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CONFORMAL VARIATIONS OF THE SPECTRAL ZETA FUNCTION OF THE LAPLACIAN

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Abstract. This work raises and addresses a question about the behaviour of the variations of the spectral zeta function, $\zeta_g(s)$, of the Laplacian, Δ_g, on a closed connected smooth Riemannian manifold, (M,g), at any point $s = s_0$. We introduce a certain distributional integral kernel and compute a second variation formula of $\zeta_g(s)$ on closed homogeneous Riemannian manifolds under volume-preserving conformal metric perturbations in terms of the kernel. Some criticality conditions for the spectral variations are found.

Keywords: Laplacian; spectral zeta function; conformal variation; Casimir energy; homogeneous manifolds.

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1. Introduction

The spectral zeta function introduced by Minakshisundaram and Pleijel in [13] and denoted by $\zeta_g(s)$ has been shown to encode various important spectral information. For instance, the notions of the determinant of the Laplacian and Casimir energy are defined via the spectral zeta function, see e.g [24, 9, 21, 8, 10, 23, 7, 15, 25] and [3] among other literature. Various generalisations of the spectral zeta function have also appeared in Ray and Singer [19]; Osgood, Philips and Sarnak [17], etc. In particular, variations of the spectral zeta function and the

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spectral determinant came to limelight in various works. Specifically, the work in this paper is motivated by analogous works done for the determinant of the Laplacian in [19] and [18].

More recently, Osgood, Philips and Sarnak in [17] found that among all fixed volume conformal class of metrics $\{g_\varepsilon = e^{\phi_\varepsilon} g\}$ on a Riemannian surface M, the constant curvature metric has maximal determinant. A similar result was obtained by Richardson [20] on 3-dimensional Riemannian manifolds. He found that the 3-sphere with the standard round metric is a local maximum for a fixed-volume conformal deformation of the metric. Okikiolu [16] generalised the result of Richardson to all closed odd n-dimensional Riemannian manifolds.

The work in this paper raises and addresses a related question as those of [17], [20] and [15] about the behaviour of the second variation of the spectral zeta function, but now, at any given point $s = s_0$ of the spectral zeta function. For example, "how does the Casimir energy behave under such volume-preserving conformal variation of the metric of a smooth, compact and connected n-dimensional Riemannian manifold (M, g)?" Our results are illustrated with the n-sphere.

2. **The Spectral zeta function**

Let (M, g) be a closed connected smooth Riemannian manifold. The Laplacian on $C^\infty(M)$ is the operator

$$\Delta_g : C^\infty(M) \to C^\infty(M) \tag{2.1}$$

defined in local coordinates, (see e.g. [2, 4, 6]) by

$$\Delta_g = -\operatorname{div}(\operatorname{grad}) = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x^i}\big(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j}\big). \tag{2.2}$$

The operator Δ_g extends to a self-adjoint operator on $L^2(M) \supset H^2(M) \to L^2(M)$ with compact resolvent. This implies that $\exists$ orthonormal basis $\{\psi_k\} \subset L^2(M)$ consisting of eigenfunctions, [16], such that

$$\Delta_g \psi_k = \lambda_k \psi_k \tag{2.3}$$

where the eigenvalues are listed with multiplicities:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \nearrow \infty. \tag{2.4}$$

Example of such manifold is the unit n-sphere.

Now recall the Riemann zeta function

$$\zeta_R(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \tag{2.5}$$

which converges absolutely for $\Re(s) > 1$ and admits a meromorphic continuation to the whole s-complex plane with only simple pole at $s = 1$ and has residue 1; [22, 24].

Following [21], one defines the spectral zeta function $\zeta_g(s)$, using the operator Δ_g^{-s} uniquely defined by the following properties: it is linear on $L^2(M)$ with 1-dimensional null space consisting of constant functions. This ensures that the smallest eigenvalue of Δ_g^{-s} is 0 of multiplicity 1 with corresponding eigenfunction $\frac{1}{\sqrt{V}}$ where V is the volume of M; the image of Δ_g^{-s} is contained in the orthogonal complement of constant functions in $L^2(M)$ i.e.

$$\int_M \Delta_g^{-s} \psi dV_g = 0 \; \forall \; \psi \in L^2(M) \text{ constant; and}$$

$\Delta_g^{-s}\psi_k(x) = \lambda_k^{-s}\psi_k(x)$ for all ψ_k; $k > 0$ an orthonormal basis of eigenfunction of Δ_g.

The integral kernel $\zeta_g(s,x,y)$, also known as the zeta kernel [14], of Δ_g^{-s} is given by

$$\zeta_g(s,x,y) := \sum_{k=1}^{\infty} \frac{\psi_k(x)\bar{\psi}_k(y)}{\lambda_k^s}; \; \Re(s) > \frac{n}{2}. \tag{2.6}$$

Thus by these properties, we see that Δ_g^{-s} is trace class, with trace given by

$$\zeta_g(s) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^s} = \mathrm{Tr}(\Delta_g^{-s}) = \int_M \zeta_g(s,x,x) dV; \tag{2.7}$$

which converges absolutely for $\Re(s) > \frac{n}{2}$; [22]. $\zeta_g(s)$ is known as the spectral zeta function.

Another kernel of interest here is the heat kernel.

Definition 2.1. [6]. *The heat kernel,*

$$K(t,x,y) : (0,\infty) \times M \times M \to \mathbb{R},$$

is a continuous function on $(0,\infty)\times M\times M$. *It is the so-called fundamental solution to the heat equation, i.e, it is the unique solution to the following system of equations:*

$$\left.\begin{array}{rcl}\left(\frac{\partial}{\partial t}+\Delta_x\right)K(t,x,y) & = & 0\\ \lim_{t\to 0}\int_M K(t,x,y)\psi(y)dV_y & = & \psi(x)\end{array}\right\} \tag{2.8}$$

for $t>0;\ x,y\in M$ *and* Δ_x *is the Laplacian acting on any* $\psi\ \in\ L^2(M)$, *where the limit in the second equation of (2.8) is uniform for every* $\psi\in C^\infty(M)$.

Minakshisundaram-Pleijel gave the expansion of the trace of the heat kernel as

$$\mathrm{Tr}(e^{-\Delta_g t})=\frac{1}{(4\pi t)^{n/2}}\Big\{a_0+a_1t+a_2t^2+\cdots+a_Nt^N+O\big(t^{N+1}\big)\Big\}; \tag{2.9}$$

as $t\to 0^+$; where a_j are some smooth functions on M which depend only on the geometric data at the point $x\in M$; [11].

The zeta kernel and the heat kernel are related by

$$\zeta(s,x,y)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(K(t,x,y)-\frac{1}{V})dt,$$

$\Re(s)>\frac{n}{2}$; [20].

3. Variations of the spectral zeta function

Consider how a conformal change of metric affects the Laplacian. Let (M,g) be a smooth homogeneous Riemannian manifold and $0<\rho\in C^\infty(M)$. Then the Laplacian with respect to the conformally equivalent metric $h=\rho g$ is given by

$$\Delta_h\psi=\rho^{-1}\Delta_g\psi+(1-\frac{n}{2})\rho^{-2}\mathrm{div}(\rho\nabla_g)\psi$$

with

$$\mathrm{div}(\rho\nabla_g):=g^{ij}(\partial_i\rho)\partial_j$$

where the so-called Einstein summation convention of summing over repeated indices is used; c.f: [3, 26]. Then one can see that the Casimir energy, defined by $\zeta_g(-\frac{1}{2})$, is not invariant under change of metric. Consider for instance a scaling of the metric with a constant $c > 0$, one quickly sees that $\Delta_g \mapsto \frac{1}{c}\Delta_g$ and

$$\zeta_{cg}(s) = \sum_{k=1}^{\infty} \frac{c^s}{\lambda_k^s} = c^s \zeta_g(s).$$

So, the Casimir energy changes as

$$\zeta_{cg}(-\frac{1}{2}) = c^{-\frac{1}{2}} \zeta_g(-\frac{1}{2}).$$

Consequently, it becomes of much interest to study how the Casimir energy and other points of the spectral zeta function vary under more general deformation of the metric such as conformal perturbation of the Riemannian manifold, and in fact, it is sensible to fix the volume so as to factor out this trivial scaling.

