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MULTI-COMMODITY CAPACITATED MULTI-FACILITY WEBER PROBLEM WITH PROBABILISTIC CUSTOMER LOCATIONS

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Abstract: This paper presents a rectilinear distance multi-commodity multi-facility Weber problem with capacity constraints and finds the location of facilities and assigns the amount of each commodity to each customer at minimum cost. Although this problem is NP-hard and has neither convex nor concave objective function, considering probabilistic customer points make the problem more difficult. An exact and an approximated expected distance function is applied in order to solve the problem. An alternate location–allocation heuristic method which divides the problem into multi-commodity transportation subproblem and pure allocation subproblem is implemented until no improvement is observed. A typical example is illustrated and results are then reported.

Keywords: Weber problem; probabilistic customer; heuristic.

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1. Introduction

Given the locations of J customers and their demands, the Multi-facility Weber Problem (MWP) is concerned with locating I uncapacitated facilities and allocating

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them to the given J customers in order to satisfy their demand at a minimum total cost. The problem known as uncapacitated multi-Weber problem can be formulated as:

$$\min \sum_{i=1}^{I} \sum_{j=1}^{J} w_j d(\mathbf{x}_i, \mathbf{a}_j) y_{ij}$$

s.t.

$$\sum_{i=1}^{I} y_{ij} = 1 \qquad j = 1, \dots, J$$

$$y_{ij} \in \{0,1\} \qquad i = 1, \dots, I \ , j = 1, \dots, J$$

$$x_i \in \mathbb{R}^2 \qquad i = 1, \dots, I$$

where the objective is to minimize the sum of weighted traveled distances $w_i d(x_i, a_j)$ between I new facilities $x_1, x_2, \dots, x_I \in \mathbb{R}^2$ and J existing customers $\mathcal{A} = \{a_1, a_2, \dots, a_J\} \subset \mathbb{R}^2$. The demand or importance of j-th customer is denoted by positive values $w_j \in \mathbb{R}^2$. The distance $d(\mathbf{x}_i, \mathbf{a}_j)$ is the rectilinear distance between an established facility x_i and customer a_j.

The MWP, firstly introduced by Cooper [1], indicates that the objective function is neither convex nor concave, which makes it difficult to solve exactly. However, it was shown to be NP-hard by Sherali et al. [2]. The MWP becomes the so-called (single-facility) Weber problem when I = 1; i.e., one single facility is to be located. Then, Cooper [3] solved the location-allocation problems with some heuristic methods. The MWP with capacity constraints gives rise to the Capacitated Multifacility Weber Problem (CMWP). The Euclidean distance CMWP is initially proposed by Cooper [4] and then solved by Selim [5] based on a biconvex cutting plane procedure. For the same problem, Sherali et al. [6] proposed branch-and-bound algorithms based on an allocation-space-partitioning procedure. The NP-hardness of CMWP was proved by Sherali and Nordai [7]. For the rectilinear distance CMWP, Sherali et al. [8] developed an algorithm based on the Reformulation-Linearization Technique (RLT) stated in [9] to find optimal solutions. Aras et al. [10] and [11] extended the discrete approximation strategy based on the Alternate Location-Allocation (ALA) heuristic of Cooper [3] and the p-median heuristic for the (uncapacitated) MWP by Hansen et al. [12]. Zainuddin and Salhi [13] and Luis et al. [14] introduced ALA-based heuristics for the CMWP. Brandeau and Chiu [15] considered the Cooper's ALA heuristic for creating a random sample for to statistically estimate confidence intervals for the optimal objective function value (OFV) of the MWP. Aky□uz et al. [16] focused on the statistically estimate confidence intervals for OFV of the Multi-commodity Capacitated Multi-facility Weber Problem (MCMWP) based on the works of [10] and [14]. Aky□uz et al. [17] developed three approximate solution methods for MCMWP based on the Lagrangean relaxation and discrete approximation methods.

