Available online at http://scik.org
J. Math. Comput. Sci. 7 (2017), No. 1, 39-58
ISSN: 1927-5307

NUMERICAL SOLUTION OF HAMMERSTEIN INTEGRAL EQUATION USING CHEBYSHEV WAVELET METHOD

J. IQBAL*, R. ABASS

Department of Mathematical Sciences, BGSB University, Rajouri-185234, J and K, India

Abstract. The aim of this work is to solve Hammerstein integral equations of both Fredholm as well as Volterra type by Chebyshev wavelet method which is widely applicable in engineering and technology. The Chebyshev wavelets together its properties are used to converts the problem into algebraic equations. Illustrative examples of Hammerstein type equations have been discussed to demonstrate the validity and applicability of the technique and the results have been compared with the existing method in the literature as well as with exact solution.

Keywords: Chebyshev wavelet; operational matrix of integration; product operational matrix; Hammerstein integral equations; MATLAB.

2010 AMS Subject Classification: 45B05, 47G20, 65T60.

1. Introduction

Many problems of mathematical physics can be stated in the form of integral equations. These equations also occur as reformulations of other mathematical problems such as partial differential equations and ordinary differential equations. The study of integral equations and

*Corresponding author

Received September 2, 2016

methods for solving them are very useful in many fields including many problems in mathematical physics, the dynamic model of chemical reactor, problems in control theory, and various reformulations of an elliptic partial differential equation with nonlinear boundary conditions. In this paper, we consider the nonlinear Fredholm-Hammerstein integral equations and nonlinear Volterra-Hammerstein integral equations respectively by the general forms [1, 13]

$$y(t) = f(t) + \lambda \int_0^1 K(t,x)F(x,y(x))\,dx, \tag{1.1}$$

$$y(t) = f(t) + \lambda \int_0^t K(t,x)F(x,y(x))\,dx, \tag{1.2}$$

where the parameter λ and functions $f(t), K(t,x)$ and $F(x,y(x)) \in L^2[0,1]$ are known functions and $y(t) \in C[0,1]$ is unknown function. Hammerstein integral equations appear in many areas of scientific fields like chemical kinetics, electrochemical machining, fluid dynamics, mathematical biology and plasma physics [12] . From past two decades, a broad class of numerical methods for approximating the solution of Hammerstein integral equations are known. For Fredholm-Hammerstein integral equations (1.1), the classical method of successive approximations was introduced in [23]. A variation of the Nystrom method was presented in [16]. A collocation-type method was developed in [14]. In [5], Brunner applied a collocation-type method to nonlinear Volterra-Hammerstein integral equations and integro-differential equations, and discussed its connection with the iterated collocation method. Han [9] introduced and discussed the asymptotic error expansion of a collocation-type method for Volterra-Hammerstein integral equations. The methods in [9, 14] transform a given integral equation into a system of nonlinear equations, which has to be solved with some kind of iterative method. In [14], the definite integrals involved in the solution may be evaluated analytically only in favorable cases, while in [9] the integrals involved in the solution have to be evaluated at each time step of the iteration.

Over the last couple of decades, wavelets have been studied extensively and have emerged as a powerful computational tool for attaining numerical solutions for a wide range of problems including integral, algebraic, differential, partial-differential, functional-delay and integro-differential equations. Wavelets are calculated as continuously oscillatory functions and possess attractive features: zero-mean, fast decay, short life, time-frequency representation, multi-resolution, etc. Wavelets have the ability to detect information at different scales and at different locations throughout a computational domain. Wavelets can provide a basis set in which the basis functions are constructed by dilating and translating a fixed function known as the mother wavelet. The wavelet method allows the creation of very fast algorithms when compared with the algorithms ordinarily used. Wavelets are considerably useful for solving Fredholm cum Volterra Hammerstein integral equations and provide accurate solutions.

In recent years, different types of wavelets and approximating functions based on orthogonal functions have been used to approximate the numerical solution of integral and differential equations such as Chebyshev [24], Legendre [2, 10], Daubechies [20], Alpert [12], Modifed Homotopy Perturbation [7] and Haar [17] wavelets. Among all the wavelet families, the Chebyshev wavelets have gained popularity among researchers due to their useful properties such as simple applicability, orthogonality and compact support. The main characteristic of Chebyshev wavelet is that it converts the problems into nonlinear system of algebraic equations and this system may be solved by using an appropriate numerical method. This approach used operational matrix P of integration and product operation matrix to eliminate the integral operator.

The rest of the paper is as follows: In section 2, Chebyshev wavelet, its properties, function approximations and convergence are discussed. Operational Matrix of Integration(OMI) is presented in section 3. In section 4, product operation matrix of Chebyshev wavelets have been discussed. Section 5, is devoted to present a computational method for solving Hammerstein integral equations utilizing Chebyshev wavelets and approximate the unknown function. Section 6, deals with the illustrative examples and their solutions by the proposed approach compared with exact as well as with existing literature. Finally, we conclude the article in section 7.