Definition 3.1. *(Conformal perturbation)*[2, 5]*: Let (M,g) and $(N,\tilde{g})$ be two Riemannian manifolds with*

$$g = dS_M^2 = \sum_{i,j} g_{ij} dx^i dx^j; \ \tilde{g} = dS_N^2 = \sum_{i,j} \tilde{g}_{ij} dx^i dx^j$$

respectively. A local smooth diffeomorphism $\Phi : M \to N$ is called conformal (angle - preserving) if there exists a positive function $\psi : M \to \mathbb{R}$ such that $\Phi^* \tilde{g} = \psi \cdot g$.

When $M = N$: the two metrics g and $\tilde{g}$ on M are called conformal if there exists a positive function $\Phi : M \to M$ such that $\tilde{g} = \Phi g$. In this case, we say (M,g) and $(M,\tilde{g})$ are conformal.

Now, choose

$$\phi : M \times (-c,c) \to \mathbb{R}$$

a family of functions smooth in the first variable x and real analytic in the second ε. Write

$$\phi_\varepsilon(x) = \phi(x,\varepsilon) \text{ with } \phi_0 = 0.$$

Define the corresponding family of conformal metrics g_ε such that $g_\varepsilon = e^{\phi_\varepsilon} g$, with the condition that

$$g_\varepsilon^{(1)} = \frac{\partial}{\partial \varepsilon}(g_\varepsilon)|_{\varepsilon=0} = \dot{\phi}_0 g, \ \dot{\phi}_0 \in C^\infty(M); \text{ where } \dot{\phi}_\varepsilon = \frac{\partial}{\partial \varepsilon}(\phi_\varepsilon).$$

It is well known among other properties of such perturbation that there exists a sequence of eigenvalues $\{\Lambda_k(\varepsilon)\} \subset \mathbb{R}$ (counted with multiplicities) and $\psi_k(\varepsilon)$ on $C^\infty(M)$, such that

$\Delta_{g(\varepsilon)}\psi_k(\varepsilon)=\Lambda_k(\varepsilon)\psi_k(\varepsilon)$ and $\Lambda_k(0)=\lambda_k$ where λ_k is the eigenvalue associated with the unperturbed metric g; see e.g Zelditch [26], Bando and Urakawa ([3]). One can now write the associated spectral zeta kernel of Δ_ε on M as

$$\zeta_{g_\varepsilon}(s,x,y)=\sum_{k=1}^{\infty}\frac{\psi_{k,j}(\varepsilon,x)\bar{\psi}_{k,j}(\varepsilon,y)}{\big(\Lambda_k(\varepsilon)\big)^s};\ \Re(s)>\frac{n}{2}.$$

3.1. **Change in the Laplacian.** Let $\{h=\rho g\}$ be set of Riemannian metrics on M in the conformal class. We immediately have that the volume form dV_h scales as

$$dV_h=\sqrt{\det(h)}dx=\rho^{\frac{n}{2}}\sqrt{\det(g)}dx. \tag{3.1}$$

Theorem 3.2. *Given $0<\rho\in C^\infty(M;\rho g)$, the Laplacian with respect to this conformally changed metric $h=\rho g$ is given by*

$$\Delta_h\psi=\rho^{-1}\Delta_g\psi+(1-\frac{n}{2})\rho^{-2}\mathrm{div}(\rho\nabla_g)\psi \tag{3.2}$$

where $\mathrm{div}(\rho\nabla_g)$ *is the operator defined by*

$$\mathrm{div}(\rho\nabla_g)=g^{ij}(\partial_i\rho)\partial_j \tag{3.3}$$

where the so-called Einstein summation convention of summing over repeated indices is used. That is,

$$\mathrm{div}(\rho\nabla_g)\psi:=\langle\nabla\rho,\nabla\psi\rangle_g.$$

Proof. Using (2.2) gives the formula □

Consequently, the corresponding family of Laplacians Δ_ε on $\psi\in C^\infty(M)$ are defined as

$$\Delta_\varepsilon\psi=e^{-\phi_\varepsilon}\Delta_g\psi+(1-\frac{n}{2})e^{-2\phi_\varepsilon}\mathrm{div}(e^{\phi_\varepsilon}\nabla_g)\psi. \tag{3.4}$$

It is not difficult to verify that the first order variation of the perturbed Laplacian $\Delta_0^{(1)}:=\frac{\partial}{\partial\varepsilon}\Delta_\varepsilon\big|_0$ is given by

$$\Delta_0^{(1)}=-\dot{\phi}_0\Delta_g+(1-\frac{n}{2})\langle\nabla_g\dot{\phi}_0,\nabla_g\cdot\rangle_g. \tag{3.5}$$

and that the second order variation of the perturbed Laplacian $\Delta_0^{(2)}:=\frac{\partial^2}{\partial\varepsilon^2}\Delta_\varepsilon\big|_0$ is given by

$$\begin{aligned}\Delta_0^{(2)} &= -\ddot{\phi}_0\Delta_g+(\dot{\phi}_0)^2\Delta_g+(n-2)\dot{\phi}_0\langle\nabla_g\dot{\phi}_0,\nabla_g\cdot\rangle_g\\ &+ (1-\frac{n}{2})\langle\nabla_g\ddot{\phi}_0,\nabla_g\cdot\rangle_g.\end{aligned} \tag{3.6}$$

3.2. **Changes in the volume and volume form.** From (3.1), it follows that $dV_\varepsilon = \sqrt{|g_\varepsilon|}dx = e^{\frac{n}{2}\phi_\varepsilon}\sqrt{|g|}dx = e^{\frac{n}{2}\phi_\varepsilon}dV_g$; where $|g|$ is the determinant of the metric g. Suppose now that the volume of (M, g_ε) is fixed to be a constant V for the conformal family $\{g_\varepsilon = e^{\phi_\varepsilon}g\}$. Then $\int_M dV_\varepsilon = V$, we have

$$\int_M dV_\varepsilon = \int_M e^{\frac{n}{2}\phi_\varepsilon}dV_g = V,$$

so we can observe that

$$\begin{aligned} 0 = \frac{\partial}{\partial\varepsilon}\int_M e^{\frac{n}{2}\phi_\varepsilon}dV_g &= \frac{n}{2}\int_M \frac{\partial}{\partial\varepsilon}(\phi_\varepsilon)e^{\frac{n}{2}\phi_\varepsilon}dV_g \\ &= \int_M \frac{\partial}{\partial\varepsilon}(\phi_\varepsilon)dV_\varepsilon. \end{aligned}$$

Observation 3.3.

(1.) *Since* $\int_M \dot{\phi}_\varepsilon dV_\varepsilon = 0$, *it follows that* $\int_M \dot{\phi}_0 dV_g = 0$.

(2.) *Also note that*

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial\varepsilon^2}\int_M dV_\varepsilon = \int_M \frac{\partial}{\partial\varepsilon}\left[\frac{n}{2}\frac{\partial}{\partial\varepsilon}(\phi_\varepsilon)e^{\frac{n}{2}\phi_\varepsilon}\right]dV_g = \int_M \left[\frac{n^2}{4}(\dot{\phi}_\varepsilon)^2 e^{\frac{n}{2}\phi_\varepsilon} + \frac{n}{2}\ddot{\phi}_\varepsilon e^{\frac{n}{2}\phi_\varepsilon}\right]dV_g \\ &\Rightarrow \int_M \ddot{\phi}dV_\varepsilon = -\frac{n}{2}\int_M (\dot{\phi}_\varepsilon)^2 dV_\varepsilon \text{ and } \int_M \ddot{\phi}dV_g = -\frac{n}{2}\int_M (\dot{\phi}_0)^2 dV_g. \end{aligned}$$

The following lemmas are salient in this work.

