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CORDIALITY IN THE CONTEXT OF DUPLICATION IN CROWN RELATED GRAPHS

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Abstract. Let $G = (V(G), E(G))$ be a graph and let $f : V(G) \to \{0, 1\}$ be a mapping from the set of vertices to {0,1} and for each edge $uv \in E$ assign the label $|f(u) - f(v)|$. If the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labled with 0 and the number of edges labeled with 1 differ by at most 1, then f is called a cordial labeling. We discuss cordial labeling of graphs obtained from duplication of certain graph elements in crown related graphs.

Keywords: graph labeling; cordial labeling; cordial graph.

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1. Introduction

We begin with simple, finite, undirected graph $G = (V(G), E(G))$ where $V(G)$ and $E(G)$ denotes the vertex set and the edge set respectively. For all other terminology we follow Gross [3]. We will give the brief summary of definitions which are useful for the present work.

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Definition 1.1: The *graph labeling* is an assignment of numbers to the vertices or edges or both subject to certain condition(s).

A detailed survey of various graph labeling is explained in Gallian [2].

Definition 1.2: For a graph $G = (V(G), E(G))$, a mapping $f : V(G) \to \{0,1\}$ is called a *binary vertex labeling* of G and $f(v)$ is called the *label* of the vertex v of G under f. For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0,1\}$ defined as $f^*(uv) = |f(u) - f(v)|$.

Let $v_f(0), v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0), e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.3: [1] *Duplication of a vertex* v of a graph G produces a new graph G' by adding a new vertex v' such that $N(v') = N(v)$. In other words a vertex v' is said to be duplication of v if all the vertices which are adjacent to v in G are also adjacent to v' in G'.

Definition 1.4: [6] The *crown graph* C_n^*, $n \geq 3$ is obtained from a cycle C_n by attaching a pendant edge at each vertex of the n-cycle.

Definition 1.5: [5] The *armed crown* AC_n, is a graph in which path P_2 is attached at each vertex of cycle C_n by an edge.

We define two new graph families as follows:

Definition 1.6: The *closed crown* CC_n^*, is the graph obtained from a crown by joining consecutive pendent vertices to form a cycle. The cycle is said to be the outer cycle of CC_n^*.

Definition 1.7: The *web crown* WbC_n^*, is obtained by adding a single pendent edge to each vertex of the outer cycle of CC_n^*.

Definition 1.8: A binary vertex labeling f of a graph G is called a *cordial labeling* if $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. A graph G is said to be cordial if it admits cordial labeling.

Vaidya and Dani [4] proved that the graphs obtained by duplication of an arbitrary edge of a cycle and a wheel admit a cordial labeling. Prajapati and Gajjar [7] proved that complement of wheel graph and complement of cycle graph are cordial if $n \not\equiv 4 \pmod 8$ or $n \not\equiv 7 \pmod 8$. Prajapati and Gajjar [8] proved that cordial labeling in the context of duplication of cycle graph and path graph.

2. Main Results

Theorem 2.1: The graph obtained by duplicating all the vertices of the crown C_n^* is cordial.

Proof: Let $V(C_n^*) = \{u_i, v_i/1 \leq i \leq n\}$ and $E(C_n^*) = \{u_iv_i/1 \leq i \leq n\} \cup \{u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{u_nu_1\}$. Let G be the graph obtained by duplicating all the vertices in C_n^*. Let $u_1', u_2', ..., u_n', v_1', v_2', ..., v_n'$ be the new vertices of G by duplicating $u_1, u_2, ..., u_n, v_1, v_2, ..., v_n$ respectively.