2. Wavelets and Chebyshev Wavelets

Wavelets constitute the family of functions constructed from the dilation and translation of a single function known as the Mother wavelet. When the dilation parameter a and translation parameter b vary continuously we have the following family of continuous wavelets [8]

$$\psi_{a,b(t)} = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right);\ a,\ b\ \in \mathbb{R},\ a \neq 0. \tag{2.1}$$

If we choose $a = a^{-k}$ and $b = nba^{-k}$ where $a > 1$, $b > 0$ and $n, k \in \mathbb{Z}^{+}$ then we get the following family of discrete wavelets:

$$\psi_{k,n}(t) = |a|^{-\frac{k}{2}}\,\psi(a^{k}t - nb). \tag{2.2}$$

These family of functions are a wavelet basis for $L^2(\mathbb{R})$ and makes an orthonormal basis for the special case $a = 2$ and $b = 1$.

Chebyshev wavelets $\psi_{n,m}(t) = \psi(k,m,n,t)$ have four arguments, $k = 0,1,2,.....$, $n = 1,2,.....,2^{k}$, m is the degree of Chebyshev polynomial of first kind and t denotes the normalized time. They are defined on the interval $[0,1)$ by

$$\psi_{n,m}(t) = \begin{cases} \frac{\alpha_m 2^{k/2}}{\sqrt{\pi}} T_m(2^{k+1}t - 2n + 1), & \frac{n-1}{2^k} \leq t \leq \frac{n}{2^k} \\ \\ 0, \text{ otherwise} \end{cases} \tag{2.3}$$

where

$$\alpha_m = \begin{cases} \sqrt{2}, & m = 0 \\ \\ 2, & m = 1,2,\ldots \end{cases}$$

$T_m(t)$ in (2.3) are well known Chebyshev polynomial of order m, which is orthogonal with respect to the weight function $\omega(t) = \frac{1}{\sqrt{1-t^2}}$ and satisfy the following recursive formula:

$$T_0(t) = 1$$

$$T_1(t) = t$$

$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1,2,3,......$$

Moreover, the set of Chebyshev wavelet are an orthogonal set with respect to the weight function $\omega_n(t) = \omega(2^{k+1}t - 2n + 1)$.

A function $f(t) \in L^2[0,1]$ may be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t), \tag{2.4}$$

where the wavelet coefficients of the series representation in (2.4) become

$$c_{nm} = \langle f(t), \psi_{nm}(t) \rangle_{w_n(t)}. \tag{2.5}$$

The convergence of the series (2.4), in $L^2[0,1]$, means that

$$\lim_{s_1, s_2 \to \infty} \left\| f(x) - \sum_{n=1}^{s_1} \sum_{m=0}^{s_2} c_{nm} \psi_{nm}(t) \right\| = 0.$$

If the infinite series in (2.4) is truncated then equation (2.4) can be written as

$$f(t) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(t) = C^T \Psi(t), \tag{2.6}$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by:

$$C = [c_{1,0}, c_{1,1}, \ldots, c_{1,M-1}, c_{2,0}, c_{2,1}, \ldots, c_{2,M-1}, \ldots, c_{2^{k-1},0}, \ldots, c_{2^{k-1}M-1}]^T, \tag{2.7}$$

$$\Psi(t) = [\psi_{1,0}, \psi_{1,1}, \ldots, \psi_{1,M-1}, \psi_{2,0}, \psi_{2,1}, \ldots, \psi_{2,M-1}, \ldots, \psi_{2^{k-1},0}, \ldots, \psi_{2^{k-1},M-1}]^T. \tag{2.8}$$

In the same way, a function of two variable $K(x,t) \in L^2([0,1] \times [0,1])$ may be approximated as:

$$K(x,t) \approx \Psi^T(x) \mathbf{K} \Psi(t), \tag{2.9}$$

where $\mathbf{K}$ is $2^{k-1}M \times 2^{k-1}M$ matrix, with $\mathbf{K}_{ij} = (\psi_i(x), (K(x,t), \psi_j(t)))$.

The integration of the product of two Chebyshev wavelets vector functions with respect to the weight function $W_n(t)$, is derived as

$$\int_0^1 W_n(t) \psi(t) \psi^T(t) dt = I, \tag{2.10}$$

where I is the identity.

Also the integer powers of a function may be approximated as

$$[y(t)]^p = [Y^T\Psi(t)]^p = Y_p^{*T}\Psi(t), \tag{2.11}$$

where Y_p^* is a $2^{k-1}\times 1$ matrix, whose elements are nonlinear combinations of the elements of the vector Y. Y_p^* is called the operational vector for production of the p^{th} power of the function $y(t)$ [3, 8, 18].