Lemma 3.4. *The following properties hold. (i). The expectation values of the commutator* $[\Delta_g, \dot{\phi}_0]$ *with respect to eigenfunctions is zero; and (ii).* $\mathrm{Tr}(\Delta_g \circ \dot{\phi}_0 e^{-t\Delta_g}) = \mathrm{Tr}(\dot{\phi}_0 \Delta_g e^{-t\Delta_g})$ *where* $A \circ B$ *denotes composition of the two operators* A *and* B.

Proof. (i). Let ψ_k be an orthonormal basis of eigenfunction of Δ_g, then the expectation value of $[\Delta_g, \dot{\phi}_0]$ on ψ_k is

$$\begin{aligned} \left\langle[\Delta_g, \dot{\phi}_0]\right\rangle_{\psi_k} &:= \langle(\Delta_g\dot{\phi}_0 - \dot{\phi}_0\Delta_g)\psi_k, \psi_k\rangle_g \\ &= \langle\Delta_g\dot{\phi}_0\psi_k, \psi_k\rangle_g - \langle\dot{\phi}_0\Delta_g\psi_k, \psi_k\rangle_g \\ &= \langle\dot{\phi}_0\psi_k, \Delta_g\psi_k\rangle_g - \langle\Delta_g\psi_k, \dot{\phi}_0\psi_k\rangle_g \\ &= 0, \end{aligned}$$

and (ii).

$$\begin{aligned}\mathrm{Tr}(\Delta_g \circ \dot{\phi}_0 e^{-t\Delta_g}) &= \sum_{k=0}^{\infty} \langle \Delta_g \circ \dot{\phi}_0 e^{-t\Delta_g} \psi_k, \psi_k \rangle_g = \sum_{k=0}^{\infty} \langle \dot{\phi}_0 e^{-t\Delta_g} \psi_k, \Delta_g \psi_k \rangle_g \\ &= \sum_{k=0}^{\infty} \langle \dot{\phi}_0 e^{-t\lambda_k} \psi_k, \lambda_k \psi_k \rangle_g = \sum_{k=0}^{\infty} \langle \dot{\phi}_0 \lambda_k e^{-t\lambda_k} \psi_k, \psi_k \rangle_g \\ &= \sum_{k=0}^{\infty} \langle \dot{\phi}_0 \Delta_g e^{-t\Delta_g} \psi_k, \psi_k \rangle_g = \mathrm{Tr}(\dot{\phi}_0 \Delta_g e^{-t\Delta_g})\end{aligned}$$

the lemma follows □

Lemma 3.5. [6]. $\Delta_g e^{-t\Delta_g} = e^{-t\Delta_g}\Delta_g$ *for all smooth functions on* M.

3.3. **Changes in the spectral zeta function.** We need the following lemma for our proof of the variational formula for the spectral zeta function.

Lemma 3.6.

$$\mathrm{Tr}(\dot{\phi}_0 e^{-t\Delta g}) = \int_M \dot{\phi}_0(x) K(t,x,x) dV_g(x)$$

and

$$\mathrm{Tr}(\mathrm{div}(\dot{\phi}_0 \nabla_g) e^{-t\Delta g}) = \frac{1}{2} \int_M \mathrm{div}(\dot{\phi}_0 \nabla_g) K(t,x,x) dV_g(x).$$

Proof. Let ψ_k be orthonormal basis of eigenfunctions of Δ_g, we have

$$\begin{aligned}\mathrm{Tr}(\dot{\phi}_0 e^{-t\Delta g}) &= \sum_{k=0}^{\infty} \langle \dot{\phi}_0 e^{-t\Delta_g} \psi_k, \psi_k \rangle_{L^2} \\ &= \sum_{k=0}^{\infty} \int_M \dot{\phi}_0(x) e^{-\lambda_k t} \psi_k(x) \cdot \bar{\psi}_k(x) dV_g(x) \\ &= \int_M \dot{\phi}_0(x) \cdot \sum_{k=0}^{\infty} e^{-\lambda_k t} (\psi_k(x))^2 dV_g(x) \\ &= \int_M \dot{\phi}_0(x) K(t,x,x) dV_g(x).\end{aligned}$$

Similarly,

$$\begin{aligned}
\operatorname{Tr}(\operatorname{div}(\dot{\phi}_0\nabla_g)e^{-t\Delta g}) &= \sum_{k=0}^{\infty}\langle \operatorname{div}(\dot{\phi}_0\nabla_g)\cdot e^{-t\Delta g}\psi_k,\psi_k\rangle_{L^2}\\
&= \sum_{k=0}^{\infty}\langle \sum_{i,j=1}^{n} g^{ij}\partial_j\dot{\phi}_0\partial_i\cdot e^{-t\Delta g}\psi_k,\psi_k\rangle_{L^2}\\
&= \sum_{i,j=1}^{n}\sum_{k=0}^{\infty}\langle g^{ij}\partial_j\dot{\phi}_0\partial_i\cdot e^{-t\Delta g}\psi_k,\psi_k\rangle_{L^2}\\
&= \sum_{i,j=1}^{n}\sum_{k=0}^{\infty}\int_M g^{ij}\partial_j\dot{\phi}_0(x)\int_M \partial_i K(t,x,y)\psi_k(y)dV_g(y)\big|_{x=y}\cdot\bar{\psi}_k(x)dV_g(x);
\end{aligned}$$

where $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial_j = \frac{\partial}{\partial x_j}$ act on functions of x and y respectively.

Now, using the symmetry of $K(t,x,y)$ i.e $K(t,x,y) = K(t,y,x)$ which implies that

$$\partial_i K(t,x,x) = \big[\partial_i K(t,x,y) + \partial_i K(t,y,x)\big]\big|_{x=y} = 2\partial_i K(t,x,y)\big|_{x=y}$$

we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}\langle \operatorname{div}(\dot{\phi}_0\nabla_g)\cdot e^{-t\Delta g}\psi_k,\psi_k\rangle_{L^2}\\
= \;& \sum_{i,j=1}^{n}\sum_{k=0}^{\infty}\int_M g^{ij}\partial_j\dot{\phi}_0(x)\int_M \partial_i K(t,x,y)\big|_{x=y}\psi_k(y)dV_g(y)\cdot\bar{\psi}_k(x)dV_g(x)\\
\Rightarrow \;& \operatorname{Tr}(\operatorname{div}(\dot{\phi}_0\nabla_g)e^{-t\Delta g}) = \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=0}^{\infty}\int_M g^{ij}\partial_j\dot{\phi}_0(x)\partial_i e^{-\lambda_k t}(\psi_k(x))^2 dV_g(x)\\
= \;& \frac{1}{2}\int_M \sum_{i,j=1}^{n} g^{ij}\partial_j\dot{\phi}_0(x)\partial_i\cdot\sum_{k=0}^{\infty}e^{-\lambda_k t}(\psi_k(x))^2 dV_g(x) = \frac{1}{2}\int_M \operatorname{div}(\dot{\phi}_0\nabla_g)K(t,x,x)dV_g(x).
\end{aligned}$$

□

Theorem 3.7. *Let (M,g) be smooth, compact and connected Riemannian manifold and Δ_g the Laplacian on it with eigenvalues $\{\lambda_k\}$ listed according to their multiplicities. Let*

$$\{g_\varepsilon = e^{\phi_\varepsilon}g\}$$

be a family of volume-preserving conformal metrics. Then the spectral zeta function of Δ_ε, given by

$$\zeta_{g_\varepsilon}(s) = \sum_{k=1}^{\infty}\frac{1}{\big(\Lambda_k(\varepsilon)\big)^s} \tag{3.7}$$

varies as

$$(3.8)\qquad \zeta_g^{(1)}(s) = s\int_M \dot\phi_0(x)\zeta_g(s,x,x)dV_g + \frac{1}{2}(\frac{n}{2}-1)s\int_M (\Delta_g\dot\phi_0(x))\zeta(s+1,x,x)dV_g$$

(c.f: [19] *and* [20]*). We denote this variation evaluated at* $\varepsilon = 0$ *by* $\zeta_g^{(1)}(s)$*.*

Proof. Recall

$$\zeta_{g_\varepsilon}(s) = \frac{1}{\Gamma(s)}\int_0^\infty (\mathrm{Tr}(e^{-t\Delta_\varepsilon})-1)t^{s-1}dt$$

so,

$$(3.9)\qquad \zeta_g^{(1)}(s) = \frac{\partial}{\partial\varepsilon}\zeta_{g_\varepsilon}(s)|_{\varepsilon=0} = \frac{\partial}{\partial\varepsilon}\big|_{\varepsilon=0}\Big(\frac{1}{\Gamma(s)}\int_0^\infty (\mathrm{Tr}(e^{-t\Delta_\varepsilon})-1)t^{s-1}dt\Big).$$

In line with Ray and Singer [19], one gets

$$\begin{aligned} \zeta_g^{(1)}(s) &= -\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\big(\Delta_\varepsilon^{(1)}e^{-t\Delta_g}\big)t^s dt \\ &= -\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\big(\big[-\dot\phi_0\Delta_g + (1-\frac{n}{2})\mathrm{div}(\dot\phi_0\nabla_g\big](e^{-t\Delta_g}\big)t^s dt \end{aligned}$$

where we have used the variation of Δ_ε in (3.5).