Then $V(G) = \{u_i, v_i, u_i', v_i'/1 \leq i \leq n\}$ and $E(G) = \{u_iv_i, v_iu_i', u_iv_i'/1 \leq i \leq n\} \cup \{u_nu_1, u_n'u_1, u_nu_1'\} \cup \{u_iu_{i+1}, u_i'u_{i+1}, u_iu_{i+1}'/1 \leq i \leq n-1\}$. Therefore $|V(G)| = 4n$ and $|E(G)| = 6n$. Define a vertex labeling $f : V(G) \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \{u_i, u'_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } x \in \{v_i, v'_i\},\ i \in \{1, 2, \ldots, n-1, n\}. \end{cases}$$

Thus $v_f(1) = 2n$ and $v_f(0) = 2n$. The induced edge labeling $f^* : E(G) \to \{0, 1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } \ e \in \{u_iv_i, v_iu'_i, u_iv'_i\},\ i \in \{1, 2, \ldots, n-1, n\}; \\ 0 & \text{if } \ e \in \{u_iu_{i+1}, u'_iu_{i+1}, u_iu'_{i+1}\},\ i \in \{1, 2, \ldots, n-2, n-1\}; \\ 0 & \text{if } \ e \in \{u_nu_1,\ u'_nu_1,\ u_nu'_1\}. \end{cases}$$

Thus $e_f(1) = 3n$ and $e_f(0) = 3n$. Therefore f satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial.

Theorem 2.2: The graph obtained by duplicating all the rim vertices of the crown C_n^* is cordial.

Proof: Let $V(C_n^*) = \{u_i, v_i/1 \leq i \leq n\}$ and $E(C_n^*) = \{u_iv_i/1 \leq i \leq n\} \cup \{u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{u_nu_1\}$. Let G be the graph obtained by duplicating all the rim vertices in C_n^*. Let $u_1', u_2', ..., u_n'$ be the new vertices of G by duplicating $u_1, u_2, ..., u_n$ respectively. Then $V(G) = \{u_i, v_i, u_i'/1 \leq i \leq n\}$ and $E(G) = \{u_iv_i, v_iu_i'/1 \leq i \leq n\} \cup \{u_iu_{i+1}, u_i'u_{i+1}, u_iu_{i+1}'/1 \leq i \leq n-1\} \cup \{u_nu_1, u_n'u_1, u_nu_1'\}$. Therefore $|V(G)| = 3n$ and $|E(G)| = 5n$. Using parity of n, we have the following cases:

Case 1: Let n be even. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = u_i,\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } x = v_i,\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = u'_i,\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 1 & \text{if } x = u'_i,\ i \in \{2,4,\ldots,n-2,n\}. \end{cases}$$

Thus $v_f(1) = \frac{3n}{2}$ and $v_f(0) = \frac{3n}{2}$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e = u_iv_i,\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } e \in \{v_iu'_i, u_iu'_{i+1}\},\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 1 & \text{if } e = u'_iu_{i+1},\ i \in \{2,4,\ldots,n-4,n-2\}; \\ 0 & \text{if } e = v_iu'_i,\ i \in \{2,4,\ldots,n-2,n\}; \\ 0 & \text{if } e = u_iu_{i+1},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } e = u'_iu_{i+1},\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 0 & \text{if } e = u_iu'_{i+1},\ i \in \{2,4,\ldots,n-4,n-2\}; \\ 0 & \text{if } e \in \{u_nu_1, u_nu'_1\} \\ 1 & \text{if } e = u'_nu_1. \end{cases}$$

Thus $e_f(1) = \frac{5n}{2}$ and $e_f(0) = \frac{5n}{2}$.

Case 2: Let n be odd. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = u_i,\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } x = v_i,\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = u'_i,\ i \in \{1,3,\ldots,n-2,n\}; \\ 1 & \text{if } x = u'_i,\ i \in \{2,4,\ldots,n-3,n-1\}. \end{cases}$$

Thus $v_f(1) = \frac{3n-1}{2}$ and $v_f(0) = \frac{3n+1}{2}$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e = u_iv_i,\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } e = v_iu'_i,\ i \in \{1,3,\ldots,n-2,n\}; \\ 1 & \text{if } e = u'_iu_{i+1},\ i \in \{2,4,\ldots,n-3,n-1\}; \\ 1 & \text{if } e = u_iu'_{i+1},\ i \in \{1,3,\ldots,n-4,n-2\}; \\ 0 & \text{if } e \in \{v_iu'_i, u_iu'_{i+1}\},\ i \in \{2,4,\ldots,n-3,n-1\}; \\ 0 & \text{if } e = u_iu_{i+1},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } e = u'_iu_{i+1},\ i \in \{1,3,\ldots,n-4,n-2\}; \\ 0 & \text{if } e \in \{u_nu_1, u'_nu_1, u_nu'_1\}. \end{cases}$$

Thus $e_f(1) = \frac{5n-1}{2}$ and $e_f(0) = \frac{5n+1}{2}$.