Since the truncated Chebyshev wavelets series can be an approximate solution of singular integral equations, one has an error function $E(t)$ for $f(t)$ as follows:

$$E(t) = \left| f(t) - C^T\Psi(t) \right|.$$

3. Operational Matrix of Integration(OMI)

In this section, we will first derive the operational matrix P of integration [21, 22, 24] which help us in dealing with the concerned problems Hammerstein integral equations . First we construct the matrix P for $k=2$ and $M=3$. In this case, the six basis functions are given by

$$\left.\begin{aligned} \psi_{1,0} &= \tfrac{2}{\sqrt{\pi}}, \\ \psi_{1,1} &= 2\sqrt{\tfrac{2}{\pi}}(4t-1), \\ \psi_{1,2} &= 2\sqrt{\tfrac{2}{\pi}}\left((4t-1)^2-1\right) \end{aligned}\right\} \text{for } t\in[0,1/2), \tag{3.1}$$

$$\left.\begin{aligned} \psi_{2,0} &= \tfrac{2}{\sqrt{\pi}}, \\ \psi_{2,1} &= 2\sqrt{\tfrac{2}{\pi}}(4t-3), \\ \psi_{2,2} &= 2\sqrt{\tfrac{2}{\pi}}\left((4t-3)^2-1\right) \end{aligned}\right\} \text{for } t\in(1/2,1]. \tag{3.2}$$

By integrating the above defined basis (3.1) and (3.2) from 0 to t and using wavelet coefficient, we obtain

$$\int_0^t \psi_{1,0}(t)dt = \begin{cases} \frac{2}{\sqrt{\pi}}t, & t \in [0, \frac{1}{2}) \\ \frac{1}{\sqrt{\pi}}t, & t \in (\frac{1}{2}, 1] \end{cases} = \left[\frac{1}{4}, \frac{1}{4\sqrt{2}}, 0, \frac{1}{2}, 0, 0\right]^T \Psi_6(t),$$

$$\int_0^t \psi_{1,1}(t)dt = \begin{cases} 2\sqrt{\frac{2}{\sqrt{\pi}}}, & t \in [0, \frac{1}{2}) \\ 0, & t \in (\frac{1}{2}, 1] \end{cases} = \left[-\frac{1}{8\sqrt{2}}, 0, \frac{1}{16}, 0, 0, 0\right]^T \Psi_6(t),$$

$$\int_0^t \psi_{1,2}(t)dt = \begin{cases} 2\sqrt{\frac{2}{\sqrt{\pi}}}, \left(\frac{32}{3}t^3 - 8t^2 + t\right) & t \in [0, \frac{1}{2}) \\ -\frac{1}{3}\sqrt{\frac{2}{\pi}}, & t \in (\frac{1}{2}, 1] \end{cases} = \left[-\frac{1}{6\sqrt{2}}, -\frac{1}{8}, 0, -\frac{1}{3\sqrt{2}}, 0, 0\right]^T \Psi_6(t),$$

$$\int_0^t \psi_{2,0}(t)dt = \begin{cases} 0, & t \in [0, \frac{1}{2}) \\ \frac{2}{\sqrt{\pi}}t - \frac{1}{\pi}, & t \in (\frac{1}{2}, 1] \end{cases} = \left[0, 0, 0, \frac{1}{4}, \frac{1}{4\sqrt{4}}, 0\right]^T \Psi_6(t),$$

$$\int_0^t \psi_{2,1}(t)dt = \begin{cases} 0, & t \in [0, \frac{1}{2}) \\ 2\sqrt{\frac{2}{\pi}}(2t^2 - 3t) + 1, & t \in (\frac{1}{2}, 1] \end{cases} = \left[0, 0, 0, -\frac{1}{8\sqrt{2}}, 0, -\frac{1}{16}\right]^T \Psi_6(t),$$

$$\int_0^t \psi_{2,2}(t)dt = \begin{cases} 0, & t \in [0, \frac{1}{2}) \\ 2\sqrt{\frac{2}{\pi}}\left(\frac{32}{3}t^3 - 24t^2 + 17t - \frac{23}{6}\right), & t \in (\frac{1}{2}, 1] \end{cases} = \left[0, 0, 0, -\frac{1}{6\sqrt{2}}, -\frac{1}{8}, 0\right]^T \Psi_6(t).$$

The integration of the vector $\Psi(t)$, defined in (2.8), can be obtained as

$$\int_0^t \Psi(t)dt = P\Psi(t), \tag{3.3}$$

where P is the $2^{k-1}M \times 2^{k-1}M$ operational matrix of integration [3, 4] determined as follows.

$$P_{6\times 6} = \frac{1}{4}\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 2 & 0 & 0 \\ -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{3} & -\frac{1}{2} & 0 & -\frac{2\sqrt{2}}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{3} & -\frac{1}{2} & 0 \end{bmatrix}. \tag{3.4}$$