So,

$$\begin{aligned} \zeta_g^{(1)}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\big(\dot\phi_0\Delta_g e^{-t\Delta_g}\big)t^s dt - (1-\frac{n}{2})\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\big(\mathrm{div}(\dot\phi_0\nabla_g e^{-t\Delta_g}\big)t^s dt \\ &= -\frac{1}{\Gamma(s)}\int_0^\infty \frac{\partial}{\partial t}\mathrm{Tr}\big(\dot\phi_0 e^{-t\Delta_g}\big)t^s dt + (\frac{n}{2}-1)\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\big(\mathrm{div}(\dot\phi_0\nabla_g e^{-t\Delta_g}\big)t^s dt. \end{aligned}$$

Integrating by parts in the first term, gives

$$\begin{aligned} \zeta_g^{(1)}(s) &= \frac{s}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\big(\dot\phi_0(e^{-t\Delta_g}-\frac{1}{V})\big)t^{s-1}dt \\ &+ (\frac{n}{2}-1)\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\big(\mathrm{div}(\dot\phi_0\nabla_g(e^{-t\Delta_g}-\frac{1}{V})\big)t^s dt \end{aligned}$$

where $\frac{1}{V}$ denotes $f\mapsto \frac{1}{V}\int_M f dV$ and V is the volume of (M,g).

Hence to complete the proof of Theorem (3.7), using lemma (3.6), we have the variation of the zeta function as

$$\begin{aligned} \zeta_g^{(1)}(s) &= \frac{s}{\Gamma(s)}\int_0^\infty\int_M \dot\phi_0(x)(K(t,x,x)-\frac{1}{V}))dV_g t^{s-1}dt \\ &+ \frac{1}{2}(\frac{n}{2}-1)\frac{1}{\Gamma(s)}\int_0^\infty\int_M \mathrm{div}(\dot\phi_0\nabla_g)(K(t,x,x)-\frac{1}{V}))dV_g t^s dt. \end{aligned}$$

Since

$$\int_M \left(\dot{\phi}_0(x) K(t,x,x) - \frac{1}{V} \right) dV_g(x) \to 0$$

decays exponentially fast as $t \to \infty$. Also, recognizing that $\frac{1}{\Gamma(s)} = \frac{s}{\Gamma(s+1)}$ we have

$$\begin{aligned} \zeta_g^{(1)}(s) &= s \int_M \dot{\phi}_0(x) \left\{ \frac{1}{\Gamma(s)} \int_0^\infty (K(t,x,x) - \frac{1}{V})) t^{s-1} dt \right\} dV_g \\ &+ \frac{1}{2}(\frac{n}{2} - 1) s \int_M \operatorname{div}(\dot{\phi}_0 \nabla_g) \left\{ \frac{1}{\Gamma(s+1)} \int_0^\infty (K(t,x,x) - \frac{1}{V}) t^s dt \right\} dV_g. \end{aligned}$$

Therefore,

$$\begin{aligned} \zeta_g^{(1)}(s) &= s \int_M \dot{\phi}_0(x) \zeta_g(s,x,x) dV_g \\ &+ \frac{1}{2}(\frac{n}{2} - 1) s \int_M \operatorname{div}(\dot{\phi}_0 \nabla_g) \zeta(s+1,x,x) dV_g. \end{aligned}$$

Thus,

$$\zeta_g^{(1)}(s) = s \int_M \dot{\phi}_0(x) \zeta_g(s,x,x) dV_g + \frac{1}{2}(\frac{n}{2} - 1) s \int_M (\Delta_g \dot{\phi}_0(x)) \zeta_g(s+1,x,x) dV_g$$

which completes the proof. □

Therefore by equation (3.8), the Casimir energy has the first order variation at $\varepsilon = 0$, $\mathrm{FP}[\zeta_g^{(1)}(s)]\big|_{s=-\frac{1}{2}}$, given by

$$\begin{aligned} \mathrm{FP}[\zeta_g^{(1)}(s)]\big|_{s=-\frac{1}{2}} &= -\frac{1}{2} \int_M \dot{\phi}_0(x) \mathrm{FP}[\zeta_g(s,x,x)]\big|_{s=-\frac{1}{2}} dV_g \\ &- \frac{1}{4}(\frac{n}{2} - 1) \int_M (\Delta_g \dot{\phi}_0(x)) \mathrm{FP}[\zeta_g(s+1,x,x)]\big|_{s=-\frac{1}{2}} dV_g. \end{aligned} \tag{3.10}$$

where FP is the usual Finite Part function defined by

$$\mathrm{FP}[f](s) := \begin{cases} f(s) \text{ if } s \text{ is not a pole} \\ \lim_{\varepsilon \to 0} \left(f(s+\varepsilon) - \frac{\text{Residue}}{\varepsilon} \right), \text{ if } s \text{ is a pole} \end{cases} \tag{3.11}$$

see for example ([8]).

Definition 3.8. *The metric g is called a critical point of the Casimir energy $\zeta_g(-\frac{1}{2})$ with respect to all variations $\{g_\varepsilon = e^{\phi_\varepsilon} g\}$, if the variation $\zeta_g^{(1)}(-\frac{1}{2})$ vanishes for all g_ε.*

Another result of this work is the following:

Theorem 3.9. *Let Δ_ε be the Laplacian on (M, g_ε) with zeta kernel $\zeta_g(s,x,y)$. Then, g is a critical point of the Casimir energy $\zeta_g(-\frac{1}{2})$ for all constant-volume conformal variations of the metric if* $\mathrm{FP}[\zeta_g^{(1)}(-\frac{1}{2},x,x)]$ *is constant in x.*

Proof. By the definition of critical point above and the variation of the Casimir energy (3.10), consider the function

$$F_{\dot\phi_0}(x) := (-\frac{1}{2}\dot\phi_0(x)\mathrm{FP}[\zeta_g(s,x,x)] - \frac{1}{4}(\frac{n}{2}-1)(\Delta_g\dot\phi_0)\mathrm{FP}[\zeta_g(s+1,x,x)])\big|_{s=-\frac{1}{2}} \tag{3.12}$$

where of course,

$$\zeta_g(s,x,x)\big|_{s=-\frac{1}{2}} = \frac{1}{\Gamma(s)}\int_0^\infty \big[K(t,x,x) - \frac{1}{V}\big]t^{s-1}dt\big|_{s=-\frac{1}{2}}.$$

We have a critical point if

$$\int_M F_{\dot\phi_0}(x)\mathrm{d}V_x = 0 \;\forall\dot\phi_0 \in C^\infty(M) \text{ such that } \int_M \dot\phi_0(x)\mathrm{d}V_x = 0. \tag{3.13}$$

Now, suppose $\mathrm{FP}[\zeta_g(-\frac{1}{2},x,x)]$ is constant. Then one gets

$$\begin{aligned}\int_M F_{\dot\phi_0}\mathrm{d}V_x &= -\frac{1}{2}\mathrm{FP}[\zeta_g(-\frac{1}{2},x,x)]\int_M \dot\phi_0(x)\mathrm{d}V_x \\ &- \frac{1}{4}(\frac{n}{2}-1)\int_M(\Delta_g\dot\phi_0(x))\mathrm{FP}[\zeta_g(\frac{1}{2},x,x)]\mathrm{d}V_x \\ &= -\frac{1}{4}(\frac{n}{2}-1)\int_M(\Delta_g\dot\phi_0(x))\mathrm{FP}[\zeta_g(\frac{1}{2},x,x)]\mathrm{d}V_x\end{aligned}$$

since $\int_M \dot\phi_0(x)\mathrm{d}V_x = 0$. Now by the self-adjointness of Δ_g, we have

$$\int_M F_{\dot\phi_0}(x)\mathrm{d}V_x = -\frac{1}{4}(\frac{n}{2}-1)\int_M \dot\phi_0(x)\Delta_g\mathrm{FP}[\zeta_g(\frac{1}{2},x,x)]\mathrm{d}V_x = 0$$

since the Laplacian of a constant function is zero. □

We get a corollary by considering homogeneous manifolds.

