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# $\sigma$-CONVERGENT DIFFERENCE SEQUENCE SPACES OF SECOND ORDER DEFINED BY ORLICZ FUNCTION

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**Abstract.** In this paper, we introduce the sequence space $V_\sigma(M,p,r,\triangle^2)$, where $M$ is an Orlicz function, $p=(p_m)$ is any sequence of strictly positive real numbers and $r \geq 0$ and study some of the properties and inclusion relations that arise on the said space.

**Keywords:** invariant mean; paranorm; Orlicz function and difference sequences.

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## 1. Introduction

Let N, R and C be the sets of all natural, real and complex numbers respectively.

We write

$$\omega = \{x=(x_k) : x_k \in R \ or \ C\ \},$$

the space of all real or complex sequences.

Let $\ell_\infty$, $c$ and $c_0$ denote the Banach spaces of bounded, convergent and null sequences respectively.

The following subspaces of $\omega$ were first introduced and discussed by Maddox [12-13].

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$\ell(p) = \{x \in \omega : \sum_k |x_k|^{p_k} < \infty\}$,
$\ell_\infty(p) = \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\}$,
$c(p) = \{x \in \omega : \lim_k |x_k - l|^{p_k} = 0, \text{ for some } l \in C\}$,
$c_0(p) = \{x \in \omega : \lim_k |x_k|^{p_k} = 0\}$,
where $p = (p_k)$ is a sequence of stricly positive real numbers.

The concept of paranorm is closely related to linear metric spaces.It is a generalization of that of absolute value.(see[13])

Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if for all $x, y, z \in X$,

(PI) $g(x) = 0 \; if \; x = \theta$,

(P2) $g(-x) = g(x)$,

(P3) $g(x+y) \leq g(x) + g(y)$,

(P4) If $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda \; (n \to \infty)$ and $x_n, a \in X$ with $x_n \to a \; (n \to \infty)$ , in the sense that $g(x_n - a) \to 0 \; (n \to \infty)$ , in the sense that $g(\lambda_n x_n - \lambda a) \to 0 \; (n \to \infty)$.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri[10] used the idea of Orlicz functions to construct the sequence space

$$\ell_M = \{x \in \omega : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty, \text{for some } \rho > 0\}$$

The space $\ell_M$ is a Banach space with the norm

$$||x|| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1\}$$

The space $\ell_M$ is closely related to the space $\ell_p$ which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

An Orlicz function $M$ is said to satisfy $\triangle_2$ condition for all values of $x$ if there exists a constant $K > 0$ such that $M(Lx) \leq KLM(x)$ for all values of $L > 1$.

A sequence space $E$ is said to be solid or normal if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequence of scalars $(\alpha_k)$ with $|\alpha_k| < 1$ for all $k \in N$.

For Orlicz function and related results see([2],[8],[17]).

Let $\sigma$ be an injection on the set of positive integers N into itself having no finite orbits and T be the operator defined on $\ell_\infty$ by $T(x_k) = (x_{\sigma(k)})$.
A positive linear functional $\Phi$, with $||\Phi|| = 1$, is called a $\sigma$-mean or an invariant mean if $\Phi(x) = \Phi(Tx)$ for all $x \in \ell_\infty$.
A sequence $x$ is said to be $\sigma$-convergent, denoted by $x \in V_\sigma$, if $\Phi(x)$ takes the same value, called $\sigma - \lim x$, for all $\sigma$-means $\Phi$. We have

$$V_\sigma = \{x = (x_k) : \sum_{m=1}^{\infty} t_{m,n}(x) = L \text{ uniformly in n, L} = \sigma - \lim x\},$$

where for $m \geq 0, n > 0$.

$$t_{m,n}(x) = \frac{x_k + x_{\sigma(k)} + ..... + x_{\sigma^m(k)}}{m+1}, \text{and } t_{-1,n} = 0.$$

where $\sigma^m(k)$ denotes the m$^{th}$ iterate of $\sigma$ at n. In particular, if $\sigma$ is the translation, a $\sigma$-mean is often called a Banach limit and $V_\sigma$ reduces to f, the set of almost convergent sequences.

Subsequently the spaces of invariant mean and Orlicz function have been studied by various authors. See( [1],[11],[15],[16],[19]).

