Available online at http://scik.org

J. Math. Comput. Sci. 7 (2017), No. 1, 119-131

ISSN: 1927-5307

APPLICATIONS OF ADOMIAN DECOMPOSITION METHOD FOR WAVE EQUATIONS

KRISHNA B. CHAVARADDI[1,*], RAMESH B. KUDENATTI[2], PAVITRA TEGGINAMANI[1], SUMA K. NALAVADE[1]

[1]Department of Mathematics, S.S. Government First Grade College, Naragund- 582 207, Karnataka, India

[2]Department of Mathematics, Central College Campus, Bangalore University, Bengaluru-560 001, India

Abstract: In this study, an application of Adomian decomposition method (ADM) is applied to solve the wave equations with non-local boundary conditions. If the equations considered have a solution in terms of the series expansion of known function, this powerful method catches the exact solution. Comparison is made between the exact solution and Adomian decomposition method (ADM). The results reveal that the differential transform method is very effective and simple.

Keywords: Adomian decomposition method (ADM); wave equation; non-local boundary conditions; numerical examples

2010 AMS Subject Classification: 35L05.

1. Introduction

Many physics and engineering problems are modeled by partial differential equations. In many instances these equations are nonlinear and exact solutions are difficult to obtain. Numerical methods were developed over a period of time in order to find approximate solutions to these nonlinear equations. However, numerical solutions are insufficient to determine general properties of certain systems of equations and thus analytical and semi-analytical methods have been developed. These methods have transformed numerical analysis and we are now able to provide both qualitative and quantitative analysis to complex mathematical problems.

The wave equation for a plane wave traveling in the x direction is

*Corresponding author

Received August 30, 2016

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

where v is the phase velocity of the wave and y represents the variable which is changing as the wave passes. This is the form of the wave equation which applies to a stretched string or a plane electromagnetic wave. The mathematical description of a wave makes use of partial derivatives.

In two dimensions, the wave equation takes the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

which could describe a wave on a stretched membrane.

Suppose we want to consider a horizontal string of length L that has been tightly stretched between two points at $x = 0$ and $x = L$. Because the string has been tightly stretched we can assume that the slope of the displaced string at any point is small. So just what does this do for us? Let's consider a point *x* on the string in its equilibrium position, *i.e.* the location of the point at $t = 0$. As the string vibrates this point will be displaced both vertically and horizontally, however, if we assume that at any point the slope of the string is small then the horizontal displacement will be very small in relation to the vertical displacement. This means that we can now assume that at any point *x* on the string the displacement will be purely vertical. So, let's call this displacement $u(x,t)$.

We are going to assume, at least initially, that the string is not uniform and so the mass density of the string, $\rho(x)$ may be a function of *x*.

Next we are going to assume that the string is perfectly flexible. This means that the string will have no resistance to bending. This in turn tells us that the force exerted by the string at any point *x* on the endpoints will be tangential to the string itself. This force is called the **tension** in the string and its magnitude will be given by $T(x,t)$.

Finally, we will let $Q(x,t)$ represent the vertical component per unit mass of any force acting on the string.

Provided we again assume that the slope of the string is small the vertical displacement of the string at any point is then given by,

$$\rho(x)\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(T(x,t)\frac{\partial u}{\partial x}\right)+\rho(x)Q(x,t)$$

This is a very difficult partial differential equation to solve so we need to make some further simplifications.

First, we're now going to assume that the string is perfectly elastic. This means that the magnitude of the tension, $T(x,t)$, will only depend upon how much the string stretches near *x*. Again, recalling that we're assuming that the slope of the string at any point is small this means that the tension in the string will then very nearly be the same as the tension in the string in its equilibrium position. We can then assume that the tension is a constant value, $T(x,t)=T_0$.