Definition 3.10. *([4]). A Riemannian manifold (M,g) is called homogeneous if for any two points $x,y \in M$, there exists an isometry $I : M \to M$ with $I(x) = y$. That is to say, I acts transitively on M. More generally, a smooth Riemannian manifold (M,g) endowed with transitive smooth action of a Lie group G is called a G-homogeneous manifold.*

A nice class of homogeneous manifolds comes from quotients of Lie groups with left-invariant metrics. For example, the natural action of $SO(n+1)$ on the n-sphere S^n is transitive, hence $S^n \approx SO(n+1)/SO(n)$ is a homogeneous manifold; see e.g [1, 2] and [5].

Corollary 3.11. *The metrics on homogeneous smooth Riemannian manifolds are critical points of the variation of the Casimir energy $\zeta_g(-\frac{1}{2})$ under fix-volume conformal variation of the metric.*

Proof. Since $\zeta_g(s,x,x)$ is an invariant under isometries on homogeneous manifolds, one can map any point to another point via isometry. Hence, $\zeta_g(s,x,x)$ is a constant □

Corollary 3.12. *The round metric g on the n-dimensional unit sphere, S^n, is a critical point for the Casimir energy $\zeta_g(-\frac{1}{2})$ over the constant-volume conformal class $\{g_\varepsilon = e^{\phi_\varepsilon} g\}$.*

Proof. This follows from the fact that S^n is a homogeneous manifold. □

We say that the metric g on M is critical for the heat kernel $K_g(t,x,x)$ at the time t, if for any volume-preserving deformation $\{g_\varepsilon = e^{\phi_\varepsilon} g\}$,

$$\frac{d}{d\varepsilon} K_\varepsilon(t,x,x)|_{\varepsilon=0} = 0. \tag{3.14}$$

The spectral zeta function of the Laplacian Δ_ε for $\Re(s) > \frac{n}{2}$ is

$$\zeta_{g_\varepsilon}(s) = \sum_{k=1}^{\infty} \frac{1}{\Lambda_k^s(\varepsilon)} = \frac{1}{\Gamma(s)} \int_0^\infty (K_\varepsilon(t,x,x) - \frac{1}{V}) t^{s-1} dt. \tag{3.15}$$

For sufficiently large $\Re(s)$, one has

$$\frac{d}{d\varepsilon} \zeta_{g_\varepsilon}(s)|_{\varepsilon=0} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{d}{d\varepsilon} \big(K_\varepsilon(t,x,x)\big|_{\varepsilon=0}\big) t^{s-1} dt. \tag{3.16}$$

If g is critical for the heat kernel at any time $t > 0$, then its derivative in ε vanishes at $\varepsilon = 0$ for all s, so also

$$\lim_{s \to -1/2} \big[\frac{d}{d\varepsilon} \zeta_{g_\varepsilon}(s)|_{\varepsilon=0}\big] = 0. \tag{3.17}$$

Hence, g is a critical point for the variation of the Casimir energy $\zeta_g(-\frac{1}{2})$ of the Laplacian on S^n. Consequently, we write this result as the lemma below:

Lemma 3.13. *If g is a critical metric for the heat kernel at any time $t > 0$, then it is also a critical metric for the Casimir energy $\zeta_g(-\frac{1}{2})$ under all volume-preserving conformal perturbations $\{g_\varepsilon\}$.*

Proposition 3.14. *The following conditions hold on all closed homogeneous Riemannian manifolds (M,g):*

(1.) *The metric g is critical for the heat kernel at any time $t > 0$ under all volume-preserving conformal deformations.*

(2.) *The metric g is critical for the Casimir energy $\zeta_g(-\frac{1}{2})$ under all volume-preserving conformal deformations.*

(3.) *For all $t > 0$, $K_\varepsilon(t,x,x)$ is constant on M.*

Proof. The proposition follows from the lemma (3.13) above. □

4. Hessians of $\zeta_g(s)$ on Homogeneous manifolds

We now compute the second order variation of the spectral zeta function.

Theorem 4.1. *Let M be a closed homogeneous manifold with the canonical metric g scaled to volume V. Let $\{g_\varepsilon = e^{\phi_\varepsilon} g\}$ be a family of volume-preserving conformal metrics on M where*

$$\int_M \dot{\phi}_0(x)dV_g(x) = 0 \text{ and } \int_M (\dot{\phi}_0(x))^2 dV_g(x) > 0.$$

Then the second order variation, $\zeta_g^{(2)}(s)$, of the spectral zeta function $\zeta_{g_\varepsilon}(s)$ on M at $\varepsilon = 0$ is given by

$$\begin{aligned}
\zeta_g^{(2)}(s) &= s\int_M\int_M \dot{\phi}_0(x)\Psi_{s-1}(x,y)\dot{\phi}_0(y)dV_g(x)dV_g(y) \\
&- (1-\frac{n}{2})s\int_M\int_M \dot{\phi}_0(x)\Psi_s(x,y)(\Delta_g\dot{\phi}_0(y))dV_g(x)dV_g(y) \\
&+ \frac{(n-2)^2}{16}s\int_M\int_M (\Delta_g\dot{\phi}_0(x))\Psi_{s+1}(x,y)(\Delta_g\dot{\phi}_0(y))dV_g(x)dV_g(y)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{8}(n+2)^2 s\frac{1}{V}\int_M\int_M \dot{\phi}_0(x)\dot{\phi}_0(y)\zeta(s,x,y)dV_g(x)dV_g(y)\\
&- \frac{1}{8}(n-2)^2 s\zeta_n(s+1)\frac{1}{V}\int_M \dot{\phi}_0(x)(\Delta_g\dot{\phi}_0(x))dV_g(x)\\
&+ (1-\frac{n}{2})s\zeta_n(s)\frac{1}{V}\int_M (\dot{\phi}_0(x))^2 dV_g(x),
\end{aligned} \tag{4.1}
$$

where for $\Re(s)$ sufficiently large, we introduce the integral kernel Ψ_s and define it to be

$$
\Psi_s(x,y) = \frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty (K(u,x,y)-\frac{1}{V})(K(v,x,y)-\frac{1}{V})(u+v)^{s-1}\mathrm{d}u\mathrm{d}v. \tag{4.2}
$$

Proof. Using that $\zeta_{g_\varepsilon}(s) = \mathrm{Tr}(\Delta_\varepsilon^{-s})$, let P_ε be the projection with respect to the metric g_ε onto the kernel of Δ_ε. Then, since the kernel for all values of ε is the constant functions, we get for $\Re(s) < -1$,

$$
\mathrm{Tr}\Big(\dot{P}_\varepsilon\Delta_\varepsilon^{-s-1}\Big) = -\mathrm{Tr}\Big(P_\varepsilon\dot{P}_\varepsilon\Delta_\varepsilon^{-s-1}\Big) = 0, \tag{4.3}
$$

is true for $\Re(s)$ sufficiently large. So,

$$
\zeta_{g_\varepsilon}(s) = \mathrm{Tr}\Big(\Delta_\varepsilon^{-s} - P_\varepsilon\Big) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\mathrm{e}^{-t\Delta_\varepsilon} - P_\varepsilon\Big)t^{s-1}\mathrm{d}t.
$$

