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A CHARACTERIZATION OF ONE-SIDED BEST SIMULTANEOUS L_1-APPROXIMATION

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Abstract. In this paper, we give a simplified proof of the characterization theorem of a one-sided L_1-approximation in [2] and the continuity of the function $F \to C_{M(F)}(F)$ in [3].

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1. Introduction

Let X be a compact Hausdorff space, $C(X)$ be the space of real-valued continuous functions on X. For each $f \in C(X)$, define $\|f\|_1 = \int_X |f| d\mu$, where μ is an admissible measure defined on X, that is, $\mu(O) > 0$ for every non-empty open set $O \subset X$. Let $C_1(X)$ be the space $C(X)$ equipped with the norm $\|\cdot\|_1$.

Let F be a compact subset of $C_1(X)$ and M be a finite-dimensional linear subspace of $C_1(X)$. Define

$$M(F) = \{g \in M \mid g \leq f \text{ for all } f \in F\} = \bigcap_{f \in F} \{g \in M \mid g \leq f\},$$

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where $g \leq f$ if and only if $g(x) \leq f(x)$ almost everywhere in X. Denote by $d(F, M(F)) := \inf_{g \in M(F)} \max_{f \in F} \|f - g\|_1$ and $C_{M(F)} = \{g \in M(F) \mid \max_{f \in F} \|f - g\|_1 = d(F, M(F))\}$. The elements of $C_{M(F)}(F)$ are called one-sided simultaneous best L_1-approximants of F in M. The set $M(F)$ is a closed convex subset of the finite dimensional subspace M of X. In order to ensure that the set $M(F)$ is non-empty, it is enough to assume that M contains a strictly positive function. In fact, $M(F) \neq \emptyset$ if and only if M contains a positive function. Every bounded set has a one-sided best simultaneous approximation in the closed convex set $M(F)$ of M and the set function $F \to C_{M(F)}(F)$ is continuous on $B[C_1(X)]$ with a condition that the sets $M(\cdot)$ are equal, where $B[C_1(X)]$ is the space of non-empty bounded subsets in the space $C_1(X)$ and $C[C_1(X)]$ the family of non-empty compact subsets in the space $C_1(X)$.

Definition. The Hausdorff metric on $B[C_1(X)]$ is defined by

$$H(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}.$$

The motivation is the one-sided best approximation of an element, studied by [1] and the parametric approximation in [4].

2. Main results

Theorem 1. *Let M and $M(F)$ be as defined above and $g^* \in M(F)$. Then $g^* \in C_{M(F)}(F)$ if and only if $\sup_{g \in M(F)} \int_X g d\mu = \int_X g^* d\mu$.*

Proof. Assume that $g^* \in C_{M(F)}(F)$. Then, for each $g \in M(F)$, $\max_{f \in F} \|f - g^*\|_1 \leq \max_{f \in F} \|f - g\|_1$. Hence $\max_{f \in F} \int_X |f - g^*| d\mu \leq \max_{f \in F} \int_X |f - g| d\mu$.

$\Leftrightarrow \max \int_X (f - g^*) d\mu \leq \max_{f \in F} \int_X (f - g) d\mu$

$\Leftrightarrow \max_{f \in F} \int_X f d\mu - \int_X g^* d\mu \leq \max_{f \in F} \int_X f d\mu - \int_X g d\mu$

$\Leftrightarrow \int_X g d\mu \leq \int_X g^* d\mu$. Thus $\sup_{g \in M(F)} \int_X g d\mu = \int_X g^* d\mu$.