From both the cases we can conclude $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial.

Theorem 2.3: The graph obtained by duplicating all the outer rim vertices of the closed crown CC_n^* is cordial.

Proof: Let $V(CC_n^*) = \{u_i, v_i / 1 \leq i \leq n\}$ and $E(CC_n^*) = \{u_iv_i / 1 \leq i \leq n\} \cup \{u_nu_1, v_nv_1\} \cup \{u_iu_{i+1}, v_iv_{i+1} / 1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the outer rim vertices in CC_n^*. Let $v'_1, v'_2, ..., v'_n$ be the new vertices of G by duplicating $v_1, v_2, ..., v_n$ respectively. Then $V(G) = \{u_i, v_i, v'_i / 1 \leq i \leq n\}$ and $E(G) = \{u_iv_i, u_iv'_i / 1 \leq i \leq n\} \cup \{u_nu_1, v_nv_1, v'_nv_1, v_nv'_1\} \cup \{u_iu_{i+1}, v_iv_{i+1}, v'_iv_{i+1}, v_iv'_{i+1} / 1 \leq i \leq n-1\}$. Therefore $|V(G)| = 3n$ and $|E(G)| = 6n$. Using parity n, we have the following cases:

Case 1: Let n be even. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \{u_i, v_i\},\ i \in \{1,3,\dots,n-3,n-1\}; \\ 0 & \text{if } x \in \{u_i, v_i\},\ i \in \{2,4,\dots,n-2,n\}; \\ 0 & \text{if } x = v'_i,\ i \in \{1,3,\dots,n-3,n-1\}; \\ 1 & \text{if } x = v'_i,\ i \in \{2,4,\dots,n-2,n\}. \end{cases}$$

Thus $v_f(1) = \dfrac{3n}{2}$ and $v_f(0) = \dfrac{3n}{2}$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e \in \{u_iu_{i+1}, v_iv_{i+1}\},\ i \in \{1,2,\dots,n-2,n-1\}; \\ 1 & \text{if } e = u_iv'_i,\ i \in \{1,2,\dots,n-1,n\}; \\ 0 & \text{if } e \in \{v'_iv_{i+1}, v_iv'_{i+1}\},\ i \in \{1,2,\dots,n-2,n-1\}; \\ 0 & \text{if } e = u_iv_i,\ i \in \{1,2,\dots,n-1,n\}; \\ 1 & \text{if } e \in \{u_nu_1, v_nv_1\}; \\ 0 & \text{if } e \in \{v'_nv_1, v_nv'_1\}. \end{cases}$$

Thus $e_f(1) = 3n$ and $e_f(0) = 3n$.

Case 2: Let n be odd. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \{u_i, v_i\},\ i \in \{1,3,\dots,n-2,n\}; \\ 0 & \text{if } x \in \{u_i, v_i\},\ i \in \{2,4,\dots,n-3,n-1\}; \\ 0 & \text{if } x = v'_i,\ i \in \{1,3,\dots,n-2,n\}; \\ 1 & \text{if } x = v'_i,\ i \in \{2,4,\dots,n-3,n-1\}. \end{cases}$$

Thus $v_f(1) = \frac{3n+1}{2}$ and $v_f(0) = \frac{3n-1}{2}$. The induced edge labeling $f^*: E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e \in \{u_iu_{i+1}, v_iv_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 1 & \text{if } e = u_i{v'}_i,\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } e \in \{{v'}_iv_{i+1}, v_i{v'}_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } e = u_iv_i,\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } e \in \{u_nu_1, v_nv_1\}; \\ 1 & \text{if } e \in \{v'_nv_1, v_nv'_1\}. \end{cases}$$

Thus $e_f(1) = 3n$ and $e_f(0) = 3n$.