In all of works mentioned above it is assumed that customers have predetermined locations. Whereas, in real world customers need to be assumed to have random locations. For example, police stations may be assumed to have random locations, because the crime cannot be known deterministically. Probabilistic WPs was initially introduced in [18]. Katz and Cooper [19] minimized the total expected cost of the Euclidian distance of customers which are distributed bivariate normally independently. Katz and Cooper [20] considered the same problem focusing on the bivariate exponential and bivariate symmetric exponential distributions for customer locations. Weslowsky [21] extended the same problem using the bivariate normal and bivariate uniform and bivariate symmetric exponential distribution functions for customer locations in the rectilinear space. Cooper [22] generated a mathematical programming model for the transportation-location problem and solved proposed an exact and a heuristic solution algorithm. Özkısacık et al. [23] proposed the probabilistic MWP (PMWP) and generated a heuristic solution method based on the principle of vector quantization. Durmaz et al. [24] proposed the probabilistic capacitated MWP (PCMWP) which customers have independent bivariate normal distributions coordinates and introduced three discrete approximation heuristics for the problem. Altınel et al. [25] considered the CMWP where customer locations follow the bivariate random distribution function for four distance functions, Euclidian, Squared Euclidian, rectilinear and weighted norm.

In this paper, Probabilistic Multi-commodity Capacitated Multi-facility Weber

Problem (PMCMWP) in the rectilinear space is proposed. Based on our knowledge, there is no work published in this field. Since computing the probabilistic distance is impossible, we applied the expected value of the distance function. Two procedures, exact and approximation, to calculate the probabilistic distance function and an alternative location allocation heuristic algorithm are presented in this paper.

The rest of this article is organized as follows. The probabilistic programming and expected distance computing are given in Section 2. An alternative location allocation heuristic algorithm is presented in Section 3. In section 4 a typical numerical example is introduced. A conclusion and future research schemes are presented in Section5.

2. The PMCMWP

2.1. The probabilistic programming

Let K be the number of commodities that each facility can distribute to the customers and respectively $\mathbf{a}_j = (a_{j1}, a_{j2})^T$ and $\mathbf{x}_i = (x_{i1}, x_{i2})^T$ represent the coordinates of customer *j* and facility *i*. It is supposed that the coordinates of the customer follow the bivariate normal distributions function and the coordinates of the facility is unknown. Then distance between customer *j* and facility *i* is denoted by $d(\mathbf{x}_i, \mathbf{a}_j)$ considered as rectilinear distance, namely:

$$d(\mathbf{x}_i, \mathbf{a}_j) = |x_{i1} - a_{j1}| + |x_{i2} - a_{j2}|.$$

The capacity of facility *i* and demand of customer *j* for commodity *k* are given by s_{ik} and q_{jk}, respectively. Note that in this problem the balance condition (i.e., the equality of total demand and total supply for each commodity) namely $\sum_{i=1}^{I} s_{ik} = \sum_{j=1}^{J} q_{jk}$ for $k = 1, \ldots, K$ is met. Additionally, a capacity limitation on every connection (i, j) is predetermined by u_{ij}. w_{ijk} is the unknown amount of commodity *k* transported from facility *i* to customer *j* with the unit transportation cost c_{ijk} per unit distance. The mathematical formulation of the PMCMWP can be stated as:

$$\min Z = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} w_{ijk}\, c_{ijk}\, d(\mathbf{x}_i, \mathbf{a}_j) \tag{1}$$

s.t. $$\sum_{j=1}^{J} w_{ijk} = s_{ik}, \qquad i = 1, \dots, I; k = 1, \dots, K \tag{2}$$

$$\sum_{i=1}^{I} w_{ijk} = q_{jk}, \qquad j = 1, \dots, J; k = 1, \dots, K \tag{3}$$

$$\sum_{k=1}^{K} w_{ijk} \le u_{ij}, \qquad i = 1, \dots, I; j = 1, \dots, J \tag{4}$$

$$w_{ijk} \ge 0 \qquad i = 1, \dots, I; j = 1, \dots, J; k = 1, \dots, K \tag{5}$$

Observe that when the allocation w_{ijk} is given, the MCMWP reduces to the pure location problem which can be separable to I WPs. On the other side, when the locations of the facilities are known, the MCMWP becomes the ordinary multi-commodity transportation problem. Note that when $K = 1$ we deal with a CMWP. Sherali and Nordai [7] proved that CMWP is NP-hard even if all customers are located on a straight line. Consequently, MCMWP should be expected to be difficult and PMCMWP even more difficult which resulted in the researchers' promotion for accurate and efficient heuristics. In this paper we would like to consider the independent random variables with known probability distributions. Then, the PMCMWP we wish to solve consists of finding the facility locations that minimize the expected cost.

$$E[Z] = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} w_{ijk}\, c_{ijk}\, E\big[d(\mathbf{x}_i, \mathbf{a}_j)\big] \tag{6}$$

subject to constraints (2) to (5).