For general case, we have

$$P_{2^{k-1}M\times 2^{k-1}M} = \frac{1}{2^k}\begin{bmatrix} L & F & F & \dots & F \\ O & L & F & \ddots & \vdots \\ O & O & L & \ddots & F \\ \vdots & \ddots & \ddots & \ddots & F \\ O & \dots & O & O & L \end{bmatrix}, \tag{3.5}$$

where L, F and O are $M \times M$ matrices given by

$$L = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & 0 & \dots & 0 & 0 & 0 \\ -\frac{1}{8\sqrt{2}} & 0 & \frac{1}{8} & 0 & \dots & 0 & 0 & 0 \\ -\frac{1}{6\sqrt{2}} & -\frac{1}{4} & 0 & \frac{1}{12} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\frac{1}{\sqrt{2}(M-1)(M-3)} & 0 & 0 & 0 & \dots & -\frac{1}{4(M-3)} & 0 & -\frac{1}{4(M-1)} \\ -\frac{1}{2\sqrt{2}M(M-2)} & 0 & 0 & 0 & \dots & 0 & -\frac{1}{4(M-2)} & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} \sqrt{2} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ -\frac{\sqrt{2}}{3} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ -\frac{\sqrt{2}}{15} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\sqrt{2}}{M(M-2)} & 0 & 0 & \dots & 0 \end{bmatrix}$$

and

$$O = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

4. Product Operation Matrix of Chebyshev Wavelets

The following properties of the product of two Chebyshev wavelet function vectors is also used for solving differential a well as integral equations:

$$C^T\Psi(t)\Psi^T(t) \approx \tilde{C}\Psi^T(t), \tag{4.1}$$

where C and $\Psi(t)$ are give in Eq.(2.7) and (2.8), respectively and $\tilde{C}$ is $(2^{k-1}M) \times (2^{k-1}M)$ product operational matrix [11]. To illustrate the calculation procedures, we choose $M = 3$, $k = 2$ and using $\Psi(t)$ similarly to Eq.(2.8),we have

$$\Psi(t)\Psi^T(t)=\begin{bmatrix}\psi_{10}\psi_{10} & \psi_{10}\psi_{11} & \psi_{10}\psi_{12} & \psi_{10}\psi_{20} & \psi_{10}\psi_{21} & \psi_{10}\psi_{22}\\ \psi_{11}\psi_{10} & \psi_{11}\psi_{11} & \psi_{11}\psi_{12} & \psi_{11}\psi_{20} & \psi_{11}\psi_{21} & \psi_{11}\psi_{22}\\ \psi_{12}\psi_{10} & \psi_{12}\psi_{11} & \psi_{12}\psi_{12} & \psi_{12}\psi_{20} & \psi_{12}\psi_{21} & \psi_{12}\psi_{22}\\ \psi_{20}\psi_{10} & \psi_{20}\psi_{11} & \psi_{20}\psi_{12} & \psi_{20}\psi_{20} & \psi_{20}\psi_{21} & \psi_{20}\psi_{22}\\ \psi_{21}\psi_{10} & \psi_{21}\psi_{11} & \psi_{21}\psi_{12} & \psi_{21}\psi_{20} & \psi_{21}\psi_{21} & \psi_{21}\psi_{22}\\ \psi_{22}\psi_{10} & \psi_{22}\psi_{11} & \psi_{22}\psi_{12} & \psi_{22}\psi_{20} & \psi_{22}\psi_{21} & \psi_{22}\psi_{22}\end{bmatrix}. \tag{4.2}$$

As we know, the support of $\psi_{m,n}$, the entries of vector $\Psi(t)$ are the intervals $\left[\frac{n-1}{2^k},\frac{n}{2^k}\right]$, therefore $\psi_{ij}\psi_{kl}=0$ if $i\neq k$. We also have

$$\psi_{i0}\psi_{ij}=\tfrac{2}{\sqrt{\pi}}\psi_{ij},$$
$$\psi_{i1}\psi_{i1}=\tfrac{2}{\sqrt{\pi}}\psi_{i0}+\sqrt{\tfrac{2}{\pi}}\,\psi_{i2}.$$

If we retain only the elements of $\Psi(t)$, then we have

$$\Psi(t)\Psi^T(t)=\frac{1}{\sqrt{\pi}}\begin{bmatrix}2\psi_{10} & 2\psi_{11} & 2\psi_{12} & 0 & 0 & 0\\ 2\psi_{11} & 2\psi_{10}+\sqrt{2}\psi_{12} & \sqrt{2}\psi_{11} & 0 & 0 & 0\\ 2\psi_{12} & \sqrt{2}\psi_{11} & 2\psi_{10} & 0 & 0 & 0\\ 0 & 0 & 0 & 2\psi_{20} & 2\psi_{21} & 2\psi_{22}\\ 0 & 0 & 0 & 2\psi_{21} & 2\psi_{20}+\sqrt{2}\psi_{22} & \sqrt{2}\psi_{21}\\ 0 & 0 & 0 & 2\psi_{22} & \sqrt{2}\psi_{21} & 2\psi_{22}\end{bmatrix}.$$

Therefore the 6×6 matrix $\tilde{C}$ in Eq.(4.1) can be written as

$$\tilde{C} = \begin{bmatrix} B_1 & 0 \\ \\ 0 & B_2 \end{bmatrix}, \tag{4.3}$$

where $B_i,\ i = 1,2$, are 3×3 matrices given by

$$B_i = \frac{1}{\sqrt{\pi}} \begin{bmatrix} 2c_{i0} & 2c_{i1} & 2c_{i2} \\ \\ 2c_{i1} & 2c_{i0} + \sqrt{2}c_{i2} & \sqrt{2}c_{i1} \\ \\ 2c_{i2} & \sqrt{2}c_{i1} & 2c_{i0} \end{bmatrix}, \tag{4.4}$$

where $c_{i,d}$, $d = 0,1,2$ taken from Eq.(2.7).