From both the cases we can conclude $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial.

Theorem 2.4: The graph obtained by duplicating all the vertices of the web crown WbC_n^* is cordial.

Proof: Let $V(WbC_n^*) = \{u_i, v_i, w_i / 1 \leq i \leq n\}$ and $E(WbC_n^*) = \{u_iv_i, v_iw_i / 1 \leq i \leq n\} \cup \{u_nu_1, v_nv_1\} \cup$

$\{u_iu_{i+1}, v_iv_{i+1} / 1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the vertices in WbC_n^*. Let $u'_1, u'_2, \ldots, u'_n, v'_1, v'_2, \ldots, v'_n, w'_1, w'_2, \ldots, w'_n$ be the new vertices of G by duplicating $u_1, u_2, \ldots, u_n, v_1, v_2,$
$\ldots, v_n, w_1, w_2, \ldots, w_n$ respectively. Then $V(G) = \{u_i, v_i, w_i, u'_i, v'_i, w'_i / 1 \leq i \leq n\}$ and
$E(G) = \{u_iv_i, u_iv'_i, v_iw_i, w_iv'_i, v_iw'_i, u'_iv_i / 1 \leq i \leq n\} \cup \{u_nu_1, v_nv_1, v'_nv_1, v_nv'_1, u'_nu_1, u_nu'_1\} \cup$
$\{u_iu_{i+1}, v_iv_{i+1}, v'_iv_{i+1}, v_iv'_{i+1}, u'_iu_{i+1}, u_iu'_{i+1} / 1 \leq i \leq n-1\}$. Therefore $|V(G)| = 6n$ and $|E(G)| = 12n$. Define a vertex labeling $f: V(G) \to \{0,1\}$ as follows:

$$f(x)=\begin{cases}1 & \text{if } x\in\{u_i,v_i,v'_i\},\ i\in\{1,2,\dots,n-1,n\};\\ 0 & \text{if } x\in\{w_i,u'_i,w'_i\},\ i\in\{1,2,\dots,n-1,n\}.\end{cases}$$

Thus $v_f(1)=3n$ and $v_f(0)=3n$. The induced edge labeling $f^*:E(G)\to\{0,1\}$ is $f^*(uv)=|f(u)-f(v)|$, for every edge $e=uv\in E$. Therefore

$$f^*(e)=\begin{cases}1 & \text{if } e\in\{v_iw_i,v_iw'_i,w_iv'_i,u'_iv_i\},\ i\in\{1,2,\dots,n-1,n\};\\ 1 & \text{if } e\in\{u'_iu_{i+1},u_iu'_{i+1}\},\ i\in\{1,2,\dots,n-2,n-1\};\\ 0 & \text{if } e\in\{u_iv_i,u_iv'_i\},\ i\in\{1,2,\dots,n-1,n\};\\ 0 & \text{if } e\in\{u_iu_{i+1},v_iv_{i+1},v'_iv_{i+1},v_iv'_{i+1}\},\ i\in\{1,2,\dots,n-2,n-1\};\\ 1 & \text{if } e\in\{u'_nu_1,u_nu'_1\};\\ 0 & \text{if } e\in\{u_nu_1,v_nv_1,v'_nv_1,v_nv'_1\}.\end{cases}$$

Thus $e_f(1)=6n$ and $e_f(0)=6n$. Therefore f satisfies the conditions $|v_f(1)-v_f(0)|\le 1$ and $|e_f(1)-e_f(0)|\le 1$. So, f admits cordial labeling on G. Hence G is cordial.

Theorem 2.5: The graph obtained by duplicating all the outer rim vertices of the web crown WbC_n^* is cordial.