2.2. Exact method for expected distance computing

Generally, the expected distance for given facility coordinate values

$$E\big[d(\mathbf{x}_i, \mathbf{a}_j)\big] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d(\mathbf{x}_i, \mathbf{a}_j) f_j(\mathbf{a}_j)\, da_{j1}\, da_{j2} \tag{7}$$

where $\mathbf{a}_j = (\mathrm{a}_{j1}, \mathrm{a}_{j2})$ is a random variable with a_{j1} and a_{j2} being independent normal random variables having means μ_{j1} and μ_{j2} and the standard deviations $\sigma_{j1} = \sigma_{j2} = \sigma_j$ with respect to $\mathbf{a}_j$ around $\boldsymbol{\mu} = (\mu_{j1}, \mu_{j2})$. So, expression (7) can be written equivalently as

$$E\left[d\left(\mathbf{x}_i,\mathbf{a}_j\right)\right]=\sum_{p=1}^{2}\int_{-\infty}^{\infty}\left|x_{ip}-a_{jp}\right|f_{jp}\left(a_{jp}\right)da_{jp}$$
$$=\sum_{p=1}^{2}\left(\int_{-\infty}^{x_{ip}}\left(x_{ip}-a_{jp}\right)f_{jp}\left(a_{jp}\right)da_{jp}+\int_{x_{ip}}^{\infty}\left(a_{jp}-x_{ip}\right)f_{jp}\left(a_{jp}\right)da_{jp}\right) \tag{8}$$

Applying integration by parts we have

$$E\left[d\left(\mathbf{x}_i,\mathbf{a}_j\right)\right]=\sum_{p=1}^{2}\left(2\int_{-\infty}^{x_{ip}}F_{jp}\left(a_{jp}\right)da_{jp}+\mu_{jp}-x_{ip}\right) \tag{9}$$

where $F_{jp}\left(a_{jp}\right)=\Pr\left(X_{jp}\le x_{jp}\right)$. Wesolowsky [21] considered rectilinear distance probabilistic Weber problem for bivariate normal, exponential and uniform location distributions. Altınel et al. [25] motivated the work of [21] and proved the convexity of the obtained expected distance. Suppose random variable a_{jp} is distributed uniformly within $\left(\alpha_{jp},\beta_{jp}\right)$ with distribution function

$$F_{jp}\left(a_{jp}\right)=\begin{cases}0 & \text{if } a_{jp}<\alpha_{jp}\\ \dfrac{a_{jp}-\alpha_{jp}}{\beta_{jp}-\alpha_{jp}} & \text{if } \alpha_{jp}\le a_{jp}\le\beta_{jp}\\ 1 & \text{if } a_{jp}>\beta_{jp}\end{cases} \tag{10}$$

Then

$$E\left[d\left(\mathbf{x}_i,\mathbf{a}_j\right)\right]=\sum_{p=1}^{2}\left(2\int_{-\infty}^{x_{ip}}F_{jp}\left(a_{jp}\right)da_{jp}+\mu_{jp}-x_{ip}\right)$$
$$=\begin{cases}\displaystyle\sum_{p=1}^{2}\left(\frac{\alpha_{jp}+\beta_{jp}}{2}-x_{ip}\right) & \text{if } x_{ip}<\alpha_{jp}\\ \displaystyle\sum_{p=1}^{2}\left(\frac{\left(x_{ip}-\alpha_{jp}\right)^2}{\beta_{jp}-\alpha_{jp}}+\frac{\alpha_{jp}+\beta_{jp}}{2}-x_{ip}\right) & \text{if } \alpha_{jp}\le x_{ip}\le\beta_{jp}\\ \displaystyle\sum_{p=1}^{2}\left(x_{ip}-\frac{\alpha_{jp}+\beta_{jp}}{2}\right) & \text{if } x_{ip}>\beta_{jp}\end{cases} \tag{11}$$

for $i=1,\ldots,I;j=1,\ldots,J;p=1,2$. Whereas the expected distance in each region is convex individually, the whole expected distance function in nonconvex.