For general case, $\tilde{C}$ is a $2^k M \times 2^k M$ matrix in the form as

$$\tilde{C} = \begin{bmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & \dots & B_{2^k} \end{bmatrix}, \tag{4.5}$$

for B_i, $i = 1,2,\dots,2^k$ taken from Eq.(4.4).

5. Description of the Proposed Method

In this section, we will use the operational matrix of integration and product operation matrix of Chebyshev wavelets to solve Hammerstein integral equation of Fredholm and Volterra type. Firstly, we consider Fredholm Hammerstein integral equation as:

$$y(t) = f(t) + \int_0^1 K(t,x)[y(x)]^p \, dx, \tag{5.1}$$

where $f \in L^2[0,1]$, $K \in L^2([0,1]\times[0,1])$ and y is an unknown function [6] and $p \in Z^+$. Now we approximate $f(t)$, $y(t)$, $k(t,x)$ and $[y(t)]^p$ in the following way:

$$f(t) \simeq \Psi(t)^T F \, y(t) \simeq \Psi(t)^T Y, \tag{5.2}$$

$$K(t,x) \simeq \Psi(t)^T \mathbf{K} \Psi(t) \text{ and } [y(t)]^p \simeq \Psi(t)^T Y^*, \tag{5.3}$$

where Y^* is a column vector function of the elements of the vector Y [3, 4, 18].

By substituting the approximation function mentioned in Eq.(5.2) and Eq.(5.3) into Eq.(5.1) we obtain

$$\begin{aligned}
\Psi(t)^T \mathbf{Y} &= \Psi(t)^T F + \int_0^1 \Psi^T(t)\mathbf{K}\Psi(x)\Psi^T(x) Y^* \, dx \\
&= \Psi(t)^T F + \Psi^T(t) K \left(\int_0^1 \Psi(x)\Psi(x) \, dx \right) Y^* \\
&= \Psi(t)^T (F + \mathbf{K} Y^*),
\end{aligned}$$

then the required non-linear system of algebraic equation become

$$Y - \mathbf{K} Y^* = F. \tag{5.4}$$

Secondly, we consider the following Volterra Hammerstein type integral equation

$$y(t) = f(t) + \int_0^t K(t,x)[y(x)]^p \, dx, \tag{5.5}$$

In the light of (4.1),(5.2) and (5.3) we have:

$$\begin{aligned}
\int_0^t K(t,x)[y(x)]^p \, dx &\cong \int_0^t \Psi(t)^T \mathbf{K} \Psi(x)^T Y^* \, dx \\
&= \Psi^T(t) \mathbf{K} \int_0^t \Psi(x)\Psi^T(x) Y^* dx \\
&= \Psi^T(t) \mathbf{K} \int_0^t \tilde{Y}^{*T} \Psi(x) dx \\
&= \Psi^T(t) \mathbf{K} \tilde{Y}^{*T} P \Psi(t).
\end{aligned}$$

Then

$$\Psi^T(x_i)Y = f(t) + \Psi^T(t)\mathbf{K}\tilde{Y}^{*T}P\Psi(t). \tag{5.6}$$

By evaluating this equation in $2^{k-1}M$ points $\{t_i\}_{i=1}^{2^{k-1}M}$ in the interval $[0,1]$ we have a system of nonlinear equations:

$$\Psi^T(x_i)Y = F(t_i) + \Psi^T(t_i)\mathbf{K}\tilde{Y}^{*T}P\Psi(t_i), i = 1,2,\ldots,2^{k-1}M. \tag{5.7}$$

The nonlinear system of algebraic equations (5.4) and (5.7) can be solve by Newton's methods using mathematical software MATLAB.

6. Numerical experiments and discussion

In this section, we implement the proposed method for solving Hammerstein integral equation to achieve the effectiveness, the validity, the accuracy and support our theoretical discussion in the above sections. we consider four examples of Hammerstein integral equation of Fredohlm and Volterra type. All computations have been done with the software package MATLAB 2013a. The numerical results achieved by the proposed method are shown in Table 6.1-6.4 and graphically shown in Figure 6.1-6.4.

Example 6.1 Consider the Fredholm Hammerstein integral of second kind as [5]

$$y(t) = f(t) + \int_0^1 K(t,x)[y(x)]^2 dx, \quad 0 \le t \le 1, \tag{6.1}$$

where

$$f(t) = 1 + 3\sin^2(t) \quad \text{and} \quad K(t,x) = \begin{cases} -3\sin(t-x), & \text{for } x \in [0,1] \\ 0, & \text{for } x \in [t,1]. \end{cases}$$

The computational results obtained by the proposed method at $k = 2, M = 3$ and $k = 2, M = 4$ together with the exact solution $y(t) = \cos(t)$ of Example 6.1 are tabulated in Table 6.1 and graphically shown in Figure 6.1. Also we compare the obtained result with the result obtained by semi-orthogonal spline wavelets [15].

