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ON RICCI SOLITONS IN KENMOTSU MANIFOLDS WITH THE SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this paper, we study 3-dimensional Kenmotsu manifolds with the semi-symmetric non-metric connection. We obtain some results on Ricci solitons in Kenmotsu manifolds with the semi-symmetric non-metric connection satisfying the conditions $\widetilde{C}(\xi,X).\widetilde{S}=0$, $\widetilde{H}(\xi,X).\widetilde{S}=0$ and $\widetilde{P}(\xi,X).\widetilde{C}=0$, where $\widetilde{C}$ is the quasi-conformal curvature tensor, $\widetilde{S}$ is the Ricci tensor, $\widetilde{P}$ is the projective curvature tensor and $\widetilde{H}$ is the conharmonic curvature tensor. We also show that Ricci solitons are shrinking and expanding.

Keywords: Kenmotsu manifolds; Ricci solitons; semi-symmetric non-metric connection.

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1. Introduction

In 1969, Tanno studied almost contact Riemannian manifolds [18]. Later Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions and this manifold is known as Kenmotsu manifold [12]. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension [19]. De and Pathak studied

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some curvature conditions on 3-dimensional Kemotsu manifolds [6]. Yıldız and Çetinkaya studied Kenmotsu manifolds satisfying the same curvature conditions [23].

In [10], authors use a 2-dimensional Ricci soliton to illustrate the behaviour of mass under Ricci flow. According to [3], a Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M,g). A Ricci soliton is a triple (g,V,λ) where g is a Riemannian metric, V is a vector field and λ is a real scalar such that

$$L_V g + 2S + 2\lambda g = 0. \tag{1.1}$$

Here S is a Ricci tensor of M and L_V denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding if λ is negative, zero and positive, respectively. Nagaraja and Premalatha obtained some results on Ricci solitons in Kenmotsu manifolds using quasi-conformal, conharmonic and projective curvature tensors satisfying

$$R(\xi,X).\widetilde{C}=0,\ P(\xi,X).\widetilde{C}=0,\ H(\xi,X).S=0 \text{ and } \widetilde{C}(\xi,X).S=0$$

curvature conditions [15]. Bagewadi proved conditions for Ricci solitons in Kenmotsu manifolds to be shrinking, steady, and expanding [3]. Yıldız and Çetinkaya proved that a Kenmotsu manifold with the semi-symmetric non-metric connection satisfying $\widetilde{R}(X,Y).R=0$ is an η-Einstein manifold, and a Kenmotsu manifold with respect to the semi-symmetric non-metric connection satisfying $\widetilde{R}(X,Y).\widetilde{R}=0$ is locally isometric to the hyperbolic space $H^n(-1)$ [23].

In 1932, Hayden [11] introduced the idea of semi-symmetric metric connections on a Riemannian manifold $(M,\ g)$. Also in 2008, authors [14] studied Riemannian manifolds with a semi-symmetric metric connection satisfying some semisymmetry conditions and in 2011, authors [24] obtained some semisymmetry conditions on Riemannian manifolds. A semi-symmetric connection $\widetilde{\nabla}$ is said to be a semi-symmetric metric connection if

$$\widetilde{\nabla} g = 0.$$

A relation between the semi-symmetric metric connection and the Levi-Civita connection ∇ of $(M,\ g)$ was given by Yano [21] as

$$\widetilde{\bigtriangledown}_X Y = \nabla_X Y + u(Y)X - g(X,Y)p$$

where $u(X) = g(X,p)$. The study of a semi-symmetric metric connection $\overline{\bigtriangledown}$ satisfying

$$\overline{\bigtriangledown} g \neq 0$$

was initiated by Prvanovic [16] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [2]. This connection is said to be a semi-symmetric non-metric connection [17].

The paper is organized as follows: In section 2, we give some basic notions used in this study. In section 3, we introduce $f-$Kenmotsu manifolds. In the next section, we study 3-dimensional Kenmotsu manifolds with the semi-symmetric non-metric connection. We obtain some results on Ricci solitons in Kenmotsu manifolds with the semi-symmetric non-metric connection satisfying the conditions $\widetilde{C}(\xi,X).\widetilde{S}=0$, $\widetilde{P}(\xi,X).\widetilde{C}=0$ and $\widetilde{H}(\xi,X).\widetilde{S}=0$ where $\widetilde{C}$ is quasi-conformal curvature tensor, $\widetilde{S}$ is Ricci tensor, $\widetilde{P}$ is projective curvature tensor and $\widetilde{H}$ is conharmonic curvature tensor. We also show that Ricci solitons are shrinking and expanding.