2.3. Approximation method for expected distance computing

Since customers are distributed uniformly in the specific ranges, we generate a large enough random numbers for customer j by $\tilde{\mathbf{a}}_j^r = \left(\tilde{a}_{j1}^r, \tilde{a}_{j2}^r\right)^T, r = 1, \dots, R$ where R is the number of produced random numbers. Note that $\tilde{a}_{jp}^r \in \left[\alpha_{jp}, \beta_{jp}\right], r = 1, \dots, R$ and $p = 1{,}2$. Then approximated customer points can be calculated by the average value of coordinates of the generated random points $\bar{\mathbf{a}}_j = \left(\bar{a}_{j1}, \bar{a}_{j2}\right)^T$ where $\bar{a}_{jp} = \frac{1}{R}\sum_{r=1}^{R} \tilde{a}_{jp}^r$ for $p = 1{,}2$. Next, the rectilinear distance between point $\mathbf{x}_i$ and $\bar{\mathbf{a}}_j$ is computed by

$$E\left[d\left(\mathbf{x}_i, \mathbf{a}_j\right)\right] \cong \left|x_{i1} - \bar{a}_{j1}\right| + \left|x_{i2} - \bar{a}_{j2}\right| \tag{12}$$

Generally the PMCMWP is formulated by

$$\min E[Z] = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} w_{ijk}\, c_{ijk}\, E\left[d\left(\mathbf{x}_i, \mathbf{a}_j\right)\right]$$

subject to constraints (2) – (5).

3. Alternate location–allocation (ALA) heuristic

Location-allocation problems (LAPs) optimally site a set of facilities and allocate the customer demands to the facilities in order to meet the capacity and demand restrictions at minimum total cost. Therefore, any LAP becomes a pure multi-facility location problem when an allocation scheme is predetermined. Conversely, when facility locations are given, it becomes a pure allocation problem. Starting at an initial set of facility locations, Cooper [3] applied firstly ALA for the MWP, which simply consists of the solution of the location and allocation problems alternately, until no further improvement is possible. ALA terminates with a local optimum solution which is no better locations can be found concerning the current allocations, and no better allocations can be found given the current locations. The capacitated version (CALA) of CMWP is then presented in [4]. However, in the allocation phase each customer is assigned to the nearest facility in case of the MWP, which becomes the solution of an ordinary transportation problem for the CMWP and the solution of the expected distance multi-commodity transportation problem (EMTP):

$$\min Z = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\bar{c}_{ijk}w_{ijk} \tag{13}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} w_{ijk} = s_{ik}, \qquad i = 1,\dots,I; k = 1,\dots,K \tag{14}$$

$$\sum_{i=1}^{I} w_{ijk} = q_{jk}, \qquad j = 1,\dots,J; k = 1,\dots,K \tag{15}$$

$$\sum_{k=1}^{K} w_{ijk} \leq u_{ij}, \qquad i = 1,\dots,I; j = 1,\dots,J \tag{16}$$

$$w_{ijk} \geq 0 \qquad i = 1,\dots,I; j = 1,\dots,J; k = 1,\dots,K \tag{17}$$

The unit expected transportation cost is specified as $\bar{c}_{ijk} = c_{ijk}\, E\left[d\left(\mathbf{x}_i, \mathbf{a}_j\right)\right]$ where $\mathbf{x}_i$ was the generated randomly initial facility locations. Here the allocation values for the generated facility locations are enhanced. As stated above, when the allocation w_{ijk} is available, the MCMWP reduces to the pure location problem which can be separable to *I* WPs. So, we deal with *I* probabilistic Weber problems (PWP) by