The idea of Difference sequence sets

$$X_\triangle = \{x = (x_k) \in \omega : \triangle x = (x_k - x_{k+1}) \in X\},$$

where $X = \ell_\infty,\ c$ or $c_0$ was introduced by Kizmaz [9].
Kizmaz [9] defined the sequence spaces,

$$\ell_\infty(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in \ell_\infty\},$$

$$c(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c\},$$

$$c_0(\triangle)=\{x=(x_k)\in\omega:(\triangle x_k)\in c_0\},$$

where $\triangle x=(x_k-x_{k+1})$. These are Banach spaces with the norm

$$||x||_{\triangle}=|x_1|+||\triangle x||_{\infty}.$$

After then Mikael [14] defined the sequence spaces,

$$\ell_\infty(\triangle^2)=\{x=(x_k)\in\omega:(\triangle^2 x_k)\in\ell_\infty\},$$

$$c(\triangle^2)=\{x=(x_k)\in\omega:(\triangle^2 x_k)\in c\},$$

$$c_0(\triangle^2)=\{x=(x_k)\in\omega:(\triangle^2 x_k)\in c_0\},$$

and showed that these are Banach spaces with norm

$$||x||_{\triangle}=|x_1|+|x_2|+||\triangle^2 x||_{\infty}.$$

For difference sequences see([3-5],[8],[9]).

Recently Ebadullah[6] introduced and studied the sequence space

$$V_\sigma(M,p,r)=\{x=(x_k):\sum_{m=1}^{\infty}\frac{1}{m^r}[M(\frac{|t_{m,n}(x)|}{\rho})]^{p_m}<\infty \text{ uniformly in n, } \rho>0\}.$$

Where $M$ is an Orlicz function, $p=(p_m)$ is any sequence of strictly positive real numbers and $r\geq 0$ .

After then Ebadullah[7] introduced the sequence space

$$V_\sigma(M,p,r,\triangle)=\{x=(x_k):\sum_{m=1}^{\infty}\frac{1}{m^r}[M(\frac{|t_{m,n}(\triangle x)|}{\rho})]^{p_m}<\infty \text{ uniformly in n, } \rho>0\}.$$

and discussed the following sequence spaces ;

For $M(x)=x$ we get

$$V_\sigma(p,r,\triangle)=\{x=(x_k):\sum_{m=1}^{\infty}\frac{1}{m^r}|t_{m,n}(\triangle x)|^{p_m}<\infty \text{ uniformly in n}\}$$

For $p_m = 1$, for all m, we get

$$V_\sigma(M,r,\triangle) = \{x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|t_{m,n}(\triangle x)|}{\rho})] < \infty \text{ uniformly in n}, \rho > 0\}$$

For r = 0 we get

$$V_\sigma(M,p,\triangle) = \{x = (x_k) : \sum_{m=1}^{\infty} [M(\frac{|t_{m,n}(\triangle x)|}{\rho})]^{p_m} < \infty \text{ uniformly in n}, \rho > 0\}$$

For $M(x) = x$ and r=0 we get

$$V_\sigma(p,\triangle) = \{x = (x_k) : \sum_{m=1}^{\infty} |t_{m,n}(\triangle x)|^{p_m} < \infty \text{ uniformly in n}, \rho > 0\}$$

For $p_k = 1$, for all m and r=0, we get

$$V_\sigma(M,\triangle) = \{x = (x_k) : \sum_{m=1}^{\infty} [M(\frac{|t_{m,n}(\triangle x)|}{\rho})] < \infty \text{ uniformly in n}, \rho > 0\}$$

For $M(x) = x$, $p_m = 1$, for all m and r=0, we get

$$V_\sigma(\triangle x) = \{x = (x_k) : \sum_{m=1}^{\infty} |t_{m,n}(\triangle x)| < \infty \text{ uniformly in n}\}.$$

## 2. Main results

In this article we introduce the sequence space

$$V_\sigma(M,p,r,\triangle^2) = \{x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|t_{m,n}(\triangle^2 x)|}{\rho})]^{p_m} < \infty \text{ uniformly in n}, \rho > 0\}.$$

Where $M$ is an Orlicz function, $p = (p_m)$ is any sequence of strictly positive real numbers and $r \geq 0$ .