Now the decay in the integrand allows us to bring the derivative inside, and we obtain:

$$
\begin{aligned}
\frac{\partial}{\partial\varepsilon}\big(\zeta_{g_\varepsilon}(s)\big) &= -\frac{s}{\Gamma(s+1)}\int_0^\infty \mathrm{Tr}\Big(\dot{\Delta}_\varepsilon\big(\mathrm{e}^{-t\Delta_\varepsilon} - P_\varepsilon\big)\Big)t^s\mathrm{d}t - \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}(\dot{P})t^s\,\mathrm{d}t\\
&= -\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\dot{\Delta}_\varepsilon\big(\mathrm{e}^{-t\Delta_\varepsilon} - P_\varepsilon\big)\Big)t^s\mathrm{d}t.
\end{aligned} \tag{4.4}
$$

Hence at $\varepsilon = 0$, we get

$$
\zeta_g^{(1)}(s) = -\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\Delta_0^{(1)}\big(\mathrm{e}^{-t\Delta_0} - P\big)\Big)t^s\mathrm{d}t
$$

($P \equiv P_0$) which we can check agrees with the first-order variation of the spectral zeta function (3.9).

Differentiating (4.4) a second time, we get

$$
\begin{aligned}
\frac{\partial^2}{\partial\varepsilon^2}\big(\zeta_{g_\varepsilon}(s)\big) &= -s\mathrm{Tr}\Big(\ddot{\Delta}_\varepsilon\Delta_\varepsilon^{-s-1}\Big) - s\mathrm{Tr}\Big(\dot{\Delta}_\varepsilon(\frac{\partial}{\partial\varepsilon}(\Delta_\varepsilon^{-s-1})\Big)\\
&= -\frac{s}{\Gamma(s+1)}\int_0^\infty \mathrm{Tr}\Big(\ddot{\Delta}_\varepsilon\mathrm{e}^{-t\Delta_\varepsilon}\Big)t^s\mathrm{d}t\\
&- \frac{s}{\Gamma(s+1)}\int_0^\infty \mathrm{Tr}\Big(\dot{\Delta}_\varepsilon\big(\frac{\partial}{\partial\varepsilon}(\mathrm{e}^{-t\Delta_\varepsilon}) - P_\varepsilon)\big)\Big)t^s\mathrm{d}t.
\end{aligned} \tag{4.5}
$$

By Duhamel's formula, see e.g [4, 6],

$$\frac{\partial}{\partial\varepsilon}\mathrm{e}^{-t\Delta_\varepsilon} = -\int_0^t \mathrm{e}^{-u\Delta_\varepsilon}\dot{\Delta}_\varepsilon \mathrm{e}^{-(t-u)\Delta_\varepsilon}\mathrm{d}u$$

for times $0 < u < t < T$.

Thus at $\varepsilon = 0$, we have

$$\begin{aligned}\zeta_g^{(2)}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty\int_0^t \mathrm{Tr}\Big(\Delta_0^{(1)}\mathrm{e}^{-u\Delta_g}\Delta_0^{(1)}\mathrm{e}^{-(t-u)\Delta_g}\Big)t^s\mathrm{d}u\mathrm{d}t\\ &- \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\Delta_0^{(2)}\big(\mathrm{e}^{-t\Delta_g}-P_0\big)\Big)t^s\mathrm{d}t.\end{aligned}$$

We write this for simplicity as

$$\begin{aligned}\zeta_g^{(2)}(s) &= \mathrm{var}_1(s)+\mathrm{var}_2(s) \text{ where}\\ \mathrm{var}_1(s) &:= \frac{1}{\Gamma(s)}\int_0^\infty\int_0^t \mathrm{Tr}\Big(\Delta_0^{(1)}\mathrm{e}^{-u\Delta_g}\Delta_0^{(1)}\mathrm{e}^{-(t-u)\Delta_g}\Big)t^s\mathrm{d}u\mathrm{d}t \text{ and}\\ \mathrm{var}_2(s) &:= -\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\Delta_0^{(2)}\mathrm{e}^{-t\Delta_g}-P_0\Big)t^s\mathrm{d}t.\end{aligned}$$

For easier handling of the computation of the terms of this second-order variation, we rewrite the operator Δ_ε defined by (3.4) in terms of the Laplacian as follows. Define the operator

$$\mathscr{G}_{\dot{\phi}_0} = \langle\nabla\dot{\phi}_0,\nabla\cdot\rangle_g \tag{4.6}$$

and immediately observe that for any $f\in C^\infty(M)$

$$\mathscr{G}_{\dot{\phi}_0}f = \frac{1}{2}(\Delta\dot{\phi}_0)f+\frac{1}{2}\dot{\phi}_0\Delta f-\frac{1}{2}(\Delta\circ\dot{\phi}_0)f. \tag{4.7}$$

Thus, for any $\psi\in C^\infty(M)$, $\Delta_\varepsilon\psi$ can be written as

$$\begin{aligned}\Delta_\varepsilon\psi &= e^{-\phi_\varepsilon}\Delta_g\psi+\frac{1}{2}(1-\frac{n}{2})e^{-\phi_\varepsilon}(\Delta_g\phi_\varepsilon)\psi\\ &+ \frac{1}{2}(1-\frac{n}{2})e^{-\phi_\varepsilon}\phi_\varepsilon(\Delta_g\psi)-\frac{1}{2}(1-\frac{n}{2})e^{-\phi_\varepsilon}(\Delta_g\circ\phi_\varepsilon)\psi.\end{aligned} \tag{4.8}$$

Note that $\mathscr{G}_1\circ\mathscr{G}_2$ denotes composition of operators; e.g

$$(e^{-\phi_\varepsilon}(\Delta_g\circ\phi_\varepsilon)\psi)(x) = e^{-\phi_\varepsilon}\Delta_g(\phi_\varepsilon(x)\psi(x)).$$

Similarly, one can re-write (3.5) as

$$\Delta_0^{(1)} = -\frac{1}{2}(\frac{n}{2}+1)\dot{\phi}_0\Delta_g-\frac{1}{2}(\frac{n}{2}-1)(\Delta_g\dot{\phi}_0)+\frac{1}{2}(\frac{n}{2}-1)\Delta_g\circ\dot{\phi}_0. \tag{4.9}$$

4.1. **Computation of** $\mathrm{var}_1(s)$**.** We now compute the term

$$\mathrm{var}_1(s) \;:=\; \frac{1}{\Gamma(s)}\int_0^\infty\int_0^t \mathrm{Tr}\Big(\Delta_0^{(1)}\mathrm{e}^{-u\Delta_g}\Delta_0^{(1)}\mathrm{e}^{-(t-u)\Delta_g}\Big)t^s\mathrm{d}u\mathrm{d}t. \tag{4.10}$$

First we make a few notes. In what follows, we will often drop the subscript g of Δ_g and write $K(u,x,y)$ and $K(t-u,y,x)$ for the kernels of the operators $\mathrm{e}^{-u\Delta_g}$ and $\mathrm{e}^{-(t-u)\Delta_g}$ respectively.