2. Preliminaries

Let M be a $3-$dimensional differentiable manifold with an almost contact structure(ϕ,ξ,η,g) satisfying

$$\begin{aligned} &\eta(\xi)=1, \phi\xi=0, && \eta(\phi X)=0, \\ &g(X,\phi Y)=-g(\phi X,Y), && \phi^2(X)=-X+\eta(X)\xi, \\ &g(X,\xi)=\eta(X), && g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y) \end{aligned} \tag{2.1}$$

for any vector fields X, $Y \in \chi(M)$, where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is the Riemannian metric. Then M is called an almost contact manifold. For an almost contact manifold M, it follows that [22]

$$\begin{aligned} (\nabla_X\phi)Y &= \nabla_X\phi Y - \phi(\nabla_X Y), \\ (\nabla_X\eta)Y &= \nabla_X\eta(Y) - \eta(\nabla_X Y). \end{aligned} \tag{2.2}$$

Also the semi-symmetric non-metric connection on a Kenmotsu manifold is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \eta(Y) X \tag{2.3}$$

where η is a 1-form, $\widetilde{\nabla}$ is semi-symmetric non-metric connection and $\bigtriangledown$ is Riemann connection [1].

And a relation between the scalar curvature of M with the Riemannian connection and the semi-symmetric non-metric connection is given by [23]

$$\tilde{r} = r - (n-1)(n-2). \tag{2.4}$$

Let R be Riemann curvature tensor, S Ricci curvature tensor, Q Ricci operator, r scalar curvature and $\{e_1, ..., e_n\}$ be orthonormal basis of $T_P(M)$. $\forall X, Y \in \chi(M)$ it follows that [7]

$$S(X,Y) = \sum_{i=1}^{n} g(R(e_i, X) Y, e_i), \tag{2.5}$$

$$Q(X) = -\sum_{i=1}^{n} R(e_i, X) e_i \tag{2.6}$$

$$r = -\lambda n - (n-1) \tag{2.7}$$

and

$$S(X,Y) = g(Q(X), Y). \tag{2.8}$$

If the Ricci tensor S of an f-Kenmotsu manifold M satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta \eta(X) \eta(Y) \tag{2.9}$$

where α, β are certain scalars, then M is said to be η Einstein manifold. If $\beta = 0$, then M manifold is Einstein manifold [5].

Also Ricci tensor S of an Kenmotsu manifold M with the semi-symmetric non-metric connection and Ricci operator $\widetilde{Q}$ of any vector field X satisfies the conditions [23]

$$\widetilde{S}(X,Y) = -(\lambda+2) g(X,Y) + \eta(X) \eta(Y) \tag{2.10}$$

$$\widetilde{Q}X = -(\lambda+2) X + \eta(X) \xi. \tag{2.11}$$

In a three dimensional Riemann manifold the curvature tensor R is described as

$$\begin{aligned} R(X,Y)Z &= S(Y,Z)X - g(X,Z)QY + g(Y,Z)QX \\ &\quad - S(X,Z)Y - \frac{\tau}{2}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{2.12}$$

where S is the Ricci tensor, Q is the Ricci operator and τ is the scalar curvature for 3-dimensional M manifold [22].

On the other hand, let M be an n-dimensional Riemannian manifold with the Riemannian connection ∇. A linear connection $\widetilde{\nabla}$ on M is said to be a semi-symmetric connection if its torsion tensor $\widetilde{T}$ of the connection $\widetilde{\nabla}$ satisfies

$$\widetilde{T}(X,Y) = \eta(Y)X - \eta(X)Y \tag{2.13}$$

where η is a non-zero 1-form and $\widetilde{T} \neq 0$.

If moreover $\widetilde{\nabla} g = 0$ then the connection is called a semi-symmetric metric connection. If $\widetilde{\nabla} g \neq 0$ then the connection is called a semi-symmetric non-metric [23].

For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Projective curvature tensor P is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - S(X,Z)Y\} \tag{2.14}$$

for any $X,Y,Z \in \chi(M)$, where R is the curvature tensor and S is the Ricci tensor of M [13]. If $P(X,Y)\xi = 0$ for any $X,Y \in \chi(M)$, M manifold is called $\xi-$projective flat [22].