Proof: Let $V(WbC_n^*)=\{u_i,v_i,w_i/1\le i\le n\}$ and $E(WbC_n^*)=\{u_iv_i,v_iw_i/1\le i\le n\}\cup\{u_nu_1,v_nv_1\}\cup\{u_iu_{i+1},v_iv_{i+1}/1\le i\le n-1\}$. Let G be the graph obtained by duplicating all the outer rim vertices in WbC_n^*. Let $v'_1,v'_2,...,v'_n$ be the new vertices of G by duplicating $v_1,v_2,...,v_n$ respectively. Then $V(G)=\{u_i,v_i,w_i,v'_i/1\le i\le n\}$ and $E(G)=\{u_iv_i,u_iv'_i,v_iw_i,w_iv'_i/1\le i\le n\}\cup\{u_nu_1,v_nv_1,v'_nv_1,v_nv'_1\}\cup\{u_iu_{i+1},v_iv_{i+1},v'_iv_{i+1},v_iv'_{i+1}/1\le i\le n-1\}$. Therefore $|V(G)|=4n$ and $|E(G)|=8n$. Define a vertex labeling $f:V(G)\to\{0,1\}$ as follows:

$$f(x)=\begin{cases}1 & \text{if } x\in\{u_i,w_i\},\ i\in\{1,2,\dots,n-1,n\};\\ 0 & \text{if } x\in\{v_i,v'_i\},\ i\in\{1,2,\dots,n-1,n\}.\end{cases}$$

Thus $v_f(1) = 2n$ and $v_f(0) = 2n$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e \in \{u_iv_i, u_iv'_i, v_iw_i, w_iv'_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } e \in \{u_iu_{i+1}, v_iv_{i+1}, v'_iv_{i+1}, v_iv'_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } e \in \{u_nu_1, v_nv_1, v'_nv_1, v_nv'_1\}. \end{cases}$$

Thus $e_f(1) = 4n$ and $e_f(0) = 4n$. Therefore f satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial.

Theorem 2.6: The graph obtained by duplicating all the vertices other than the pendent vertices of the armed crown AC_n is cordial.

Proof: Let $V(AC_n) = \{u_i, v_i, w_i / 1 \leq i \leq n\}$ and $E(AC_n) = \{u_iv_i, v_iw_i / 1 \leq i \leq n\} \cup \{u_nu_1\} \cup \{u_iu_{i+1} / 1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the vertices except the pendent vertices in AC_n. Let $u'_1, u'_2, ..., u'_n, v'_1, v'_2, ..., v'_n$ be the new vertices of G by duplicating $u_1, u_2, ..., u_n, v_1, v_2, ..., v_n$ respectively. Then $V(G) = \{u_i, v_i, w_i, u'_i, v'_i / 1 \leq i \leq n\}$ and $E(G) = \{u_nu_1, u'_nu_1, u_nu'_1\} \cup \{u_iv_i, v_iw_i, w_iv'_i, u_iv'_i, u'_iv_i / 1 \leq i \leq n\} \cup \{u_iu_{i+1}, u'_iu_{i+1}, u_iu'_{i+1} / 1 \leq i \leq n-1\}$. Therefore $|V(G)| = 5n$ and $|E(G)| = 8n$. Using parity of n, we have the following cases:

Case 1: Let n be even. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \{u_i, v'_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } x = w_i,\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 0 & \text{if } x \in \{v_i, u'_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = w_i,\ i \in \{2,4,\ldots,n-2,n\}. \end{cases}$$

Thus $v_f(1) = \frac{5n}{2}$ and $v_f(0) = \frac{5n}{2}$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e)=\begin{cases}1 & \text{if } e=v_iw_i,\ i\in\{1,3,\dots,n-3,n-1\};\\ 1 & \text{if } e={v'}_iw_i,\ i\in\{2,4,\dots,n-2,n\};\\ 1 & \text{if } e=u_iv_i,\ i\in\{1,2,\dots,n-1,n\};\\ 1 & \text{if } e\in\{{u'}_iu_{i+1},u_i{u'}_{i+1}\},\ i\in\{1,2,\dots,n-2,n-1\};\\ 0 & \text{if } e=v_iw_i,\ i\in\{2,4,\dots,n-2,n\};\\ 0 & \text{if } e={v'}_iw_i,\ i\in\{1,3,\dots,n-3,n-1\};\\ 0 & \text{if } e\in\{u_i{v'}_i,{u'}_iv_i\},\ i\in\{1,2,\dots,n-1,n\};\\ 0 & \text{if } e=u_iu_{i+1},\ i\in\{1,2,\dots,n-2,n-1\};\\ 0 & \text{if } e=u_nu_1;\\ 1 & \text{if } e\in\{u'_nu_1,u_nu'_1\}.\end{cases}$$

Thus $e_f(1)=4n$ and $e_f(0)=4n$.