If A and B are differential operators, then their compositions with smoothing operators are bounded, see e.g [12, 19]. So by the vanishing of trace on commutators of bounded operators, and by change of variables,

$$\begin{aligned}\int_0^t \mathrm{Tr}\Big(Ae^{-u\Delta}Be^{-(t-u)\Delta}\Big)du &= \int_0^t \mathrm{Tr}\Big(Be^{-(t-u)\Delta}Ae^{-u\Delta}\Big)du\\ =\int_0^t \mathrm{Tr}\Big(Ae^{-(t-u)\Delta}Be^{-u\Delta}\Big)du &= \int_0^t \mathrm{Tr}\Big(Be^{-u\Delta}Ae^{-(t-u)\Delta}\Big)du.\end{aligned} \tag{4.11}$$

Observe that for any $f\in C^\infty(M)$,

$$\int_M \mathrm{div}(\dot\phi_0\nabla_g)fdV_g(x) = \int_M\langle\nabla_g\dot\phi_0,\nabla_g f\rangle_g dV_g(x) = \int_M(\Delta_g\dot\phi_0)fdV_g(x). \tag{4.12}$$

We denote $K(u,x,y)-\frac{1}{V}$ by $\tilde K_u$ and $K(v,x,y)-\frac{1}{V}$ by $\tilde K_v$. For $f_1,f_2\in C^\infty(M)$ we also define the notation

$$\mathrm{Tr}[f_1\tilde K_u f_2\tilde K_v] := \int_M\int_M f_1(x)\tilde K_u f_2(y)\tilde K_v dV_g(x)dV_g(y). \tag{4.13}$$

Using (4.9), we get the terms inside the trace in the formula for $\mathrm{var}_1(s)$. The terms are simplified further via the following lemmata.

Lemmata 4.2.

$$\mathrm{Tr}[\dot\phi_0(\Delta\tilde K_u)(\Delta\dot\phi_0)\tilde K_v] = -\frac{\partial}{\partial u}\mathrm{Tr}[\dot\phi_0\tilde K_u(\Delta\dot\phi_0)\tilde K_v]$$

$$\mathrm{Tr}[\dot\phi_0(\Delta\tilde K_u)\dot\phi_0(\Delta\tilde K_v)] = \frac{\partial^2}{\partial u\partial v}\mathrm{Tr}[\dot\phi_0\tilde K_u\dot\phi_0\tilde K_v]$$

$$\mathrm{Tr}[\dot\phi_0(\Delta\tilde K_u)\Delta\circ\dot\phi_0\tilde K_v] = \frac{\partial^2}{\partial u^2}\mathrm{Tr}[\dot\phi_0\tilde K_u\dot\phi_0\tilde K_v]$$

$$\mathrm{Tr}[\Delta\circ\dot\phi_0\tilde K_u\Delta\circ\dot\phi_0\tilde K_v] = \frac{\partial^2}{\partial u\partial v}\mathrm{Tr}[\dot\phi_0\tilde K_u\dot\phi_0\tilde K_v].$$

Proof. Use (4.11) and the properties of K as a solution operator to the heat equation (2.8) □

By a change of coordinates (u,t) to $(u,v) := (u,t-u)$, the double integral in $\mathrm{var}_1(s)$ becomes a double integral over the first quadrant in u and v. Further, because the derivatives we obtain through applying Lemmata (4.2) are applied to functions that are constant in the other variable, these derivatives carry through under the coordinate change to derivatives with respect to u and v.

Finally, collecting like terms gives

$$
\begin{aligned}
\mathrm{var}_1(s) &= \frac{1}{8}(n^2+4)\frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty \frac{\partial^2}{\partial u\partial v}\mathrm{Tr}[\dot\phi_0\tilde K_u\dot\phi_0\tilde K_v](u+v)^s\mathrm{d}u\mathrm{d}v \\
&- \frac{1}{8}(n^2-4)\frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty \frac{\partial^2}{\partial u^2}\mathrm{Tr}[\dot\phi_0\tilde K_u\dot\phi_0\tilde K_v](u+v)^s\mathrm{d}u\mathrm{d}v \\
&+ (1-\frac{1}{2})\frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty \frac{\partial}{\partial u}\mathrm{Tr}[\dot\phi_0\tilde K_u(\Delta\dot\phi_0)\tilde K_v](u+v)^s\mathrm{d}u\mathrm{d}v \\
&+ \frac{1}{16}(n-2)^2\frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty \mathrm{Tr}[(\Delta\dot\phi_0)\tilde K_u(\Delta\dot\phi_0)\tilde K_v](u+v)^s\mathrm{d}u\mathrm{d}v \\
&:= T_1+T_2+T_3+T_4 \qquad (4.14)
\end{aligned}
$$

where we have used (4.11) to treat terms involving $\frac{\partial^2}{\partial u^2}$ and $\frac{\partial^2}{\partial v^2}$ as like terms and those involving $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ similarly.

To simplify the T_1 term, we proceed as follows.

$$
T_1 := \frac{1}{8}(n^2+4)\frac{1}{\Gamma(s)}\int_0^\infty\int_0^\infty \frac{\partial^2}{\partial u\partial v}\mathrm{Tr}[\dot\phi_0\tilde K_u\dot\phi_0\tilde K_v](u+v)^s\mathrm{d}u\mathrm{d}v
$$

integrating by parts in u and using the fact that

$$
\mathrm{Tr}\Big(K(t,x,y)-\frac{1}{V}\Big)\to 0 \qquad (4.15)
$$

exponentially fast as time $t\to\infty$, we obtain

$$
\begin{aligned}
T_1 &= -\frac{1}{8}(n^2+4)\frac{1}{\Gamma(s)}\int_0^\infty \frac{\partial}{\partial v}\mathrm{Tr}[\dot\phi_0(\delta(x,y)-\frac{1}{V})\dot\phi_0\tilde K_v]v^s\mathrm{d}v \\
&- \frac{1}{8}(n^2+4)\frac{s}{\Gamma(s)}\int_0^\infty\int_0^\infty \frac{\partial}{\partial v}\mathrm{Tr}[\dot\phi_0\tilde K_u\dot\phi_0\tilde K_v](u+v)^{s-1}\mathrm{d}u\mathrm{d}v.
\end{aligned}
$$

Since K_0 is the Dirac δ distribution and $\tilde{K} = K - \frac{1}{V}$, this is

$$\begin{aligned} T_1 &= -\frac{1}{8}(n^2+4)\frac{1}{\Gamma(s)}\int_0^\infty \frac{\partial}{\partial v}\mathrm{Tr}[(\dot{\phi}_0(x))^2 K_v]v^s \mathrm{d}v \\ &+ \frac{1}{8}(n^2+4)\frac{1}{\Gamma(s)}\frac{1}{V}\int_0^\infty \frac{\partial}{\partial v}\mathrm{Tr}[\dot{\phi}_0(x)\dot{\phi}_0(y)\tilde{K}_v]v^s \mathrm{d}v \\ &- \frac{1}{8}(n^2+4)\frac{s}{\Gamma(s)}\int_0^\infty\int_0^\infty \frac{\partial}{\partial v}\mathrm{Tr}[\dot{\phi}_0\tilde{K}_u\dot{\phi}_0\tilde{K}_v](u+v)^{s-1}\mathrm{d}u\mathrm{d}v. \end{aligned}$$

Similarly, integrating by parts in v and using $\tilde{K} = K - \frac{1}{V}$, we obtain:

$$\begin{aligned} T_1 &:= \frac{1}{8}(n^2+4)\frac{s}{\Gamma(s)}\int_0^\infty \mathrm{Tr}[(\dot{\phi}_0(x))^2 K_v]v^{s-1}\mathrm{d}v \\ &- \frac{1}{8}(n^2+4)\frac{s}{\Gamma(s)}\frac{1}{V}\int_0^\infty \mathrm{Tr}[\dot{\phi}_0(x)\dot{\phi}_0(y)\tilde{K}_v]v^{s-1}\mathrm{d}v \\ &+ \frac{1}{8}(n^2+4)\frac{s}{\Gamma(s)}\int_0^\infty \mathrm{Tr}[(\dot{\phi}_0)^2 K_u]u^{s-1}\mathrm{d}u \\ &- \frac{1}{8}(n^2+4)\frac{s}{\Gamma(s)}\frac{1}{V}\int_0^\infty \mathrm{Tr}[\dot{\phi}_0(x)\tilde{K}_u\dot{\phi}_0(y)]u^{s-1}\mathrm{d}u \\ &+ \frac{1}{8}(n^2+4)\frac{s}{\Gamma(s-1)}\int_0^\infty\int_0^\infty \mathrm{Tr}[\dot{\phi}_0\tilde{K}_u\dot{\phi}_0\tilde{K}_v](u+v)^{s-2}\mathrm{d}v\mathrm{d}v \end{aligned}$$

Now expanding the trace and combining terms, we get finally

$$\begin{aligned} T_1 &= \frac{1}{8}(n^2+4)s\int_M\int_M \dot{\phi}_0(x)\Psi_{s-1}(x,y)\dot{\phi}_0(y)dV_g(x)dV_g(y) \\ &- \frac{1}{4}(n^2+4)s\frac{1}{V}\int_M\int_M \dot{\phi}_0(x)\dot{\phi}_0(y)\zeta(s,x,y)dV_g(x)dV_g(y) \\ &+ \frac{1}{4}(n^2+4)s\zeta_n(s)\frac{1}{V}\int_M (\dot{\phi}_0(x))^2 dV_g(x) \end{aligned}$$

where $\Psi_s(x,y)$ is defined in (4.2).