Now we define the sequence spaces as follows;

For $M(x) = x$ we get

$$V_\sigma(p, r, \triangle^2) = \{x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} |t_{m,n}(\triangle^2 x)|^{p_m} < \infty \text{ uniformly in n}\}$$

For $p_m = 1$, for all m, we get

$$V_\sigma(M, r, \triangle^2) = \{x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} [M(\frac{|t_{m,n}(\triangle^2 x)|}{\rho})] < \infty \text{ uniformly in n, } \rho > 0\}$$

For r = 0 we get

$$V_\sigma(M, p, \triangle^2) = \{x = (x_k) : \sum_{m=1}^{\infty} [M(\frac{|t_{m,n}(\triangle^2 x)|}{\rho})]^{p_m} < \infty \text{ uniformly in n, } \rho > 0\}$$

For $M(x) = x$ and r=0 we get

$$V_\sigma(p, \triangle^2) = \{x = (x_k) : \sum_{m=1}^{\infty} |t_{m,n}(\triangle^2 x)|^{p_m} < \infty \text{ uniformly in n, } \rho > 0\}$$

For $p_k = 1$, for all m and r=0, we get

$$V_\sigma(M, \triangle^2) = \{x = (x_k) : \sum_{m=1}^{\infty} [M(\frac{|t_{m,n}(\triangle^2 x)|}{\rho})] < \infty \text{ uniformly in n, } \rho > 0\}$$

For $M(x) = x$, $p_m = 1$, for all m and r=0, we get

$$V_\sigma(\triangle^2 x) = \{x = (x_k) : \sum_{m=1}^{\infty} |t_{m,n}(\triangle^2 x)| < \infty \text{ uniformly in n}\}.$$

**Theorem 2.1.** The sequence space $V_\sigma(M, p, r, \triangle^2)$ is a linear space over the field C of complex numbers.

**Proof.** Let $x, y \in V_\sigma(M, p, r, \triangle^2)$ and $\alpha, \beta \in C$ then there exists positive numbers $\rho_1$ and $\rho_2$ such that

$$\sum_{m=1}^{\infty} \frac{1}{m^r} [M(\frac{|t_{m,n}(\triangle^2 x)|}{\rho_1})]^{p_m} < \infty,$$

and

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|t_{m,n}(\triangle^2 y)|}{\rho_2})]^{p_m} < \infty$$

uniformly in n.

Define $\rho_3$ = max$(2|\alpha|\rho_1, 2|\beta|\rho_2)$.

Since M is non decreasing and convex we have

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\alpha t_{m,n}(\triangle^2 x) + \beta t_{m,n}(\triangle^2 y)|}{\rho_3})]^{p_m}$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|\alpha t_{m,n}(\triangle^2 x)|}{\rho_3} + \frac{|\beta t_{m,n}(\triangle^2 y)|}{\rho_3})]^{p_m}$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{m^r}\frac{1}{2}[M(\frac{t_{m,n}(\triangle^2 x)}{\rho_1}) + M(\frac{t_{m,n}(\triangle^2 y)}{\rho_2})] < \infty$$

uniformly in n.

This proves that $V_\sigma(M, p, r, \triangle^2)$ is a linear space over the field C of complex numbers.

**Theorem 2.2.** For any Orlicz function M and a bounded sequence $p = (p_m)$ of strictly positive real numbers, $V_\sigma(M, p, r, \triangle^2)$ is a paranormed space with

$$g(x) = \inf_{n\geq 1}\{\rho^{\frac{p_n}{H}} : (\sum_{m=1}^{\infty} \frac{1}{m^r}[M(\frac{|t_{m,n}(\triangle^2 x)|}{\rho})]^{p_m})^{\frac{1}{H}} \leq 1, \text{ uniformly in n}\}$$

where H = max(1, sup$p_m$).

**Proof.** It is clear that $g(\triangle^2 x) = g(-\triangle^2 x)$.

Since $M(0) = 0$, we get

inf$\{\rho^{\frac{p_m}{H}}\}$ = 0, for $x = 0$

Now for $\alpha$=$\beta$=1, we get

$g(\triangle^2 x + \triangle^2 y) \leq g(\triangle^2 x) + g(\triangle^2 y)$.