Let M be an n dimensional Kenmotsu manifold admitting a Ricci soliton (g,V,λ). The conharmonic curvature tensor [7] on M is defined by

$$\begin{aligned} H(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY]. \end{aligned} \tag{2.15}$$

The quasi-conformal curvature tensor C defined by [15] is

$$\begin{aligned} \check{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY] \\ &\quad - \frac{r}{n}(\frac{a}{n-1} + 2b)[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{2.16}$$

where a, b are constants.

And the quasi-conformal curvature tensor C on a Kenmotsu manifold with the semi-symmetric non-metric connection is defined by [23]

$$\begin{aligned}\widetilde{C}(X,Y)Z \;=\;& C(X,Y)Z+\{\tfrac{(n+1)(n+2)}{n}(\tfrac{a}{n-1}+2b)\text{-}a\\ &\text{-}2(n\text{-}1)b\}\{g(Y,Z)X\text{-}g(X,Z)Y\}+2(n\text{-}1)b\{g(Y,Z)\eta(X)\xi\\ &\text{-}g(X,Z)\eta(Y)\xi\}+2\{a+2(n\text{-}1)b\}\{\eta(Y)\eta(Z)X\text{-}\eta(X)\eta(Z)Y\}.\end{aligned} \tag{2.17}$$

3. Kenmotsu Manifolds with the semi-symmetric non-metric connection

Blair and Tripathi studied on contact metric manifolds [4], [20]. Duggal ve Sahin studied semi-symmetric manifolds and Ricci semi-symmetric manifolds [9]. In this section, we study and obtain results on Ricci solitons in Kenmotsu manifolds with respect to semi-symmetric non-metric connection satisfying some curvature conditions where $\tilde{B}$ is C-Bochner curvature tensor, $\widetilde{S}$ is Ricci tensor, $\widetilde{C}$ is quasi-conformal curvature tensor, $\widetilde{P}$ is Weyl-projective curvature tensor, $\widetilde{H}$ is conharmonic curvature tensor, $\widetilde{R}$ is Riemann curvature tensor and $\overline{\widetilde{P}}$ is pseudo projective curvature tensor. The Ricci soliton is said to be shrinking, steady or expanding if λ is negative, zero or positive, respectively.

4. Ricci Soliton in a Kenmotsu Manifold with the Semi-symmetric Non-metric Connection Satisfynig $\widetilde{C}(\xi,X).\widetilde{S}=0$

A Kenmotsu manifold with respect to the semi-symmetric non-metric connection satisfies the condition

$$\widetilde{C}(\xi,X).\widetilde{S}=0. \tag{4.1}$$

Using (2.10) we obtain

$$\begin{aligned}&\eta(\widetilde{C}(\xi,X)Y)\eta(Z)+\eta(\widetilde{C}(\xi,X)Z)\eta(Y)\\ &\quad=(\lambda+2)[g(\widetilde{C}(\xi,X)Y,Z)+g(Y,\widetilde{C}(\xi,X)Z)].\end{aligned} \tag{4.2}$$

Using (2.10), (2.11), (2.12) and (2.16) and taking $X=\xi,\ Y=X$ and $Z=Y$ in (4.2) we obtain

$$\begin{aligned}\eta(\widetilde{C}(\xi,X)Y) &= \text{-}[b(2\lambda\text{+}1)\text{+}2a+(\frac{a}{n-1}\text{+}2b)(\frac{r}{n}\text{-}\frac{(n\text{+}1)(n\text{+}2)}{n})][g(X,Y)\\ &\quad -\eta(Y)\eta(X)].\end{aligned} \tag{4.3}$$

Now from (2.10), (2.11), (2.12) and (2.16) it can be easily found that

$$\begin{aligned}C(X,Y)Z &= [a\text{+}2b(\lambda\text{+}1)\text{+}\frac{r}{n}(\frac{a}{n-1}\text{+}2b)][g(X,Z)Y\text{-}g(Y,Z)X]\\ &\quad \text{+}b[\eta(Y)\eta(Z)X\text{-}\eta(X)\eta(Z)Y\text{+}g(Y,Z)\eta(X)\xi\text{-}g(X,Z)\eta(Y)\xi].\end{aligned} \tag{4.4}$$