$$\min Z_i = \min \sum_{j=1}^{J} \bar{\bar{c}}_{ij}\, E\left[d\left(\mathbf{x}_i, \mathbf{a}_j\right)\right], \quad i = 1,\dots,I \tag{18}$$

where $\bar{\bar{c}}_{ij} = \sum_{k=1}^{K} w_{ijk}c_{ijk}$. Aky□uz et al. [16] and [17] extended Cooper's ALA heuristic [3] for the multi-commodity version (MCALA) heuristic. Here, we propose a probabilistic version of MCALA (PMCALA) sequentially which consists of the alternate solutions of the *I* PWPs (18) and EMTP (13)–(17). At the first, facility locations are generated randomly. Then, in each iteration the allocation clusters for each generated facility are determined form EMTP (13)–(17). Next, the facility locations are obtained from *I* probabilistic Weber problems with expected distance function. The steps of this algorithm are specified in Algorithm 1. The algorithm stops when in the allocation step, the assignment clusters set in two successive iterations remains to be the same or in the location step, the facility points remain unchanged. Actually with these conditions, a local minimum is achieved. Since the algorithm starts with the initial solutions of facility locations, we select the mean coordinates of *I* customers arbitrarily among *J* existing demand points as the initial facility locations (i.e. $\mathbf{x}_i = (\mu_{i1}, \mu_{i2}),\ i \in \{1, \dots, J\}$). In fact there are $\binom{J}{I}$ number of possibilities for initial solution sets.

Algorithm 1

Step 1: ***Initialization***

Select randomly I points from the J demand points as the initial facility locations:

$\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_I$.

Step 2: ***Computing expected distance***

Compute $E\left[d\left(\mathbf{x}_i, \mathbf{a}_j\right)\right]$

REPEAT

Step 3: ***Allocation***

Compute $\bar{c}_{ijk}$

Solve the EMTP, (13)–(17)

Determine w_{ijk}

Step 4: ***Location***

For $i = 1, \dots, I$ *do*

Compute $\bar{\bar{c}}_{ij}$

Solve the Weber location problem i, (18) *using determined* w_{ijk}

Determine $\mathbf{x}_i = (x_{i1}, x_{i2})^T$

End for

UNTIL *a stopping condition is satisfied.*

4. Numerical example

For a better understanding of the proposed model and showing the effectiveness of the proposed solution approaches, we explain an example consists of locating 3 new facilities to assign 2 commodities among 8 customers whom their location follows the independent bivariate normal distribution function in the rectilinear space. We applied the LINGO optimization software to solve subproblems, i.e., (13)-(17) and (18), and implemented the heuristic algorithm in Visual basic 6.0 to find the results of the proposed MCALA algorithm. The programs were executed on a desktop computer equipped with a 2.20 GH Intel Pentium Dual processor and 2 GB RAM.

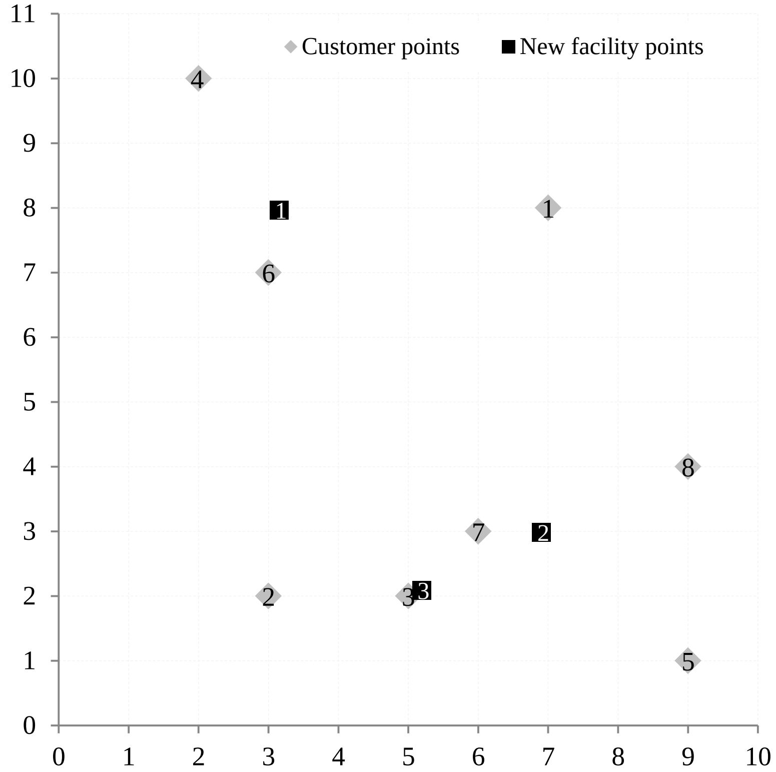
Table 1. Data for the sample problem.