Further, in most cases the only external force that will act upon the string is gravity and if the string light enough the effects of gravity on the vertical displacement will be small and so will also assume that $Q(x,t)=0$. This leads to

$$\rho\frac{\partial^2 u}{\partial t^2}=T_0\frac{\partial^2 u}{\partial x^2}$$

If we now divide by the mass density and define,

$$c^2=\frac{T_0}{\rho}$$

we arrive at the 1-D **wave equation**,

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

In this Chapter, we deal with non- classical initial boundary value problems that is, the solution of hyperbolic differential equations with non- local boundary specifications. These non- local conditions arise mainly when the data on the boundary cannot be measured directly. Many physical phenomena are modeled by hyperbolic initial boundary value problems with non- local boundary conditions. Hyperbolic equations with non- local integral conditions are widely used in chemistry, plasma physics, thermo-elasticity, engineering and so forth. The solutions of hyperbolic and parabolic equations with integral conditions were studied by several authors [1-10]. Numerical solutions of hyperbolic partial differential equations with integral conditions are still a major research area with widespread applications in engineering, physics and technology.

1.1 One-dimensional wave equation

We consider the following one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + q(x,t) \qquad 0 < x < 1, 0 < t < T \tag{1}$$

Subject to the initial condition:

$$u(x,0) = r(x) \qquad 0 < x < 1 \tag{2}$$

$$u_t(x,0) = s(x) \qquad 0 < x < 1 \tag{3}$$

and the non-local boundary condition:

$$u(0,t) = p(t) \qquad 0 < t < T \tag{4}$$

$$\int_0^1 u(x,t)dt = q(t) \qquad 0 < t \leq T \tag{5}$$

Where r, s, p and q are known functions, we suppose that f is sufficiently smooth to produce a smooth classical solution.

In this paper, we focus on the solutions of wave equations with non-local boundary conditions by non-perturbation analytical methods known as Adomian Decomposition method(ADM) and also to compare these solutions with exact solution focusing on accuracy, convergence and computational efficiency.

2. Adomian Decomposition method

The Adomian decomposition method has been applied [11-14] for solving a large classes of linear and non linearordinary and partial differential equations with approximate solutions which converges rapidly to accurate solutions. In recent years, many papers were devoted to the problem of approximate of one-dimensional wave equation with non-local boundary conditions. The motivation of this work is to apply the decomposition method for solving the one dimensional wave equation with an integral boundary condition. It is well known in the literature that this algorithm provides solution in rapidly convergent series. The implementation of the Adomian method has shown reliable results in that few terms only are needed to obtain accurate solutions.

Consider equation (1) to (5) written in the form

$$L_{tt}(u) = L_{xx}(u) + q(x,t) \tag{6}$$

Where the differential operators are given as;

$$L_{tt}(.) = \frac{\partial^2 (.)}{\partial t^2} \quad \text{And} \quad L_{xx} = \frac{\partial^2}{\partial x^2}$$

The inverse operator L_{tt}^{-1} is these for considered a two-fold integral operator defined by

$$L_{tt}^{-1} = \int_0^t \int_0^t (.) dt\, dt$$

Operating with L_{tt}^{-1} on equation (6), it then follows that:

$$L_{tt}^{-1}(L_{tt}(u)) = L_{tt}^{-1}(L_{xx}(u)) + L_{tt}^{-1}(q(x,t)) \qquad (7)$$

And special initial condition yield:

$$u(x,t) = r(x) + ts(x) + L_{tt}^{-1}(L_{xx}(u(x,t))) + L_{tt}^{-1}(q(x,t)) \qquad (8)$$

The decomposition method assumes an infinite series solution for unknown function $u(x,t)$ given by:

$$u(x,t) = \sum_{k=0}^{\infty} u_k \qquad (9)$$

Where the components u_k $(k = 0,1,2,3,............)$ are determined recursively by using the relation:

$$u_0 = r(x) + t\, s(x) + L_{tt}^{-1}(q(x,t)) \qquad (10)$$

And

$$u_{k+1} = L_{tt}^{-1}(L_{xx}(u_k)) \qquad (11)$$

If the series converges in a suitable way, then the general solution is obtained as:

$$u(x,t) = \lim_{n \to \infty} \sum_{k=0}^{\infty} u_k(x,t) .$$

In the next section, we solve numerical examples by Adomian Decomposition method for testing the convergence of the method.