Table 6.1: Comparison of the approximate solution of Example 6.1 with exact and the Ref.[15] at different scales.

t	$OurResult$ $(k=2, M=3)$	$OurResult$ $(k=2, M=4)$	$Ref.[15]$ $(M=4)$	$Exact$
0.0	1.000000	1.000000	1.000000	1.000000
0.1	0.995204	0.995008	0.995012	0.995004
0.2	0.983211	0.983086	0.983077	0.983095
0.3	0.955495	0.955329	0.955324	0.955336
0.4	0.921190	0.921064	0.921066	0.921061
0.5	0.877698	0.877579	0.877575	0.877583
0.6	0.825426	0.825339	0.825343	0.825336
0.7	0.765012	0.764855	0.764859	0.764842
0.8	0.696884	0.696701	0.696694	0.696707
0.9	0.621803	0.621611	0.621603	0.621619

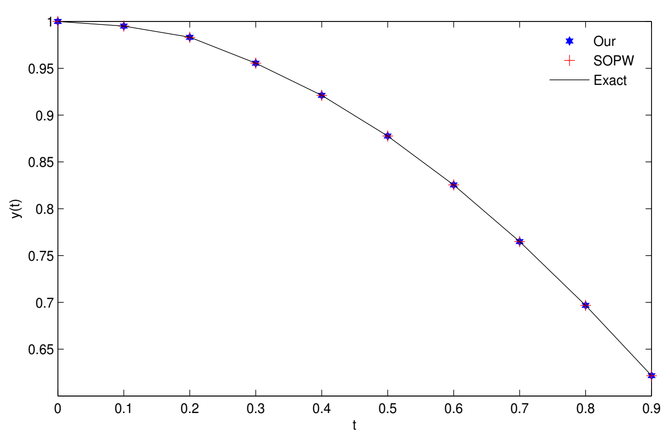

Fig.6.1 Comparison of Exact Solution with Approximate Solution and Ref.[15] for Example 6.1 at $k=2, M=4$.

Example 6.2 Consider the Fredholm Hammerstein integral of second kind [19]

$$y(t) = f(t) + \int_0^1 K(t,x)[y(x)]^3 dx, \quad 0 \le t \le 1, \tag{6.2}$$

where

$$f(t) = e^t - \frac{(1+2e^3)t}{9} \text{ and } K(t,x) = tx,$$

with exact solution $y(t) = e^t$. Table 6.2 and Figure 6.2 show the numerical results for Example 6.2 in comparison with wavelet Glaerkin methods [19].

Table 6.2: Comparison of the approximate solution of Example 6.2 with exact and the Ref.[19] at different scales.

t	$OurResult$ $(k=2, M=3)$	$OurResult$ $(k=2, M=4)$	$Ref.$[19] $(k=2, M=4)$	$Exact$
0.0	0.999943	0.999999	0.999956	1.000000
0.1	1.105225	1.105176	———	1.105171
0.2	1.221326	1.221409	1.221391	1.221403
0.3	1.349941	1.349865	———	1.349859
0.4	1.491965	1.491831	1.491845	1.491825
0.5	1.648849	1.648729	———	1.648721
0.6	1.822245	1.822126	1.822157	1.822119
0.7	2.013867	2.013759	———	2.013753
0.8	2.225639	2.225547	2.225517	2.225541
0.9	2.459698	2.459609	———	2.459603

Example 6.3 Consider the Volterra Hammerstein integral of second kind [19]

$$y(t) = f(t) + \int_0^t K(t,x)[y(x)]^2 dx, \quad 0 \le t \le 1, \tag{6.3}$$

where

$$f(t) = \left(1 - \frac{11}{9}t + \frac{2}{3}t^2 - \frac{1}{3}t^3 - \frac{2}{9}t^4\right) In(t+1) - \frac{1}{3}(t+t^3)(In(t+1))^2 - \frac{11}{9}t^2 + \frac{5}{18}t^3 - \frac{2}{27}t^4,$$

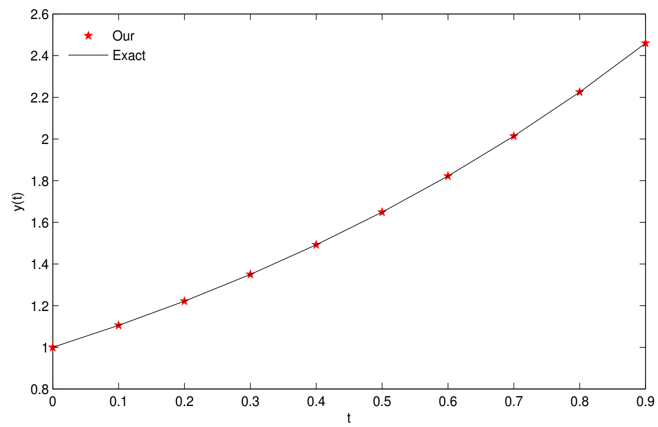

Fig.6.2 Approximate solution for Example 6.2 at $k = 2, M = 4$ with exact solution.

and $K(t,x) = tx^2$ with exact solution $y(t) = In(t+1)$. The numerical results of Example 6.3 are presented in Table 6.3 at different scale and graphically shown in Figure 6.3 at $k = 2$, $M = 4$.

