Available online at http://scik.org

J. Math. Comput. Sci. 7 (2017), No. 1, 84-91

ISSN: 1927-5307

FIXED POINT THEOREMS IN NORMAL n-NORMED SPACES

JIAN MENG[1,*], MEIMEI SONG[2]

[1]Department of Mathematics, Tianjin University of Technology, Tianjin 300384, China

[2]Department of Mathematics, Tianjin University of Technology, Tianjin 300384, China

Abstract. In this paper, we extend the definitions of normal structure and geometric properties in n-normed spaces. Fixed point theorems for nonexpansive mappings are proved via the normal structure condition in n-normed spaces. The purpose of this work is keeping continuity of fixed point theorems in n-Banach spaces.

Keywords: fixed point theorem; n-normed space; normal structure; nonexpansive mapping.

2010 AMS Subject Classification: 54G05, 46B06, 46B07.

1. Introduction

In 1964, Gähler introduced the notion of 2-normed spaces[1] and extended it to the concept of n-normed spaces[2,3,4]. Other papers dealing with n-metric spaces also gave some important results[5,10,11]. Recently, Several authors such as Iseki[5,6] and Gunawan[7,8,9] also studied some aspects of the fixed point theory and proved fixed point theorems in n-normed spaces.

In 1965, F.Browder and D.Göhde[15] independently proved that if K is a nonempty bounded closed convex sunbet of a uniformly convex Banach space, and $T : K \to K$ is a nonexpansive

*Corresponding author

E-mail address: mengjian0710@163.com.

Received November 10, 2016

mapping, then T has a fixed point in K. In 2015, K.Matsuzaki[16] showed uniform convexity's generalization in the geometric group theory and proved the fixed point property. Normal structure plays essential role in some problems of fixed point theory. Motivated by these results, our formulation of normal structure and its generalization can be applied to prove fixed point theorems for nonexpansive type mappings in normal n-normed spaces.

2. Preliminaries

In this section, we gather together some definitions and known results which will be used in section 3. Hereafter, let $(E, \| \cdot, ..., \cdot \|)$ denote an n-normed space and $C = \{c_1, ..., c_n\}$ be a set of n linearly independent vectors in E. We use $definition$ 1-$definition$ 5 in [9], and we don't make redundant narration. Before we state our fixed point theorems, we introduce the following basic definitions:

Definition 2.1 Let E be a n-normed space. The diameter of any bounded subset $K \subseteq E$, which is denote by $\delta(K)$, is defined by $\delta(K) = sup\{\| x-y, x_2, ..., x_n \| : x, y \in K\}$ for all $x_2, ..., x_n \in E$.

Definition 2.2[13] E is called uniformly convex n-normed space with respect to C: for arbitrary $\varepsilon > 0$, there is $\delta = \delta(C, \varepsilon) > 0$, such that for all $x, y \in S_C(E)$ and $\{i_2, ..., i_n\} \subseteq \{1, 2, ..., n\}$, if

$$\| \tfrac{x+y}{2}, c_{i_2}, ..., c_{i_n} \| > 1 - \delta,$$

then

$$\| x-y, c_{i_2}, ..., c_{i_n} \| < \varepsilon.$$

Here $S_C(E) = \{x \in E : \| x, c_{i_2}, ..., c_{i_n} \| = 1\}$.

If for arbitrary $C \subseteq E$, E is uniformly convex n-normed space with respect to C, then E is called uniformly convex n-normed space.

Definition 2.3 E is called uniformly convex n-normed space in every direction(n-UCED, for short) with respect to C: for arbitrary $\varepsilon > 0$ and $z \in E \setminus \{\theta\}$, there exists $\delta = \delta(C, \varepsilon, z) > 0$, such that if $x, y \in S_C(E)$, $x - y = \lambda z$,

$$\| \tfrac{x+y}{2}, c_{i_2}, ..., c_{i_n} \| > 1 - \delta,$$

then $|\lambda| < \varepsilon$.

If for arbitrary $C \subseteq E$, E is n-UCED with respect to C, then E is called uniformly convex n-normed space in every direction(n-UCED).