For the continuity of scalar multiplication let $l \neq 0$ be any complex number. Then by the definition we have

$$g(l\triangle^2 x) = \inf_{n\geq 1}\{\rho^{\frac{p_n}{H}} : (\sum_{m=1}^{\infty}\frac{1}{m^r}[M(\frac{|t_{m,n}(l\triangle^2 x)|}{\rho})]^{p_m})^{\frac{1}{H}} \leq 1, \text{ uniformly in n}\}$$

$$g(l\triangle^2 x) = \inf_{n\geq 1}\{(|l|s)^{\frac{p_n}{H}} : (\sum_{m=1}^{\infty}\frac{1}{m^r}[M(\frac{|t_{m,n}(l\triangle^2 x)|}{(|l|s)})]^{p_m})^{\frac{1}{H}} \leq 1, \text{ uniformly in n}\}$$

where $s = \frac{\rho}{|l|}$.
Since $|l|^{p_m} \leq \max(1,|l|^H)$, we have

$$g(l\triangle^2 x) \leq max(1,|l|^H)\inf_{n\geq 1}\{s^{\frac{p_n}{H}} : (\sum_{m=1}^{\infty}\frac{1}{m^r}[M(\frac{|t_{m,n}(\triangle^2 x)|}{(|l|s)})]^{p_m})^{\frac{1}{H}} \leq 1, \text{ uniformly in n}\}$$

$$g(\triangle^2 lx) \leq max(1,|l|^H)g(\triangle^2 x)$$

Therefore $g(\triangle^2 x)$ converges to zero when $g(\triangle^2 x)$ converges to zero in $V_\sigma(M,p,r,\triangle^2)$.

Now let $x$ be fixed element in $V_\sigma(M,p,r,\triangle^2)$. There exists $\rho > 0$ such that

$$g(\triangle^2 x) = \inf_{n\geq 1}\{\rho^{\frac{p_n}{H}} : (\sum_{m=1}^{\infty}\frac{1}{m^r}[M(\frac{|t_{m,n}(\triangle^2 x)|}{\rho})]^{p_m})^{\frac{1}{H}} \leq 1, \text{ uniformly in n}\}$$

.

Now

$$g(l\triangle^2 x) = \inf_{n\geq 1}\{\rho^{\frac{p_n}{H}} : (\sum_{m=1}^{\infty}\frac{1}{m^r}[M(\frac{|t_{m,n}(l\triangle^2 x)|}{\rho})]^{p_m})^{\frac{1}{H}} \leq 1, \text{ uniformly in n}\} \to 0 \text{ as } l \to 0.$$

This completes the proof.

**Theorem 2.3.** Suppose that $0 < p_m < t_m < \infty$ for each $m \in N$ and $r > 0$. Then

(a) $V_\sigma(M, p, \triangle^2) \subseteq V_\sigma(M, t, \triangle^2)$.

(b) $V_\sigma(M, \triangle^2) \subseteq V_\sigma(M, r, \triangle^2)$

**Proof.**(a) Suppose that $x \in V_\sigma(M, p, \triangle^2)$.

This implies that $[M(\frac{|t_{i,n}(\triangle^2 x)|}{\rho})]^{p_m}) \leq 1$

for sufficiently large value of i, say $i \geq m_0$ for some fixed $m_0 \in N$.

Since $M$ is non decreasing, we have

$$\sum_{m=m_0}^{\infty} [M(\frac{|t_{i,n}(\triangle^2 x)|}{\rho})]^{t_m} \leq \sum_{m=m_0}^{\infty} [M(\frac{|t_{i,n}(\triangle^2 x)|}{\rho})]^{p_m} < \infty.$$

Hence $x \in V_\sigma(M, t, \triangle^2)$.

(b) The proof is trivial.

**Corollary 2.4.** $0 < p_m \leq 1$ for each m, then $V_\sigma(M, p, \triangle^2) \subseteq V_\sigma(M, \triangle^2)$

If $p_m \geq 1$ for all m , then $V_\sigma(M, \triangle^2) \subseteq V_\sigma(M, p, \triangle^2)$.