3. Numerical examples

Example 1: Consider the following wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \ , \qquad 0 < x < 1 \ , \ 0 < t < 0.5$$

with the initial conditions:

$$\left.\begin{aligned} u(x,0) &= 0 \;, \qquad 0 \le x \le 1 \\ u_t(x,0) &= \pi\cos(\pi x)\,, \qquad 0 \le x \le 1 \end{aligned}\right\}$$

and the boundary conditions:

$$\left.\begin{aligned} &u(0,t) = p(t) = \sin(\pi t) \\ &\int_0^t u(x,t)dt = q(t) = 0 \end{aligned}\right\}$$

Substituting in equation (10) and (11) we get.

$$u_0 = r(x) + t\,s(x) + L_{tt}^{-1}(q(x,t)) = t\,(\pi\cos(\pi x))$$

$$u_{k+1} = L_{tt}^{-1}(L_{xx}(u_k)) \quad , \quad k \ge 0$$

We can then proceed to compute the first few terms of the series:

$$u_1 = L_{tt}^{-1}(L_{xx}(u_0)) = \cos(\pi x)\left(\frac{-\pi^3 t^3}{3!}\right)$$

$$u_2 = L_{tt}^{-1}(L_{xx}(u_1)) = \cos(\pi x)\left(\frac{-\pi^5 t^5}{5!}\right)$$

$$u_3 = L_{tt}^{-1}(L_{xx}(u_2)) = \cos(\pi x)\left(\frac{-\pi^7 t^7}{7!}\right)$$

And so on........

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \dots\dots\dots\dots\dots\dots$$

$$= \cos(\pi x)\left[\pi t - \frac{(\pi t)^3}{3!} + \frac{(\pi t)^5}{5!} + \frac{(\pi t)^7}{7!} - + \dots\dots\dots\dots\right]$$

$$u(x,t) = \cos(\pi x)\sin(\pi t)$$

Example 2: Consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{4}\frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < 1\,,\ 0 < t < 1.$$

with the initial condition:

$$\left.\begin{aligned} u(x,0) &= x \qquad 0 < x < 1 \\ u_t(x,0) &= e^x \qquad 0 < x < 1 \end{aligned}\right\}$$

and the boundary conditions:

$$\left.\begin{aligned} u_x(0,t) &= 2\sinh\left(\frac{t}{2}\right) && t>0 \\ u_x(1,t) &= 2e^x\left(\sinh\left(\frac{t}{2}\right)+1\right) && t>0 \end{aligned}\right\}$$

By Adomian decomposition method equation (1) yields;

$$L_{tt}(u(x,t)) - \frac{1}{4}L_{xx}(u(x,t)) = 0$$

Operating with the inverse operator L_{tt}^{-1} on both side of equation (4) and importing the corresponding initial conditions, we obtain.

$$u(x,t) = u(x,0) + t\,u_t(x,0) + \frac{1}{4}L_{tt}^{-1}(L_{xx}(u(x,t)))$$

Starting with:

$$u_0(x,t) = u(x,0) + t\,u_t(x,0) = x + t\,e^x$$

$$u_{k+1}(x,t) = \frac{1}{4}L_{tt}^{-1}[L_{xx}(u_k)], \qquad k>0$$

$$u_1(x,t) = \frac{1}{4}L_{tt}^{-1}[L_{xx}(u_0)] = 2e^x\,\frac{t^3}{2^3 3!}$$

$$u_2(x,t) = \frac{1}{4}L_{tt}^{-1}[L_{xx}(u_1)] = 2e^x\,\frac{t^5}{2^5 5!}$$

$$u_3(x,t) = \frac{1}{4}L_{tt}^{-1}[L_{xx}(u_2)] = 2e^x\,\frac{t^7}{2^7 7!}$$

By continuing the iteration we find that:

$$u_k = 2e^x\,\frac{t^{2k+1}}{2^{2k+1}(2k+1)!}$$

$$u(x,t) = x + 2e^x\sum_{k=0}^{\infty}u_k(x,t)$$

$$u(x,t) = x + 2e^x\sinh\left(\frac{t}{2}\right)$$

This converges to the exact solution.