By using (2.17) we get

$$\begin{aligned}\widetilde{C}(X,Y)Z &= [a\text{+}2b(\lambda\text{+}1)\text{+}\frac{r}{n}(\frac{a}{n\text{-}1}\text{+}2b)\text{+}a\text{+}2(n\text{-}1)b]\\ &\quad \text{-}\frac{(n\text{+}1)(n\text{+}2)}{n}(\frac{a}{n\text{-}1}\text{+}2b)][g(X,Z)Y\text{-}g(Y,Z)X\\ &\quad \text{+}b(2n\text{-}1)[g(Y,Z)\eta(X)\xi\text{-}g(X,Z)\eta(Y)\xi]\\ &\quad \text{+}[2a+b(4n\text{-}3)][\eta(Y)\eta(Z)X\text{-}\eta(X)\eta(Z)Y].\end{aligned} \tag{4.5}$$

By using (4.3) and (4.5) in (4.2) we have

$$\begin{aligned}&[-2a(3+\lambda)+b(3-2n(\lambda+2))\text{-}(\frac{a}{n\text{-}1}+2b)(\frac{r\text{-}(n+1)(n+2)}{n})]\\ &\times[2\eta(X)\eta(Y)\eta(Z)\text{-}\eta(Z)g(X,Y)\text{-}\eta(Y)g(X,Z)]=0.\end{aligned} \tag{4.6}$$

Taking $X=Y=e_i$ in (4.6) and summing over $i=1,2,...,n$ and by virtue of $a+2b(n-1)=0$ we obtain

$$[-2a(3+\lambda)+b(3-2n(\lambda+2))](1-n)\eta(Z)=0. \tag{4.7}$$

Then from $\eta(Z)\neq 0$ and $b\neq 0$ conditions we have

$$\lambda = \frac{-8n+9}{2n-4}. \tag{4.8}$$

If $n>2$ in (4.8) then $\lambda<0$; that is, the Ricci soliton is shrinking.

Hence we state the following theorem.

Theorem 4.1. *A Ricci soliton in a Kenmotsu manifold with the semi-symmetric non-metric connection satisfying*

$$\widetilde{C}(\xi,X).\widetilde{S}=0 \tag{4.9}$$

is shrinking for $n>2$.

If $a=\dfrac{n-2}{n-1}b$ in (4.1) then $\lambda=0$; that is, the Ricci soliton in Kenmotsu manifold is steady [15]. Hence we state the following result.

Result. A Ricci soliton satisfying (4.1) is steady for a Kenmotsu manifold, shrinking for a Kenmotsu manifold with the semi-symmetric non-metric connection.

5. Ricci Soliton in a Kenmotsu Manifold with the Semi-symmetric Non-metric Connection Satisfynig $\widetilde{P}(\xi,X).\widetilde{C}=0$

A Kenmotsu manifold with respect to the semi-symmetric non-metric connection satisfying the condition

$$\widetilde{P}(\xi,X).\widetilde{C}=0. \tag{5.1}$$

From (2.10), (2.12) and (2.14) we obtain

$$\begin{aligned}\widetilde{P}(X,Y)Z &= [\frac{\lambda+2}{n-1}-2](g(Y,Z)X-g(X,Z)Y)\\ &\quad+[\frac{1}{n-1}-2](\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X).\end{aligned} \tag{5.2}$$

Using (5.2) and by taking the inner product with ξ, we have

$$\begin{aligned}0 &= (\lambda\text{-}2n\text{+}4)\widetilde{C}'(Y,Z,W,X)\text{-}(\lambda\text{+}1)\eta(\widetilde{C}(Y,Z)W)\eta(X)\\ &\quad+(2n\text{-}3)\eta(\widetilde{C}(Y,Z)W)\eta(X)\text{-}(\lambda\text{-}2n\text{+}4)g(X,Y)\eta(\widetilde{C}(\xi,Z)W)\\ &\quad+(\lambda\text{+}1)\eta(Y)\eta(\widetilde{C}(X,Z)W)\text{-}(2n\text{-}3)\eta(X)\eta(Y)\eta(\widetilde{C}(\xi,Z)W)\\ &\quad\text{-}(\lambda\text{-}2n\text{+}4)g(X,Z)\eta(\widetilde{C}(Y,\xi)W)\text{+}(\lambda\text{+}1)\eta(Z)\eta(\widetilde{C}(Y,X)W)\\ &\quad\text{-}(2n\text{-}3)\eta(X)\eta(Z)\eta(\widetilde{C}(Y,\xi)W)\text{-}(\lambda\text{-}2n\text{+}4)g(X,W)\eta(\widetilde{C}(Y,Z)\xi)\\ &\quad+(\lambda\text{+}1)\eta(W)\eta(\widetilde{C}(Y,Z)X)\text{-}(2n\text{-}3)\eta(X)\eta(Z)\eta(\widetilde{C}(Y,Z)\xi).\end{aligned} \tag{5.3}$$