Table 6.3 Comparison of the approximate solution of Example 6.3 with exact and the Ref.[19] at different scales.

t	$OurResult$ $(k=2, M=3,)$	$OurResult$ $(k=2, M=4)$	$Ref.$[19] $(k=2, M=4)$	$Exact$
0.0	0.000000	0.000000	0.000000	0.000000
0.1	0.095419	0.095321	———	0.095310
0.2	0.182427	0.182332	0.182363	0.182322
0.3	0.262456	0.262377	———	0.262364
0.4	0.336564	0.336485	0.336442	0.336472
0.5	0.405532	0.405474	———	0.405465
0.6	0.470125	0.470014	0.469990	0.470004
0.7	0.530731	0.530641	———	0.530628
0.8	0.587861	0.587799	0.587803	0.587787
0.9	0.641944	0.641863	———	0.641854

Example 6.4 Consider the Volterra Hammerstein integral of second kind [19]

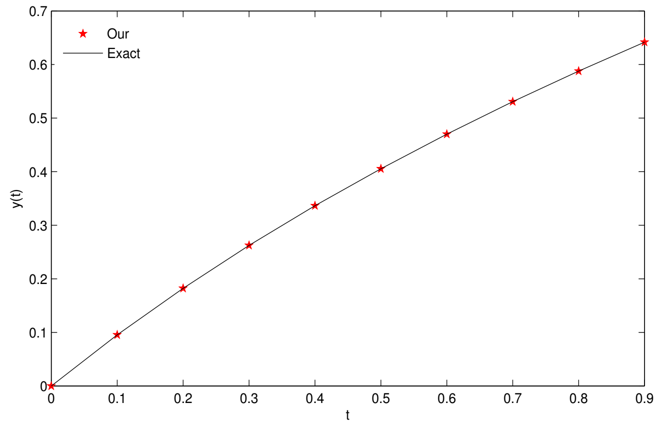

Fig.6.3 Approximate solution for Example 6.3 at $k = 2, M = 4$ with exact solution.

$$y(t) = f(t) + \int_0^t K(t,x)[y(x)]^3 dx, \quad 0 \leq t \leq 1, \tag{6.4}$$

where

$$f(t) = \frac{116}{3} e^{2t} - \left(\frac{116}{3} + 39t + 18t^2 + 9t^3 \right) e^t + 3t - 1$$

and

$$K(t,x) = -\frac{1}{3} e^{2t-x}$$

with exact solution $y(t) = 3t - 1$.

The numerical results in Table 6.4, establishes the facts given in Example 6.4. Figure 6.4 shows the comparison of numerical results and exact solutions of Example 6.4 at $k = 2, M = 4$.

t	$OurResult$ $(k=2, M=3)$	$OurResult$ $(k=2, M=4)$	$Ref.$[19] $(k=2, M=4)$	$Exact$
0.0	−0.998839	−0.999929	−0.998839	−1.000000
0.1	−0.672545	−0.695323	———	−0.700000
0.2	−0.380013	−0.399801	−0.399243	−0.400000
0.3	−0.097211	−0.099399	———	−0.100000
0.4	0.188097	0.199711	0.199299	0.200000
0.5	0.482341	0.493662	———	0.500000
0.6	0.778990	0.799803	0.800185	0.800000
0.7	1.129910	1.116001	———	1.100000
0.8	1.377521	1.399926	1.399874	1.400000
0.9	1.689989	1.700143	———	1.700000

Table 6.4 Comparison of the approximate solution of Example 6.4 with exact and the Ref.[19] at different scales.

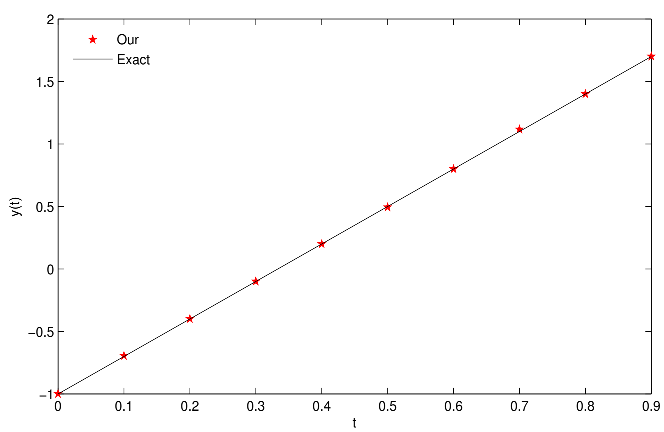

Fig.6.4 Approximate solution for Example 6.4 at $k=2, M=4$ with exact solution.