Example 3: Consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < 1 \, , \, 0 < t < 1.$$

with the initial condition:

$$\left.\begin{array}{ll} u(x,0) = \sin(\pi x) & 0 \le x \le 1 \\ u_t(x,0) = 0 & 0 \le x \le 1 \end{array}\right\}$$

And the boundary conditions:

$$\left.u(0,t) = u(1,t) = 0 \qquad t > 0 \right\}$$

Compare the result with the exact solution

$$u(x,t) = \sin(\pi x)\cos(2\pi t)$$

By Adomian decomposition method equation (1) yields;

$$L_{tt}(u(x,t)) - 4L_{xx}(u(x,t)) = 0$$

Operating with the inverse operator L_{tt}^{-1} on both side of equation (4) and importing the corresponding initial conditions, we obtain.

$$u(x,t) = u(x,0) + t\,u_t(x,0) + 4L_{tt}^{-1}(L_{xx}(u(x,t)))$$

Starting with:

$$u_0(x,t) = u(x,0) + t\,u_t(x,0) = \sin(\pi x)$$

$$u_{k+1}(x,t) = 4L_{tt}^{-1}[L_{xx}(u_k)] \, , \qquad k > 0$$

$$u_1(x,t) = 4L_{tt}^{-1}[L_{xx}(u_0)] = -\sin(\pi x)\frac{(2\pi t)^2}{2!}$$

$$u_2(x,t) = 4L_{tt}^{-1}[L_{xx}(u_1)] = \sin(\pi x)\frac{(2\pi t)^4}{4!}$$

$$u_3(x,t) = 4L_{tt}^{-1}[L_{xx}(u_2)] = \sin(\pi x)\frac{(2\pi t)^6}{6!}$$

By continuing like this:

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u(x,t) = \sin(\pi x)\cos(2\pi t)$$

This converges to the exact solution.

Example 4. Consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < 1\,,\; 0 < t < 1.$$

with the initial condition:

$$\left.\begin{aligned} u(x,0) &= \sin(\pi x) \qquad 0 \le x \le 1 \\ u_t(x,0) &= 0 \qquad\qquad\; 0 \le x \le 1 \end{aligned}\right\}$$

And the boundary conditions:

$$\left. u(0,t) = u(1,t) = 0 \qquad t > 0 \right\}$$

Compare the result with the exact solution

$$u(x,t) = \sin(\pi x)\cos(\pi t)$$

By Adomian decomposition method equation (1) yields;

$$L_{tt}(u(x,t)) - L_{xx}(u(x,t)) = 0$$

Operating with the inverse operator L_{tt}^{-1} on both side of equation (4) and importing the corresponding initial conditions, we obtain.

$$u(x,t) = u(x,0) + t\,u_t(x,0) + L_{tt}^{-1}(L_{xx}(u(x,t)))$$

Starting with:

$$u_0(x,t) = u(x,0) + t\,u_t(x,0) = \sin(\pi x)$$

$$u_{k+1}(x,t) = L_{tt}^{-1}[L_{xx}(u_k)]\,, \qquad k > 0$$

$$u_1(x,t) = L_{tt}^{-1}[L_{xx}(u_0)] = -\sin(\pi x)\frac{(\pi t)^2}{2!}$$

$$u_2(x,t) = L_{tt}^{-1}[L_{xx}(u_1)] = \sin(\pi x)\frac{(\pi t)^4}{4!}$$

$$u_3(x,t) = L_{tt}^{-1}[L_{xx}(u_2)] = \sin(\pi x)\frac{(\pi t)^6}{6!}$$

By continuing like this:

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u(x,t) = \sin(\pi x)\cos(\pi t)$$

This converges to the exact solution.