Case 2: Let n be odd. Define a vertex labeling $f:V(G)\rightarrow\{0,1\}$ as follows:

$$f(x)=\begin{cases}1 & \text{if } x\in\{u_i,{v'}_i\},\ i\in\{1,2,\dots,n-1,n\};\\ 1 & \text{if } x=w_i,\ i\in\{1,3,\dots,n-2,n\};\\ 0 & \text{if } x\in\{v_i,{u'}_i\},\ i\in\{1,2,\dots,n-1,n\};\\ 0 & \text{if } x=w_i,\ i\in\{2,4,\dots,n-3,n-1\}.\end{cases}$$

Thus $v_f(1) = \dfrac{5n+1}{2}$ and $v_f(0) = \dfrac{5n-1}{2}$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e = v_iw_i,\ i \in \{1,3,\dots,n-2,n\}; \\ 1 & \text{if } e = v'_iw_i,\ i \in \{2,4,\dots,n-3,n-1\}; \\ 1 & \text{if } e = u_iv_i,\ i \in \{1,2,\dots,n-1,n\}; \\ 1 & \text{if } e \in \{u'_iu_{i+1}, u_iu'_{i+1}\},\ i \in \{1,2,\dots,n-2,n-1\}; \\ 0 & \text{if } e = v_iw_i,\ i \in \{2,4,\dots,n-3,n-1\}; \\ 0 & \text{if } e = v'_iw_i,\ i \in \{1,3,\dots,n-2,n\}; \\ 0 & \text{if } e \in \{u_iv'_i, u'_iv_i\},\ i \in \{1,2,\dots,n-1,n\}; \\ 0 & \text{if } e = u_iu_{i+1},\ i \in \{1,2,\dots,n-2,n-1\}; \\ 0 & \text{if } e = u_nu_1; \\ 1 & \text{if } e \in \{u'_nu_1, u_nu'_1\}. \end{cases}$$

Thus $e_f(1) = 4n$ and $e_f(0) = 4n$.

From both the cases we can conclude $|v_f(1) - v_f(0)| \le 1$ and $|e_f(1) - e_f(0)| \le 1$. So, f admits cordial labeling on G. Hence G is cordial.

Theorem 2.7: The graph obtained by duplicating all the vertices of the armed crown AC_n is cordial.

Proof: Let $V(AC_n) = \{u_i, v_i, w_i / 1 \le i \le n\}$ and $E(AC_n) = \{u_iv_i, v_iw_i / 1 \le i \le n\} \cup \{u_nu_1\} \cup \{u_iu_{i+1} / 1 \le i \le n-1\}$. Let G be the graph obtained by duplicating all the vertices in AC_n. Let $u'_1, u'_2, \dots, u'_n, v'_1, v'_2, \dots, v'_n, w'_1, w'_2, \dots, w'_n$ be the new vertices of G by duplicating $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n$ respectively. Then $V(G) = \{u_i, v_i, w_i, u'_i, v'_i, w'_i / 1 \le i \le n\}$ and $E(G) = \{u_iv_i, v_iw_i, w'_iv_i, w_iv'_i, u_iv'_i, u'_iv_i / 1 \le i \le n\} \cup \{u_iu_{i+1}, u'_iu_{i+1}, u_iu'_{i+1} / 1 \le i \le n-1\} \cup \{u_nu_1, u'_nu_1, u_nu'_1\}$. Therefore $|V(G)| = 6n$ and $|E(G)| = 9n$. Using parity of n, we have the following cases:

Case 1: Let n be even. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \{u_i, v'_i\},\ i \in \{1,2,\dots,n-1,n\}; \\ 1 & \text{if } x = w'_i,\ i \in \{2,4,\dots,n-2,n\}; \\ 1 & \text{if } x = w_i,\ i \in \{1,3,\dots,n-3,n-1\}; \\ 0 & \text{if } x \in \{v_i, u'_i\},\ i \in \{1,2,\dots,n-1,n\}; \\ 0 & \text{if } x = w_i,\ i \in \{2,4,\dots,n-2,n\}; \\ 0 & \text{if } x = w'_i,\ i \in \{1,3,\dots,n-3,n-1\}. \end{cases}$$

Thus $v_f(1) = 3n$ and $v_f(0) = 3n$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e \in \{v_i w'_i, v'_i w_i\},\ i \in \{2,4,\dots,n-2,n\}; \\ 1 & \text{if } e = v_i w_i,\ i \in \{1,3,\dots,n-3,n-1\}; \\ 1 & \text{if } e = u_i v_i,\ i \in \{1,2,\dots,n-1,n\}; \\ 1 & \text{if } e \in \{u'_i u_{i+1}, u_i u'_{i+1}\},\ i \in \{1,2,\dots,n-2,n-1\}; \\ 0 & \text{if } e \in \{v_i w'_i, v'_i w_i\},\ i \in \{1,3,\dots,n-3,n-1\}; \\ 0 & \text{if } e = v_i w_i,\ i \in \{2,4,\dots,n-2,n\}; \\ 0 & \text{if } e \in \{u_i v'_i, u'_i v_i\},\ i \in \{1,2,\dots,n-1,n\}; \\ 0 & \text{if } e = u_i u_{i+1},\ i \in \{1,2,\dots,n-2,n-1\}; \\ 0 & \text{if } e = u_n u_1; \\ 1 & \text{if } e \in \{u'_n u_1, u_n u'_1\}. \end{cases}$$

Thus $e_f(1) = \dfrac{9n}{2}$ and $e_f(0) = \dfrac{9n}{2}$.

Case 2: Let n be odd. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \{u_i, v'_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } x = w'_i,\ i \in \{2,4,\ldots,n-3,n-1\}; \\ 1 & \text{if } x = w_i,\ i \in \{1,3,\ldots,n-2,n\}; \\ 0 & \text{if } x = w_i,\ i \in \{2,4,\ldots,n-3,n-1\}; \\ 0 & \text{if } x \in \{u'_i, v_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = w'_i,\ i \in \{1,3,\ldots,n-2,n\}. \end{cases}$$

Thus $v_f(1) = 3n$ and $v_f(0) = 3n$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e \in \{v_i w'_i, v'_i w_i\},\ i \in \{2,4,\ldots,n-3,n-1\}; \\ 1 & \text{if } e = v_i w_i,\ i \in \{1,3,\ldots,n-2,n\}; \\ 1 & \text{if } e = u_i v_i,\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } e \in \{u'_i u_{i+1}, u_i u'_{i+1}\},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } e \in \{v_i w'_i, v'_i w_i\},\ i \in \{1,3,\ldots,n-2,n\}; \\ 0 & \text{if } e = v_i w_i,\ i \in \{2,4,\ldots,n-3,n-1\}; \\ 0 & \text{if } e \in \{u_i v'_i, u'_i v_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } e = u_i u_{i+1},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } e = u_n u_1; \\ 1 & \text{if } e \in \{u'_n u_1, u_n u'_1\}. \end{cases}$$

Thus $e_f(1) = \dfrac{9n-1}{2}$ and $e_f(0) = \dfrac{9n+1}{2}$.

From both the cases we can conclude $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial.

Theorem 2.8: The graph obtained by duplicating all the vertices other than the rim vertices of the armed crown AC_n is cordial.