Now from (2.10), (2.11), (2.12) and (2.16) it can be easily found that

$$\begin{aligned}\widetilde{C}(X,Y)Z = \; & (2a+b)[\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y] \\ & +b[\eta(X)g(Y,Z)\xi-\eta(Y)g(X,Z)\xi] \\ & -[\frac{r}{n}(\frac{a}{n-1}+2b)+2a+2\lambda b+4b][g(Y,Z)X-g(X,Z)Y].\end{aligned} \tag{5.4}$$

Taking the inner product of () with ξ, we have

$$\eta(\widetilde{C}(X,Y)Z) = [b(2\lambda+3)+\frac{r}{n}(\frac{a}{n-1}+2b)+2a][g(X,Z)\eta(Y)-g(Y,Z)\eta(X)]. \tag{5.5}$$

Taking

$$k = b(2\lambda+3)+\frac{r}{n}(\frac{a}{n-1}+2b)+2a \tag{5.6}$$

and using (5.3) and () we obtain

$$\begin{aligned} 0 = \; & (\lambda-2n+4)\widetilde{C}'(Y,Z,W,X)+(\lambda-2n+4)kg(X,Y)g(Z,W) \\ & -(\lambda-2n+4)kg(Y,W)g(X,Z)+(2n-3)kg(X,Y)\eta(Z)\eta(W) \\ & -(2n-3)kg(X,Z)\eta(W)\eta(Y).\end{aligned} \tag{5.7}$$

Putting $X=Y$, $Y=Z$ and $Z=W$ in () and taking the inner product of () with X, we have

$$\begin{aligned} g(X,\widetilde{C}(Y,Z)W) = \; & (2a+b)[\eta(Z)\eta(W)g(X,Y)-\eta(Y)\eta(W)g(X,Z)] \\ & +b[\eta(X)\eta(Y)g(Z,W)-\eta(X)\eta(Z)g(Y,W)] \\ & -(k+b)[g(Z,W)g(X,Y)-g(Y,W)g(X,Z)].\end{aligned} \tag{5.8}$$

By using (5.7) and (5.8) we get

$$\begin{aligned} 0 = \; & [(\lambda-2n+4)(2a+b)+(2n-3)k]\eta(Z)\eta(W)g(X,Y) \\ & +[-(\lambda-2n+4)(2a+b)-(2n-3)k]\eta(Y)\eta(W)g(X,Z) \\ & +(\lambda-2n+4)b[\eta(X)\eta(Y)g(Z,W)-\eta(X)\eta(Z)g(Y,W)] \\ & +[-(\lambda-2n+4)(k+b)+(\lambda-2n+4)k]g(X,Y)g(Z,W) \\ & +[(\lambda-2n+4)(k+b)-(\lambda-2n+4)k]g(X,Z)g(Y,W).\end{aligned} \tag{5.9}$$

Taking $X = Y = e_i$ and $Z = W = e_i$ in (5.9) and summing over $i = 1, 2, ..., n$ we obtain

$$\{(\lambda - 2n + 4)[2a - b(n-2)] + (2n-3)k\}(n-1) \quad = 0. \tag{5.10}$$

Then from (5.6) and $n \neq 1,\ a + 2b(n-1) = 0$ and $b \neq 0$ conditions we have

$$\lambda \quad = \quad \frac{2n^2 - 6n + 3}{n}.$$

If $a + (n-1)b = 0$ in (5.1) then $\lambda > 0$ for $n \geq 3$; that is, the Ricci soliton is expanding.

Hence we state the following theorem.

Theorem 5.1. *A Ricci soliton in a Kenmotsu manifold with the semi-symmetric non-metric connection satisfying*

$$\widetilde{P}(\xi, X)\,.\widetilde{C} = 0$$

is expanding for $n \geq 3$.

If $a + (n-1)b = 0$ in (5.1) then $\lambda > 0$ for $n \geq 3$; that is, the Ricci soliton in Kenmotsu manifold is expanding [15]. Hence we state the following result.

Result. A Ricci soliton satisfying (5.1) is expanding for a Kenmotsu manifold and Kenmotsu manifold with the semi-symmetric non-metric connection.