Example 5: Consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{16\pi^2}\frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < 1\ ,\ 0 < t < 1.$$

with the initial condition:

$$\left.\begin{array}{ll} u(x,0) = 0 & 0 \le x \le 1 \\ u_t(x,0) = \sin(4\pi\, x) & 0 \le x \le 1 \end{array}\right\}$$

And the boundary conditions:

$$u(0,t) = u(0.5,t) = 0 \qquad t > 0 \quad \}$$

Compare the result with the exact solution

$$u(x,t) = \sin(4\pi\, x)\sin(t)$$

By Adomian decomposition method equation (1) yields;

$$L_{tt}(u(x,t)) - L_{xx}(u(x,t)) = 0$$

Operating with the inverse operator L_{tt}^{-1} on both side of equation (4) and importing the corresponding initial conditions, we obtain.

$$u(x,t) = u(x,0) + t\,u_t(x,0) + L_{tt}^{-1}(L_{xx}(u(x,t)))$$

Starting with:

$$u_0(x,t) = u(x,0) + t\,u_t(x,0) = t\sin(4\pi\, x)$$

$$u_{k+1}(x,t) = L_{tt}^{-1}[L_{xx}(u_k)]\ , \qquad k > 0$$

$$u_1(x,t) = L_{tt}^{-1}[L_{xx}(u_0)] = -\sin(4\pi\, x)\frac{(t)^3}{3!}$$

$$u_2(x,t) = L_{tt}^{-1}[L_{xx}(u_1)] = \sin(4\pi\, x)\frac{(t)^5}{5!}$$

$$u_3(x,t) = L_{tt}^{-1}[L_{xx}(u_2)] = \sin(4\pi\, x)\frac{(t)^7}{7!}$$

By continuing like this:

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u(x,t) = \sin(4\pi\, x)\sin(t)$$

This converges to the exact solution.

Example 6: Consider the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < \pi \ , \ 0 < t$$

with the initial condition:

$$\left.\begin{aligned} &u(x,0) = \sin(x) && 0 \le x \le \pi \\ &u_t(x,0) = 0 && 0 \le x \le \pi \end{aligned}\right\}$$

And the boundary conditions:

$$u(0,t) = u(\pi,t) = 0 \qquad t > 0$$

Compare the result with the exact solution

$$u(x,t) = \sin(x)\cos(t)$$

By Adomian decomposition method equation (1) yields;

$$L_{tt}(u(x,t)) - L_{xx}(u(x,t)) = 0$$

Operating with the inverse operator L_{tt}^{-1} on both side of equation (4) and importing the corresponding initial conditions, we obtain.

$$u(x,t) = u(x,0) + t\,u_t(x,0) + L_{tt}^{-1}(L_{xx}(u(x,t)))$$

Starting with:

$$u_0(x,t) = u(x,0) + t\,u_t(x,0) = \sin(x)$$

$$u_{k+1}(x,t) = L_{tt}^{-1}[L_{xx}(u_k)] \ , \qquad k > 0$$

$$u_1(x,t) = L_{tt}^{-1}[L_{xx}(u_0)] = -\sin(x)\frac{(t)^2}{2!}$$

$$u_2(x,t) = L_{tt}^{-1}[L_{xx}(u_1)] = \sin(x)\frac{(t)^4}{4!}$$

$$u_3(x,t) = L_{tt}^{-1}[L_{xx}(u_2)] = \sin(x)\frac{(t)^6}{6!}$$

By continuing like this:

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u(x,t) = \sin(x)\cos(t)$$

This converges to the exact solution.

4. Conclusion

We have applied the Adomian decomposition method for the solution of the wave equation with non-local boundary conditions. This algorithm is simple and easy to implement. The obtained results confirmed a good accuracy of the method. On the other hand, the calculations are simpler and faster than in traditional techniques.

