + 2\lambda_n(1-\lambda_n)d(\psi_n(x_n),\bar{x})d(x_n,\bar{x})\cos\theta \\
&\leq \lambda_n^2 d^2(\psi_n(x_n),\bar{x}) + (1-\lambda_n)^2 d^2(x_n,\bar{x}) \\
&\qquad + 2\lambda_n(1-\lambda_n)(d(\psi_n(x_n),\psi_n(\bar{x})) + d((\psi_n(\bar{x}),(\bar{x})))d(x_n,\bar{x})\cos\theta \\
&\leq \lambda_n^2 d^2(\psi_n(x_n),\bar{x}) + (1-\lambda_n)^2 d^2(x_n,\bar{x}) \\
&\qquad + 2\lambda_n(1-\lambda_n)(\langle \exp_{\bar{x}}^{-1}\psi_n(\bar{x}), \exp_{\bar{x}}^{-1} x_n\rangle + \rho_n d^2(x_n,\bar{x})) \\
&\leq \lambda_n^2 d^2(\psi_n(x_n),\bar{x}) + [(1-\lambda_n)^2 \\
&\qquad + 2\lambda_n(1-\lambda_n)\rho_n]d^2(x_n,\bar{x})) + 2\lambda_n(1-\lambda_n)(\langle \exp_{\bar{x}}^{-1}\psi_n(\bar{x}), \exp_{\bar{x}}^{-1} x_n\rangle) \\
&= (1 - 2\lambda_n + \lambda_n^2 + 2\lambda_n(1-\lambda_n)\rho_n)d^2(x_n,\bar{x}) + \lambda_n^2 d^2(\psi_n(x_n),\bar{x}) \\
&\qquad + 2\lambda_n(1-\lambda_n)(\langle \exp_{\bar{x}}^{-1}\psi_n(\bar{x}), \exp_{\bar{x}}^{-1} x_n\rangle) \\
&= (1-\alpha_n)d^2(x_n,\bar{x}) + \alpha_n\beta_n
\end{aligned}$$

where $\beta_n = \frac{1}{\alpha_n}(\lambda_n^2 d^2(\psi_n(x_n),\bar{x}) + 2\lambda_n(1-\lambda_n)(\langle \exp_{\bar{x}}^{-1}\psi_n(\bar{x}), \exp_{\bar{x}}^{-1} x_n\rangle))$

and $\alpha_n = 2\lambda_n + \lambda_n^2 + 2\lambda_n(1-\lambda_n)\rho_n$.

Now using given hypothesis (i) and (3.12), $\lim_{n\to\infty} \beta_n \leq 0$ and $\lim_{n\to\infty} \alpha_n = 0$. Also, by hypothesis (ii), we obtain $\sum_{n=0}^{\infty} \alpha_n = \infty$. Thus applying Lemma 2.1, (3.13) holds.

Remark 3.1. In [21], the convergence of the following viscosity approximation method was proved in the setting of Hadamard Manifold:

$$x_{n+1} = \exp_{\psi(x_n)}((1-\alpha_n)\exp^{-1}_{\psi(x_n)} T(x_n))), \quad \forall\, n \geq 0 \tag{3.14}$$

where $\gamma_n : [0,1] \to M$ is the geodesic joining $\psi(x_n)$ to $T(x_n)$ and ψ is a contraction mapping on C.

In this paper, when we take $\psi_1 = \psi_2 = \psi_3 = \ldots = \psi_n = \ldots = \psi$, $\forall\, n \in N$, we casethat is a special case of (3.1).

Remark 3.2 ([20])**.** Halpern's iteration method

$$x_{n+1} = \exp_u((1-\alpha_n)\exp_u^{-1} T(x_n)), \quad \forall\, n \geq 0$$

is also a special case of (3.1) when $\psi_1 = \psi_2 = \psi_3 = \ldots = \psi_n = \ldots = u$, $\forall\, n \in N$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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