7. Conclusion

In this paper, we have proposed an efficient and accurate method based on Chebyshev wavelets to solve both Fredholm and Volterra Hammerstein integral equations arising in different field of sciences, engineering and technology. Comparisons between our approximate solutions of the problems with its exact solutions and with the approximate solutions achieved by other methods were introduced to confirm the validity and accuracy of our scheme. The numerical experiments confirm that the Chebyshev wavelet method is superior to other existing ones and is highly accurate and can be applicable to Hammerstein integral equation. The main advantage of this Chebyshev wavelet method is that it transfers the whole scheme into a system of algebraic equations for which the computation is easy and simple. In addition, other pretty features of this scheme are its simplicity, applicability and less computational effort.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] S. Abbasbandy, Numerical solution of integral equation: Homotopy perturbation method and Adomians decomposition method, Appl. Math. Comput. 173 (2006), 493-500.

[2] H. Adibi, P. Assari, On the numerical solution of weakly singular Fredholm integral equations of the second kind using Legendre wavelets, J. Vib. Control. 17 (2011), 689-698.

[3] E. Babolian, F. Fattahzadeh, Numerical computation method in solving integral equations by using Chebyshev wavelet operational matrix of integration, Appl. Math. Comp. 188(2007), 1016-1022.

[4] E. Babolian, F. Fattahzadeh, Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration, Appl. Math. Comp. 188 (2007), 417-426.

[5] H. Brunner, Implicitly linear collocation methods for nonlinear Volterra equations, Appl.Numer. Math. 9(3-5) (1992), 235-247.

[6] L.M. Delves and J.L. Mohammed, Computational methods for integral equatos, Cambridge university press, Oxford, 1983.

[7] A. Golbabai, M. Javidi, Modifed Homotopy Perturbation method for solving non-linear Fredholm integral equations, Chaos. Sols. Fract. 40 (2009), 1408-1412.

[8] J. S. Gu, W. S. Jiang, The Haar wavelets operational matrix of integration, Int. J. Syst. Sci. 27(1996), 623-628.

[9] G. Q. Han, Asymptotic error expansion of a collocation-type method for Volterra-Hammerstein integral equations, Appl. Numer. Math, 13(5) (1993), 357-369.

[10] S. Javadi, J.Saeidian and F. Safari, Legendre wavelet method for solving Hammerstein integral equations of the second kind. Theo. Appr. Appl. 9(2) (2013), 37-55.

[11] M. T. Kajani, A. H. Vencheh, and M. Ghasemi, The Chebyshev wavelets operational matrix of integration and product operation matrix, Int. J. Comp. Math. 86 (2009), 1181-1125.

[12] H. Kaneko, R. D. Noren, B. Novaprateep, Wavelet applications to the Petrov-Galerkin method for Hammerstein equations, Appl. Numer. Math. 45 (2003), 255-273.

[13] R. P. Kanwal. Linear integral equations theory and technique, Academic Press; New York and London. 1971.

[14] K. Kumar, I. H. Sloan, A new collocation-type method for Hammerstein integral equations, Math. Comp. 48(178) (1987), 585-593.

[15] M. Lakestani, M. Razzagi, M. Dehghan, Solution of nonlinear Fredholm-Hammerstein integral equations by using semiorthogonal spline wavelets, Math. Prob. Engi. 1 (2005), 113-121.

[16] L. J. Lardy, A variation of Nyström's method for Hammerstein equations, J. Integ. Eqs. 3 (1981), 43-60.

[17] U. Lepik, E. Tamme, Solution of nonlinear Fredholm integral equations via the Haar wavelet method, Proc. Estonian Acad. Sci. Phys. Math. 56 (2007), 17-27.

[18] Y. LI, Solving a nonlinear fractional differential equation using Chebyshev wavelets, Comm. Non. Sci. Num. Simu. 15 (2010), 2284-2292.

[19] Y. Mahmoudi, Wavelet Galerkin method for numerical solution of nonlinear integral equation, Appl. Math. Comp. 167 (2005), 1119-1129.

[20] K. Maleknejad, H. Derili, The collocation method for Hammerstein equations by Daubechies wavelets, Appl. Math. Comput. 172 (2006), 846-864.

[21] M. Razzaghi, S. Yousefi, The Legendre wavelets operational matrix of integration, Int. J. Syst. Sci. 32 (2001), 495-502.

[22] M. Razzaghi, S. Yousefi, Legendre wavelets method for the solution of nonlinear problems in the calculus of variations, Mathematical and Computer Modelling 34 (2001), 45-54.

[23] F. G. Tricomi, Integral Equations, Dover Publications, New York, 1985.

[24] M. T. Kajani, A. H. Vencheh, M. Ghasemi, The Chebyshev wavelets operational matrix of integration and product operation matrix, Int. J. Comp. Math. 86(7) (2009), 1118-1125.