The main advantage found from this was that the method generates an analytic expression for the solution. The computations are easily utilized and could be completed without computer assistance if desired. This work indicates that an increase in symbolism results in an increase in the complexity of the solution, thereby losing the computational ease which has been a major advantage of this method. Furthermore, it appears that the type or strength of nonlinearity also influences the qualitative properties of the solution. The solutions generated in this work exhibit functions in the series terms that are unfavorable for use in recursive relationship.

The Adomian decomposition method can handle any kind of linear or non-linear differential equation. Solutions of ADM are compared with exact method which excellent convergent. As discussed earlier ADM have an easy computational and highly convergent system to solve either linear or non-linear problems of the real world without assumptions on changing the essential non-linear nature. The main advantage of ADM is that it can be applied directly for all types of differential and integral equations, linear or non-linear, homogeneous or non-homogeneous, with constant, or with variable coefficients. Another important advantage is that the method is capable of greatly reducing the size of computation work while still maintaining high accuracy of the numerical solution.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgement

The authors would like to thank the Principal and Higher Authority of Department of Collegiate Education, GOK for their encouragement and support.

REFERENCES

[1] Marasi, H. R., Faghiri, R., Hashimi, M. S.: A series solution of non local Hyperbolic Equations, Communications in Numerical Analysis, 2012 (2012), Article ID cna-00113.

[2] Sharma P. R. and Giriraj Methi, Analysis of Homotopy Perturbation Method for Solution of HyperbolicEquation with An integral Condition, Journal of Applied Mathematics, 5 (2) (2012), 85-95.

[3] Somayeth Nemati and Yadollah Ordokhant: A second Kind Chebychev Polynomial for the Wave Equationsubject to An integral Conservation, Journal ofInformation and Computing Science, 7(3) (2012), 163-171.

[4] Cheniguel, A.: Numerical Simulation of Two-Dimensional Diffusion Equation with Non-Local Boundary Conditions, International Mathematical Forum, 7(50) (2012), 2457-2463.

[5] Cheniguel, A: Numerical Method for Solving HeatEquation with Derivatives Boundary Conditions, Proceedings of the World Congress on Engineering and Computer Science, II, WCECS October19-21, 2011, San Francisco, USA. (2011)

[6] Cheniguel, A. and Ayadi, A: Solving Heat Equation by the Adomian Decomposition Method, Proceedings of the World Congress on Engineering, Vol I, WCE, July 6-8, 2011, London, UK. (2011)

[7] Choudhury, A. H and Deka, R. K.: Wavelet Method for numerical Solution of Wave Equation with Neumann Boundary Conditions, Proceedings of the International Multi-conference of Engineering and Computer Scientists, Vol II, IMECS 2011, March 16-18, Hong Kong. (2011)

[8] Ramezani, M., Dehghan M and Razzaghi, M.: Combined Finite Difference and Spectral Methods forthe Numerical Solution of Hyperbolic Equation with an integral Condition Numerical Methods for Partial Differential Equations, 24(1) (2008), 1-8.

[9] Saadatmandi A. and Dehghan, M: Numerical Solution of the One-Dimensional Wave Equation with Anintegral Condition, Numerical Methods for Partial Differential Equations, 23(2) (2007), 282-292.

[10] Dehghan, M.: On the Solution of an initial-boundaryValue Problem the Combines Neumann and Integral Condition for the Wave Equation, Numerical Methods for Partial Differential Equations, 21(1) (2008), 24-40.

[11] Abbaoui, K. and Cherruault, Y.: Convergence of the Adomian Method applied to Differential Equations, Comput. Math. 28 (5) (1994), 103-109.

[12] Adomian, G.: Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Dordrecht, 1994.

[13] Adomian, G. and Rach, R.: Noise terms in decomposition solution series, Computers Math. Appl., 24(11) (1992), 61-64.

[14] Adomian, G.: A review of the decomposition method in applied mathematics, J. Math. Anal. Appl. 135(1988), 501-544.