6. Ricci Soliton in a Kenmotsu Manifold with the Semi-symmetric Non-metric Connection Satisfynig $\widetilde{H}(\xi, X)\,.\widetilde{S} = 0$

A Kenmotsu manifold with respect to the semi-symmetric non-metric connection satisfies the condition

$$\widetilde{H}(\xi, X)\,.\widetilde{S} = 0. \tag{6.1}$$

Using (2.10) we obtain

$$\begin{aligned} &\eta(Z)\eta(\widetilde{H}(\xi, X)Y) + \eta(Y)\eta(\widetilde{H}(\xi, X)Z) \\ &\qquad = (\lambda + 2)[g(\widetilde{H}(\xi, X)Y, Z) + g(Y, \widetilde{H}(\xi, X)Z)]. \end{aligned} \tag{6.2}$$

From (2.10), (2.12) and (2.15) we get

$$\begin{aligned}\widetilde{H}(X,Y)Z &= \frac{2n\text{-}2\lambda\text{-}8}{n\text{-}2}[g(X,Z)Y\text{-}g(Y,Z)X]+\frac{2n\text{-}5}{n\text{-}2}[\eta(Y)\eta(Z)X \\ &\quad \text{-}\eta(X)\eta(Z)Y]+\frac{1}{n\text{-}2}[g(X,Z)\eta(Y)\xi\text{-}g(Y,Z)\eta(X)\xi].\end{aligned} \tag{6.3}$$

Putting $X=\xi,\ Y=X$ and $Z=Y$ in (6.3) we have

$$\begin{aligned}\widetilde{H}(\xi,X)Y &= 2\eta(X)\eta(Y)\xi-[\frac{2\lambda+3}{n-2}]\eta(Y)X \\ &\quad -[\frac{2n-2\lambda-7}{n-2}]g(X,Y)\xi.\end{aligned} \tag{6.4}$$

Taking the inner product of (6.4) with ξ, we have

$$\begin{aligned}\eta(\widetilde{H}(\xi,X)Y) &= 2\eta(X)\eta(Y)-[\frac{2\lambda+3}{n-2}]\eta(Y)\eta(X)-[\frac{2n-2\lambda-7}{n-2}]g(X,Y) \\ &= [\frac{2n-2\lambda-7}{n-2}][\eta(X)\eta(Y)-g(X,Y)].\end{aligned} \tag{6.5}$$

Now from (6.2) and (6.5) it can be easily found that

$$\begin{aligned}&(-4n+4\lambda-4\lambda n+2)\eta(X)\eta(Y)\eta(Z) \\ &-(-2n+2\lambda-2\lambda n+1)[\eta(Y)g(X,Z)+\eta(Z)g(X,Y)]=0.\end{aligned} \tag{6.6}$$

Taking $X=Y=e_i$ in (6.6) and summing over $i=1,2,...,n$ and by virtue of $\eta(Z)\neq 0$ and $n\neq 1$ conditions we obtain

$$\lambda = -\frac{(2n-1)}{2(n-1)}. \tag{6.7}$$

If $n>2$ in (6.7) then $\lambda<0$; that is, the Ricci soliton is shrinking.

Hence we state the following theorem.

Theorem 6.1. *A Ricci soliton in a Kenmotsu manifold with the semi-symmetric non-metric connection satisfying*

$$\widetilde{H}(\xi,X).\widetilde{S}=0 \tag{6.8}$$

is shrinking for $n>2$.

A Ricci soliton in Kenmotsu manifold satisfying (6.1) is steady [15]. Hence we state the following result.

Result. A Ricci soliton satisfying (6.1) is steady for a Kenmotsu manifold, shrinking for a Kenmotsu manifold with the semi-symmetric non-metric connection.

7. Example of a 3-dimensional Kenmotsu manifold with the semi-symmetric non-metric connection

It is calculated that the following example for 3-dimentional f-Kenmotsu Manifold in [23] and [8] by using $f=1$.

We consider the 3-dimensional manifold $M=\left\{(x,y,z)\in\mathbb{R}^3, z\neq 0\right\}$, where (x,y,z) are the standart coordinates in $\mathbb{R}^3$. The vector fields

$$e_1=z^2\frac{\partial}{\partial x},\ e_2=z^2\frac{\partial}{\partial y},\ e_3=\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1,e_3) &= g(e_2,e_3)=g(e_1,e_2)=0, \\ g(e_1,e_1) &= g(e_2,e_2)=g(e_3,e_3)=1. \end{aligned}$$

Let η be the 1-form defined by $\eta(Z)=g(Z,e_3)$ for any $Z\in\chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1)=-e_2$, $\phi(e_2)=e_1$, $\phi(e_3)=0$.