The calculation for the terms T_2, T_3 and T_4 are the same as the one for T_1, except the following lemmas which are not difficult to prove. We use the first to reduce terms involving $\Delta_x\zeta(s+1,x,y)$ to those containing $\zeta(s,x,y)$ only and the second to deal with the second derivative in u

Lemma 4.3.

$$\int_M \int_M \dot{\phi}_0(x)(\Delta_g \dot{\phi}_0(y))\zeta(s+1,x,y)dV_g(x)dV_g(y)$$
$$= \int_M \int_M \dot{\phi}_0(x)\dot{\phi}_0(y)\zeta(s,x,y)dV_g(x)dV_g(y).$$

Lemma 4.4.

$$\begin{aligned} & \frac{s}{\Gamma(s)}\int_0^\infty \int_0^\infty \frac{\partial}{\partial u}\mathrm{Tr}[\dot{\phi}_0(x)\tilde{K}_u\dot{\phi}_0(y)\tilde{K}_v](u+v)^{s-1}\mathrm{d}u\mathrm{d}v \\ = & -s\int_M \int_M \dot{\phi}_0(x)\Psi_{s-1}(x\cdot y)\dot{\phi}_0(y)dV_g(x)dV_g(y) \\ + & \ s\frac{1}{V}\int_M \int_M \dot{\phi}_0(x)\dot{\phi}_0(y)\zeta(s,x,y)dV_g(x)dV_g(y) \\ - & \ s\int_M \dot{\phi}_0(x)\dot{\phi}_0(x)\zeta(s,x,x)dV_g(x). \end{aligned} \tag{4.16}$$

Applying the lemmata (4.2, 4.3) and (4.4) gives

$$\begin{aligned} \mathrm{var}_1(s) = & \ s\int_M \int_M \dot{\phi}_0(x)\Psi_{s-1}(x,y)\dot{\phi}_0(y)dV_g(x)dV_g(y) \\ - & \ (1-\frac{n}{2})s\int_M \int_M \dot{\phi}_0(x)\Psi_s(x,y)(\Delta_g\dot{\phi}_0(y))dV_g(x)dV_g(y) \\ + & \ \frac{1}{16}(n-2)^2 s\int_M \int_M (\Delta_g\dot{\phi}_0(x))\Psi_{s+1}(x,y)(\Delta_g\dot{\phi}_0(y))dV_g(x)dV_g(y) \\ - & \ \frac{1}{8}(n+2)^2 s\frac{1}{V}\int_M \int_M \dot{\phi}_0(x)\dot{\phi}_0(y)\zeta(s,x,y)dV_g(x)dV_g(y) \\ - & \ \frac{1}{8}(n-2)^2 s\zeta_n(s+1)\frac{1}{V}\int_M \dot{\phi}_0(x)(\Delta_g\dot{\phi}_0(x))dV_g(x) \\ + & \ 2s\zeta_n(s)\frac{1}{V}\int_M (\dot{\phi}_0(x))^2 dV_g(x). \end{aligned} \tag{4.17}$$

4.2. **Computation of** $\mathrm{var}_2(s)$**.** Next, we compute the term

$$\mathrm{var}_2(s) = -\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\Delta_\varepsilon^{(2)}(e^{-t\Delta_g}-P)\Big)t^s dt. \tag{4.18}$$

Substituting the expression for $\Delta_0^{(2)}$ given in Equation (3.6) into (4.18) gives

$$
\begin{aligned}
\mathrm{var}_2(s) &= \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\ddot{\phi}_0\Delta_g(e^{-t\Delta_g}-P)\Big)t^s dt \\
&- \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big((\dot{\phi}_0)^2\Delta_g(e^{-t\Delta_g}-P)\Big)t^s dt \\
&- (n-2)\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\dot{\phi}_0\langle\nabla_g\dot{\phi}_0,\nabla_g\cdot\rangle_g(e^{-t\Delta_g}-P)\Big)t^s dt \\
&- (1-\frac{n}{2})\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\Big(\langle\nabla_g\ddot{\phi}_0,\nabla_g\cdot\rangle_g(e^{-t\Delta_g}-P)\Big)t^s dt.
\end{aligned}
$$

Using the same argument as in the calculation of $\mathrm{var}_1(s)$ we get

$$
\begin{aligned}
\mathrm{var}_2(s) &= s\int_M \ddot{\phi}_0(x)\zeta(s,x,x)dV_g(x) \\
&- s\int_M (\dot{\phi}_0(x))^2\zeta(s,x,x)dV_g(x) \\
&- (n-2)s\int_M \dot{\phi}_0(x)\langle\nabla_g\dot{\phi}_0(x),\nabla_g\zeta(s+1,x,x)\rangle_g dV_g(x) \\
&- (1-\frac{n}{2})s\int_M \langle\nabla_g\ddot{\phi}_0,\nabla_g\zeta(s+1,x,x)\rangle_g dV_g(x).
\end{aligned}
$$

The homogeneity of M implies that $\zeta(s+1,x,x)$ is constant in x, so the third and fourth terms here vanish. Further, using the identity $\int_M \ddot{\phi}_0(x)dV_g = -\frac{n}{2}\int_M(\dot{\phi}_0(x))^2 dV_g$ (by observation (3.3)), we get that $\mathrm{var}_2(s)$ simplifies to

$$
\mathrm{var}_2(s) = -\frac{(n+2)s}{2V}\zeta_n(s)\int_M(\dot{\phi}_0(x))^2 dV_g(x). \tag{4.19}
$$

Combining (4.17) and (4.19), we get that the second order variation $\zeta_g^{(2)}(s)$ of the spectral zeta function $\zeta_{g_\varepsilon}(s)$ on M is given by (4.1) for $\Re(s)$ large enough. Thus, the proof of the Theorem (4.1) is complete □

5. **Conclusion**

We make a number of concluding remarks. Firstly, we observed that our second variation formula (4.1) reduces to a well-known result for the variation of the determinant of the Laplacian on S^3. That is,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big(\zeta_g^{(2)}(s)\big)\big|_{s=0} &= \frac{1}{16}\mathrm{Tr}\big[\big(\Delta_g\dot{\phi}_0\Delta_g^{-1}\big)^2\big] - \frac{\zeta_{S^3}(1)}{8V}\langle\Delta_g\dot{\phi}_0,\dot{\phi}_0\rangle_{L^2(S^3)} \\ &- \frac{5}{8V}\langle\dot{\phi}_0,\dot{\phi}_0\rangle_{L^2(S^3)}. \end{aligned} \tag{5.1}$$

This is exactly the result of Richardson [20].

Other special values of s can be computed using the formula. For example, with the aid of Mathematica, we computed

$$\zeta_g^{(2)}(\frac{10}{3}) = 0.0797 \text{ where we choose } \dot{\phi}_0(\theta) = \frac{2}{3}\cos(3\theta) \text{ on } S^3.$$

A numerical check using 500 eigenvalues of the Laplacian on S^n confirmed this number.

We hope that the result of this paper can be used for further numerical and analytical studies of the spectral zeta function and the Casimir energy.

Conflict of Interests

The authors declare that there is no conflict of interests.

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