Conversely, let $g^* \in M(F)$ and $\sup_{g \in M(F)} \int_X g d\mu = \int_X g^* d\mu$. Then, for all $g \in M(F)$,

$\max_{f \in F} \|f - g\|_1 = \max_{f \in F} \int_X |f - g| d\mu = \max_{f \in F} \int_X (f - g) d\mu$

$= \max_{f\in F} \int_X f d\mu - \int_X g d\mu \geq \max_{f\in F} \int_X f d\mu - \int_X g^* d\mu = \max_{f\in F} \int_X (f-g^*) d\mu$
$= \max_{f\in F} \int_X |f-g^*| d\mu = \max_{f\in F} \|f-g^*\|_1$. Hence $g^* \in C_{M(F)}(F)$. □

Denote

$$Z(f-g) := \{x \in X | f(x) = g(x)\}.$$

Theorem 2. *Let M be a finite-dimensional subspace of $C_1(X)$. Suppose $\int_X g d\mu \neq 0$ for some $g \in M$ and that there is a $g_0 \in M$ such that $g_0 < f$ on X for all $f \in F$. Let $g^* \in M(F)$. Then $g^* \in C_{M(F)}(F)$ if and only if, for all $g \in M$ with $g \leq 0$ on $\bigcup_{f\in F} Z(f-g^*)$, we have $\int_X g d\mu \leq 0$.*

Proof. Let $g^* \in C_{M(F)}(F)$. Then, there is an $h \in M$ such that $\int_X h d\mu > 0$. If $\bigcup_{f\in F} Z(f-g^*) = \emptyset$, then, there is an $\epsilon > 0$ such that $g^* + \epsilon h \leq f$ for each $f \in F$. Hence, $g^* + \epsilon h \in M(F)$ and

$$\int_X (g^* + \epsilon h) d\mu = \int_X g^* d\mu + \epsilon \int_X h d\mu > \int_X g^* d\mu.$$

This contradicts the fact that $g^* \in C_{M(F)}(F)$. Therefore $\bigcup_{f\in F} Z(f-g^*) \neq \emptyset$.

Assume that there is a $g \in M$ with $g \leq 0$ on $\bigcup_{f\in F} Z(f-g^*)$ and $\int_X g d\mu > 0$. Now by hypothesis, there is a $g_0 \in M$ such that $g_0 < f$ on X for each $f \in F$. Then, $\widehat{g} = g^* - g_0 > 0$ on $\bigcup_{f\in F} Z(f-g^*)$. Hence, there is a $\delta > 0$ such that $g - \delta\widehat{g} < 0$ on $\bigcup_{f\in F} Z(f-g^*)$ and $\int_X (g-\delta\widehat{g}) d\mu = \int_X g d\mu - \delta \int_X \widehat{g} d\mu > 0$. For each $f \in C_1(X)$, let $J(f) := \{x \in X \mid f(x) < 0\}$. Then $\bigcup_{f\in F}(f-g^*) \subset J(g-\delta\widehat{g})$. The set $X \setminus J(g-\delta\widehat{g})$ is compact. Hence there is a constant $m > 0$ such that $m \leq f - g^*$ on $X \setminus J(g-\delta\widehat{g})$ and there is a constant M such that $g - \delta\widehat{g} \leq M$ on X. Let $\epsilon = \frac{m}{M}$ and $\widehat{h} = g - \delta\widehat{g}$. Then $g^* + \epsilon\widehat{h} \in M(F)$. Taking integral, we obtain $\int_X g^* d\mu < \int_X \widehat{h} d\mu$, which contradicts $g^* \in C_{m(F)}(F)$.

Conversely, let $g \in M(F)$. Now for all $x \in \bigcup_{f\in F} Z(f-g^*)$, there is an $f \in F$ such that $f(x) = g^*(x)$. Hence $g(x) \leq g^*(x)$. Therefore $g - g^* \leq 0$ on $\bigcup_{f\in F} Z(f-g^*)$. By assumption

$\int_X (g - g^*)d\mu \leq 0$ which implies that $\int_X gd\mu \leq \int_X g^*d\mu$. Thus, $\sup_{g\in M(F)} \int_X gd\mu = \int_X g^*d\mu$ and hence $g^* \in C_{M(F)}(F)$. □

Lemma 3. [2] *Let M be an $n-$dimensional subspace of $C(X)$ and assume that $\int_X gd\mu \neq 0$ for some $g \in M$. Let K be a closed subset of X with the property that, if $g \in M$ satisfies $g(x) \leq 0$ for all $x \in K$, then $\int_X gd\mu < 0$. Then there exist points $x_1, x_2, \ldots, x_k \in K$, $1 \leq k \leq n$ and positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\int_X gd\mu = \sum_{i=1}^{k} \lambda_i g(x_i)$ for each $g \in M$.*