**Theorem 2.5.** The sequence space $V_\sigma(M, p, r, \triangle^2)$ is solid.

**Proof.** Let $x \in V_\sigma(M, p, r, \triangle^2)$. This implies that

$$\sum_{m=1}^{\infty} \frac{1}{m^r} [M(\frac{|t_{m,n}(\triangle^2 x)|}{\rho})]^{p_m} < \infty.$$

Let $\alpha_m$ be a sequence of scalars such that $|\alpha_m| \leq 1$ for all $m \in N$. Then the result follows from the following inequality.

$$\sum_{m=1}^{\infty} \frac{1}{m^r} [M(\frac{|\alpha_m t_{i,n}(\triangle^2 x)|}{\rho})]^{p_m} \leq \sum_{m=1}^{\infty} \frac{1}{m^r} [M(\frac{|t_{i,n}(\triangle^2 x)|}{\rho})]^{p_m} < \infty.$$

Hence $\alpha x \in V_\sigma(M,p,r,\triangle^2)$ for all sequence of scalars $(\alpha_m)$ with $|\alpha_m| \leq 1$ for all $m \in N$ whenever $x \in V_\sigma(M,p,r,\triangle^2)$.

**Corollary 2.6.** The sequence space $V_\sigma(M,p,r,\triangle^2)$ is monotone.

**Theorem 2.7.** Let $M_1, M_2$ be Orlicz function satisfying $\triangle_2$ condition and $r, r_1, r_2 \geq 0$. Then we have

(a) If $r > 1$ then $V_\sigma(M_1,p,r,\triangle^2) \subseteq V_\sigma(M0M_1,p,r,\triangle^2)$,

(b) $V_\sigma(M_1,p,r,\triangle^2) \cap V_\sigma(M_2,p,r,\triangle^2) \subseteq V_\sigma(M_1+M_2,p,r,\triangle^2)$,

(c) If $r_1 \leq r_2$ then $V_\sigma(M,p,r_1,\triangle^2) \subseteq V_\sigma(M,p,r_2,\triangle^2)$.

**Proof.** (a) Since $M$ is continuous at 0 from right, for $\varepsilon > 0$ there exists $0 < \delta < 1$ such that $0 \leq c \leq \delta$ implies $M(c) < \varepsilon$.

If we define

$$I_1 = \{m \in N : M_1(\frac{|t_{m,n}(\triangle^2 x)|}{\rho}) \leq \delta \text{ for some } \rho > 0\},$$

$$I_2 = \{m \in N : M_1(\frac{|t_{m,n}(\triangle^2 x)|}{\rho}) > \delta \text{ for some } \rho > 0\},$$

when

$$M_1(\frac{|t_{m,n}(\triangle^2 x)|}{\rho}) > \delta$$

we get

$$M(M_1(\frac{|t_{m,n}(\triangle^2 x)|}{\rho})) \leq \{\frac{2M(1)}{\delta}\} M_1(\frac{|t_{m,n}(\triangle^2 x)|}{\rho})$$

Hence for $x \in V_\sigma(M_1,p,r,\triangle^2)$ and $r > 1$

$$\sum_{m=1}^{\infty} \frac{1}{m^r}[M0M_1(\frac{|t_{m,n}(\triangle^2 x)|}{\rho})]^{p_m} = \sum_{m \in I_1} \frac{1}{m^r}[M0M_1(\frac{|t_{m,n}(\triangle^2 x)|}{\rho})]^{p_m} + \sum_{m \in I_2} \frac{1}{m^r}[M0M_1(\frac{|t_{m,n}(\triangle^2 x)|}{\rho})]^{p_m}.$$

$$\sum_{m=1}^{\infty}\frac{1}{m^r}[M0M_1(\frac{|t_{m,n}(\triangle^2 x)|}{\rho})]^{p_m} \leq max(\varepsilon^h,\varepsilon^H)\sum_{m=1}^{\infty}\frac{1}{m^r}+max(\{\frac{2M_1}{\delta}\}^h,\{\frac{2M_1}{\delta}\}^H)$$

$$\text{where } 0 < h = \inf p_m \leq p_m \leq H = \sup_m p_m < \infty$$

## Conflict of Interests

The author declares that there is no conflict of interests.

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