Proof: Let $V(AC_n) = \{u_i, v_i, w_i / 1 \leq i \leq n\}$ and $E(AC_n) = \{u_i v_i, v_i w_i // 1 \leq i \leq n\} \cup \{u_n u_1\} \cup$

$\{u_iu_{i+1}/1 \leq i \leq n-1\}$. Let G be the graph obtained by duplicating all the vertices except the rim vertices in AC_n. Let $v'_1, v'_2, ..., v'_n, w'_1, w'_2, ..., w'_n$ be the new vertices of G by duplicating $v_1, v_2, ..., v_n, w_1, w_2, ..., w_n$ respectively. Then $V(G) = \{u_i, v_i, w_i, v'_i, w'_i/1 \leq i \leq n\}$ and $E(G) = \{u_nu_1\} \cup \{u_iv_i, v_iw_i, w'_iv_i, w_iv'_i, u_iv'_i/1 \leq i \leq n\} \cup \{u_iu_{i+1}/1 \leq i \leq n-1\}$. Therefore $|V(G)| = 5n$ and $|E(G)| = 6n$. Using parity of n, we have the following cases:

Case 1: Let n be even. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \{u_i, w'_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 1 & \text{if } x = v'_i,\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 0 & \text{if } x \in \{v_i, w_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } x = v'_i,\ i \in \{2,4,\ldots,n-2,n\}. \end{cases}$$

Thus $v_f(1) = \dfrac{5n}{2}$ and $v_f(0) = \dfrac{5n}{2}$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e = v'_iw_i,\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 1 & \text{if } e = v'_iu_i,\ i \in \{2,4,\ldots,n-2,n\}; \\ 1 & \text{if } e \in \{v_iw'_i, u_iv_i\},\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } e = v'_iw_i,\ i \in \{2,4,\ldots,n-2,n\}; \\ 0 & \text{if } e = v'_iu_i,\ i \in \{1,3,\ldots,n-3,n-1\}; \\ 0 & \text{if } e = v_iw_i,\ i \in \{1,2,\ldots,n-1,n\}; \\ 0 & \text{if } e = u_iu_{i+1},\ i \in \{1,2,\ldots,n-2,n-1\}; \\ 0 & \text{if } e = u_nu_1. \end{cases}$$

Thus $e_f(1) = 3n$ and $e_f(0) = 3n$.

Case 2: Let n be odd. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \{u_i, {w'}_i\},\ i \in \{1,2,\dots,n-1,n\}; \\ 1 & \text{if } x = {v'}_i,\ i \in \{1,3,\dots,n-2,n\}; \\ 0 & \text{if } x \in \{v_i, w_i\},\ i \in \{1,2,\dots,n-1,n\}; \\ 0 & \text{if } x = {v'}_i,\ i \in \{2,4,\dots,n-3,n-1\}. \end{cases}$$

Thus $v_f(1) = \dfrac{5n+1}{2}$ and $v_f(0) = \dfrac{5n-1}{2}$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e = {v'}_i w_i,\ i \in \{1,3,\dots,n-2,n\}; \\ 1 & \text{if } e = {v'}_i u_i,\ i \in \{2,4,\dots,n-3,n-1\}; \\ 1 & \text{if } e \in \{v_i {w'}_i, u_i v_i\},\ i \in \{1,2,\dots,n-1,n\}; \\ 0 & \text{if } e = {v'}_i w_i,\ i \in \{2,4,\dots,n-3,n-1\}; \\ 0 & \text{if } e = {v'}_i u_i,\ i \in \{1,3,\dots,n-2,n\}; \\ 0 & \text{if } e = v_i w_i,\ i \in \{1,2,\dots,n-1,n\}; \\ 0 & \text{if } e = u_i u_{i+1},\ i \in \{1,2,\dots,n-2,n-1\}; \\ 0 & \text{if } e = u_n u_1. \end{cases}$$

Thus $e_f(1) = 3n$ and $e_f(0) = 3n$.

From both the cases we can conclude $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, f admits cordial labeling on G. Hence G is cordial.

3. Conclusion

we have derived eight new results by investigating cordial labeling in the context of duplication in crown related graphs. More exploration is possible for other graph families and in the context of different graph labeling problems.

Conflict of Interests

The authors declare that there is no conflict of interests.

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