				q_{jk}		u_{ij}			s_{ik}		
	$[(\mu_{j1},\mu_{j2});\sigma_j]$	$(\alpha_{j1},\alpha_{j2})$	(β_{j1},β_{j2})	k=1	k=2	i=1	i=2	i=3		k=1	k=2
j=1	[(7,8);0.3]	(6,8)	(7,9)	10	18	25	20	25	i=1	35	41
j=2	[(3,2);0.25]	(2,4)	(1,3)	12	17	25	25	30	i=2	35	45
j=3	[(5,2);0.28]	(4,6)	(1,3)	15	18	20	25	30	i=3	32	50
j=4	[(2,10);0.4]	(1,3)	(9,11)	8	21	10	30	10			
j=5	[(9,1);0.6]	(8,10)	(0,2)	13	18	10	25	15			
j=6	[(3,7);0.6]	(2,4)	(6,8)	15	12	25	25	30			
j=7	[(6,3);0.55]	(5,7)	(2,4)	19	19	20	20	15			
j=8	[(9,4);0.5]	(8,10)	(3,5)	10	13	15	20	20			

Table 2. Unit shipment cost for the sample problem.

		j=1	j=2	j=3	j=4	j=5	j=6	j=7	j=8
i=1	k=1	57	41	48	58	41	52	46	48
	k=2	45	42	58	57	53	49	56	49
i=2	k=1	53	43	54	47	44	45	51	45
	k=2	56	44	57	43	48	46	55	40
i=3	k=1	43	42	43	55	55	43	56	40
	k=2	55	45	43	51	46	55	58	58

The mean coordinates μ_{jp} and standard deviations σ_j of customer locations, uniform distribution parameters for each coordinate of customers α_{jp} and $\beta_{j\mathrm{p}}$, capacity of facilities q_{jk}, demand of customers u_{ij} and facility-customer connection restrictions s_{ik} are given in Table 1. The unit shipment cost of commodity k from facility i to customer j is presented in Table 2. These data are randomly generated in the predetermined ranges. Solving the sample problem via presented procedure, the location of the new facilities as well as the amount of different commodities should be shipped to the customers is enhanced. The results are presented in Figure 1 and are reported in Table 3. The obtained objective function value is 34879.



i	x_i	y_i
1	3.153153	7.966601
2	6.90401	2.989082
3	5.189666	2.087433

Coordinates of new facilities

Figure 1. Customer and new facility points

Table 3. Amount of commodities shipped to customers

	i=1		*i=2*		*i=3*	
	k=1	*k=2*	*k=1*	*k=2*	*k=1*	*k=2*
j=1	3	12	7	6	0	0
j=2	0	0	0.115611	1.884389	11.88439	15.11561
j=3	0	0	10.74001	2.259989	4.259989	15.74001
j=4	8	17	0	4	0	0
j=5	6	0	7	3	0	15
j=6	15	12	0	0	0	0
j=7	3	0	0.144379	14.85562	15.85562	4.144379
j=8	0	0	10	13	0	0

5. Conclusion

In this paper a mathematical programming model for the probabilistic Weber location problem have been considered. Two expected distance function and a heuristic solution method which divides the problem into two subproblems have been presented. Results showed that using the approximation expected distance the given algorithm leads us to a local optimum solution. Applying a procedure to start the LAL algorithm efficiently in order to improve the solution quality can be an extension for this work. Other heuristic methods can also be examined for this problem. Different distance functions and different distribution functions can lead the problem to the real world problems.

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