Then using linearity of ϕ and g we have

$$\eta(e_3)=1,\quad \phi^2(Z)=-Z+\eta(Z)e_3$$

$$g(\phi Z,\phi W)=g(Z,W)-\eta(Z)\eta(W)$$

for any $Z,W\in\chi(M)$. Now, by direct computations we obtain

$$[e_1,e_2]=0,\ [e_2,e_3]=-\frac{2}{z}e_2,\ [e_1,e_3]=-\frac{2}{z}e_1.$$

From (2.7) for 3 dimensional manifold it is verified that

$$r=-3\lambda-2. \tag{7.1}$$

By using these above equations we get

$$\begin{aligned} \nabla_{e_1}e_3 &= -\frac{2}{z}e_1, \quad \nabla_{e_1}e_1 = \nabla_{e_2}e_2 = \frac{2}{z}e_3, \\ \nabla_{e_2}e_3 &= -\frac{2}{z}e_2, \quad \nabla_{e_1}e_2 = \nabla_{e_2}e_1 = \nabla_{e_3}e_1 = \nabla_{e_3}e_2 = \nabla_{e_3}e_3 = 0. \end{aligned} \tag{7.2}$$

[22]. It's known that

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{7.3}$$

By using (7.2) and (7.3) we obtain

$$\begin{aligned} &3R(e_1,e_2)e_2 = 2R(e_1,e_3)e_3 = -\frac{12}{z^2}e_1, \quad R(e_1,e_1)e_1 = R(e_1,e_2)e_3 = 0, \\ &3R(e_2,e_1)e_1 = 2R(e_2,e_3)e_3 = -\frac{12}{z^2}e_2, \quad R(e_2,e_3)e_1 = R(e_1,e_3)e_2 = 0, \\ &R(e_3,e_1)e_1 = R(e_3,e_2)e_2 = -\frac{6}{z^2}e_3, \qquad R(e_2,e_2)e_2 = R(e_3,e_3)e_3 = 0, \end{aligned} \tag{7.4}$$

From (2.5) and (7.4), it is verified that

$$6S(e_1,e_1) = 6S(e_2,e_2) = 5S(e_3,e_3) = -\frac{60}{z^2}. \tag{7.5}$$

Now from (2.14), (7.4) and (7.5) it can be easily found that

$$P(e_1,e_2)e_3 = 0,\ P(e_1,e_3)e_3 = -\frac{2}{z}e_1,\ P(e_2,e_3)e_3 = -\frac{2}{z}e_2. \tag{7.6}$$

By using (2.6) and (7.4) we obtain

$$Qe_1 = -\frac{10}{z^2}e_1, \quad Qe_2 = -\frac{10}{z^2}e_2, \quad Qe_3 = -\frac{12}{z^2}e_3. \tag{7.7}$$

Now from (2.15), (7.4), (7.5) and (7.7) it can be easily found that

$$\begin{aligned} &H(e_1,e_3)e_3 = -\frac{6}{z^2}e_1, \quad H(e_3,e_1)e_1 = \frac{16}{z^2}e_3, \\ &H(e_2,e_1)e_1 = \frac{16}{z^2}e_2, \qquad H(e_1,e_2)e_3 = H(e_1,e_3)e_2 = 0. \end{aligned} \tag{7.8}$$

By using (2.16), (6.8), (7.4), (7.5), (7.7) and taking $a = 1,\ b = 1$ we obtain

$$\begin{aligned} &C(e_1,e_3)e_3 = [-\frac{28}{z^2} + \frac{3\lambda+2}{2}]e_1, \quad C(e_3,e_1)e_1 = [-\frac{28}{z^2} + \frac{3\lambda+2}{2}]e_3, \\ &C(e_2,e_1)e_1 = [-\frac{24}{z^2} + \frac{3\lambda+2}{2}]e_2, \quad C(e_1,e_3)e_2 = C(e_1,e_2)e_3 = 0. \end{aligned} \tag{7.9}$$

Now we consider this example for semi-symmetric non-metric connection:

From (2.4) for 3 dimensional manifold it is verified that

$$\tilde{r} = -3\lambda - 4. \tag{7.10}$$

From (2.3) and (7.2), we get

$$\begin{aligned} &\widetilde{\nabla}_{e_1} e_3 = (1-\frac{2}{z})e_1, \quad \widetilde{\nabla}_{e_2} e_3 = (1-\frac{2}{z})e_2, \quad \widetilde{\nabla}_{e_3} e_1 = \widetilde{\nabla}_{e_3} e_2 = 0, \\ &\widetilde{\nabla}_{e_2} e_2 = \widetilde{\nabla}_{e_1} e_1 = \frac{2}{z} e_3, \quad \widetilde{\nabla}_{e_1} e_2 = \widetilde{\nabla}_{e_2} e_1 = 0, \quad \widetilde{\nabla}_{e_3} e_3 = e_3. \end{aligned} \tag{7.11}$$

It is known that

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z. \tag{7.12}$$

By using (7.11) and (7.12) we obtain

$$\begin{aligned} &\widetilde{R}(e_k, e_3) e_3 = (1-\frac{6}{z^2})e_k, \quad \widetilde{R}(e_1,e_1) e_1 = \widetilde{R}(e_2,e_2) e_2 = 0, \\ &\widetilde{R}\left(e_k, e_j\right) e_j = (\frac{2}{z} - \frac{4}{z^2})e_k, \quad \widetilde{R}(e_1,e_2) e_3 = \widetilde{R}(e_2,e_3) e_1 = 0, \\ &\widetilde{R}(e_3,e_k) e_k = (\frac{2}{z} - \frac{6}{z^2})e_3, \quad \widetilde{R}(e_3,e_3) e_3 = \widetilde{R}(e_1,e_3) e_2 = 0. \end{aligned} \tag{7.13}$$

where $k, j = 1,2$ and $j \neq k$ From (2.5) and (7.13), it is verified that

$$\widetilde{S}(e_1,e_1) = \widetilde{S}(e_2,e_2) = 1 + \frac{2}{z} - \frac{10}{z^2}, \ \ \widetilde{S}(e_3,e_3) = \frac{4}{z} - \frac{12}{z^2}. \tag{7.14}$$

Now from (2.14), (5.7) and (7.14) it can be easily found that

$$\widetilde{P}(e_1,e_2) e_3 = 0, \ \ \widetilde{P}(e_1,e_3) e_3 = -\frac{2}{z} e_1, \ \ \widetilde{P}(e_2,e_3) e_3 = -\frac{2}{z} e_2. \tag{7.15}$$

By using (2.6) and (7.13) we obtain

$$\widetilde{Q} e_1 = (1 + \frac{2}{z} - \frac{10}{z^2})e_1, \quad \widetilde{Q} e_2 = (1 + \frac{2}{z} - \frac{10}{z^2})e_2, \quad \widetilde{Q} e_3 = (\frac{4}{z} - \frac{12}{z^2})e_3. \tag{7.16}$$

From (2.15), (7.13), (7.14), and (7.16), it's verified that

$$\begin{aligned} &\widetilde{H}(e_1,e_2) e_3 = \widetilde{H}(e_1,e_3) e_2 = 0, \quad \widetilde{H}(e_2,e_1) e_1 = (2 + \frac{6}{z} - \frac{24}{z^2})e_2, \\ &\widetilde{H}(e_1,e_3) e_3 = (\frac{16}{z^2} - \frac{6}{z})e_1, \qquad \widetilde{H}(e_3,e_1) e_1 = (1 + \frac{8}{z} - \frac{28}{z^2})e_3. \end{aligned} \tag{7.17}$$

By using (2.16), (), (7.13), (7.14), (7.16) and taking $a=1,\ b=1$ we obtain

$$\begin{aligned}
&\widetilde{C}(e_1,e_3)e_3=(\frac{3\lambda+8}{2}+\frac{6}{z}-\frac{28}{z^2})e_1, \quad \widetilde{C}(e_1,e_2)e_3=0,\\
&\widetilde{C}(e_3,e_1)e_1=(\frac{3\lambda+6}{2}+\frac{8}{z}-\frac{28}{z^2})e_3, \quad \widetilde{C}(e_1,e_3)e_2=0, \qquad (7.18)\\
&\widetilde{C}(e_2,e_1)e_1=(\frac{3\lambda+8}{2}+\frac{6}{z}-\frac{24}{z^2})e_2.
\end{aligned}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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