Theorem 4. *Let M be an $n-$dimensional linear subspace of $C_1(X)$ such that $\int_X gd\mu \neq 0$ for some $g \in M$. Assume that there is a $g_0 \in M$ such that $g_0 < f$ for each $f \in F$. Then $g^* \in C_{M(F)}(F)$ if and only if, there are $1 \leq k \leq n$ distinct points $x_1, x_2, \ldots, x_k \in \bigcup_{f\in F} Z(f - g^*)$ and k positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\int_X gd\mu = \sum_{i=1}^{k} \lambda_i g(x_i)$ for each $g \in M$.*

Proof. Assume $g^* \in C_{M(F)}(F)$. Then, by the above theorem, for each $g \in M$ with $g \leq 0$ on $\bigcup_{f\in F} Z(f - g^*)$, we have $\int_X gd\mu \leq 0$. Let $K = \bigcup_{f\in F} Z(f - g^*)$, then by the lemma, there are points $x_1, x_2, \ldots, x_k \in K$ and positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\int_X gd\mu = \sum_{i=1}^{k} \lambda_i g(x_i)$ for each $g \in M$.

Conversely, assume there are k distinct points $x_1, x_2, \ldots, x_k \in \bigcup_{f\in F} Z(f - g^*)$ and k positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\int_X gd\mu = \sum_{i=1}^{k} \lambda_i g(x_i)$ for each $g \in M$. Then for each $g \in M(F)$, $\int_X gd\mu = \sum_{i=1}^{k} \lambda_i g(x_i)$

$\leq \sum_{i=1}^{k} \lambda_i f(x_i)$ for all $f \in F = \sum_{i=1}^{k} \lambda_i g^*(x_i) = \int_X g^*d\mu$. That is

$\int_X gd\mu = \int_X g^*d\mu$ for each $g \in M(F)$. Hence $\sup_{g\in M(F)} \int_X gd\mu = \int_X g^*d\mu$. Therefore $g^* \in C_{M(F)}(F)$. □

Theorem 5. *Let M be a finite-dimensional subspace of $C_1(X)$. For any $F, G \in B[C_1(X)]$ with $M(F) = M(G)$ and any $\epsilon > 0$, there exists $\delta > 0$ such that $H(F, G) < \delta$ implies $H(C_{M(F)}(F), C_{M(G)}(G)) < 2\epsilon$.*

Proof. For any $\epsilon > 0$, let $0 < \delta < \frac{\epsilon}{2} < \min(d(F, M(F)), d(G, M(G)))$ where $F, G \in B[C_1(X)]$ with $M(F) = M(G)$. Assume $H(F, G) < \delta$. Then, for any $x \in C_1(X)$, $|d(x, F) - d(x, G)| \leq \delta$. In fact, for any $u \in F$, there exists $v \in G$ such that $\|u - v\| < \delta$. Then $\|u - v\| - \|x - v\| \leq \|u - v\| < \delta$. Then $\|u - x\| \leq \|x - v\| + \delta$.
Thus $d(x, F) \leq d(x, G) + \delta$. Similarly, $d(x, G) \leq d(x, F) + \delta$. For any $x \in C_{M(G)}(G)$, $d(x, F) \leq d(x, G) + \delta$. Thus $d(M(F), F) \leq d(M(G), G) + \delta$. Hence $|d(M(F), F) - d(M(G), G)| \leq \delta$.
For any $z \in C_{M(F)(F)}$,
$d(z, G) \leq d(z, F) + \delta = d(M(F), F) + \delta$
$\leq d(M(G), G) + 2\delta$
$\leq d(M(G), G)(1 + \epsilon)$.
There exists $w \in C_{M(G)}(G)$ with $\|z - w\| \leq 2\epsilon$,
so $\sup_{z \in C_{M(F)}(F)} \inf_{w \in C_{M(G)}(G)} \|z - w\| \leq 2\epsilon$. Hence $H(C_{M(F)}(F), C_{M(G)}(G)) \leq 2\epsilon$. □

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