Proposition 2.4 *E is uniformly convex n-normed space with respect to C, then E is n-UCED with respect to C.*

Proof. For arbitrary $\varepsilon > 0$ and $z \in E \setminus \{\theta\}$, let $\varepsilon' = \| z, c_{i_2}, ..., c_{i_n} \| \cdot \varepsilon$. Since E is uniformly convex n-normed space with respect to C, there exists $\delta = \delta(C, \varepsilon') > 0$, such that when $x, y \in S_C(E)$, $x - y = \lambda z$,

$$\| \tfrac{x+y}{2}, c_{i_2}, ..., c_{i_n} \| > 1 - \delta,$$

then

$$\| x - y, c_{i_2}, ..., c_{i_n} \| < \varepsilon'.$$

We obtain that $| \lambda | < \varepsilon$, and hence E is n-UCED with respect to C.

Definition 2.5 Let E be an n-normed space, we define directional modulus of convexity of E with respect C, which is denote by $\delta_C^E(z, \varepsilon)$: for $\varepsilon > 0$ and $\{i_2, ..., i_n\} \subseteq \{1, 2, ..., n\}$,

$\delta_C^E(z, \varepsilon) = \inf\{1 - \frac{\|x+y, c_{i_2}, ..., c_{i_n}\|}{2} : x, y \in B_C(E), \| x - y, c_{i_2}, ..., c_{i_n} \| > \varepsilon, x - y = tz, z \in E, t \in R\}$.

Here $B_C(E) = \{x \in E : \| x, c_{i_2}, ..., c_{i_n} \| \leq 1\}$.

Proposition 2.6 *For $\varepsilon > 0$, if $\delta_C^E(z, \varepsilon) > 0$, then E is n-UCED with respect to C.*

Definition 2.7 An n-normed space E is said to have normal structure with respect to C: for every bounded closed and convex subset $K \subseteq E$($\delta(K) > 0$) with respect to C ($C \notin spanK$), there exists a element $u \in K$ and $\{i_2, ..., i_n\} \subseteq \{1, 2, ..., n\}$, such that

$$sup_{x \in K} \| x - u, c_{i_2}, ..., c_{i_n} \| < \delta(K).$$

If for arbitrary $C \subseteq E$, E has normal structure with respect to C, then E is said to have normal structure.

Definition 2.8 Let E be a n-normed space then the mapping $T : E \to E$ is said to be a nonexpansive mapping with respect to C, if

$$\| Tx - Ty, c_{i_2}, ..., c_{i_n} \| \leq \| x - y, c_{i_2}, ..., c_{i_n} \|,$$

for all $x, y \in E$ and $\{i_2, ..., i_n\} \subseteq \{1, 2, ..., n\}$.

3. Main results

Lemma 3.1. *Let E be an n-normed space, $K \subseteq E$ be nonempty bounded closed and convex subsets with respect to C, $C \notin spanK$ and T be a selfmap into K, If K has intersection property (i.e., any decreasing sequence of nonempty closed subset of K has a nonempty intersection), then there is a minimal $T-$ invariant, nonempty bounded closed and convex subset with respect to C.*

Proof. Let F_C be a set family containing all $T-$ invariant, nonempty bounded closed and convex subsets with respect to C in K. F_C is a nonempty set ($K \in F$). According to the set inclusion relation, K has intersection property, then we can use the *zorn* lemma to obtain a minimal element in F_C.

Lemma 3.2 *Assuming D is the minimal element in F_C, then $\overline{CoTD}_C = D$. Here $\overline{CoTD}_C$ is the closed convex hull of TD with respect to C.*

Proof. Since D is $T-$ invariant, $TD \subseteq D$. The assumption that D is a closed and convex subset with respect to C such that $\overline{CoTD}_C \subseteq D$, then

$$T\overline{CoTD}_C \subseteq TD \subseteq \overline{CoTD}_C,$$

so $\overline{CoTD}_C$ is $T-$ invariant. Since the minimality of D, $\overline{CoTD}_C = D$.

Theorem 3.3 *Let E be an n-normed space and $K \subseteq E$ be nonempty bounded closed and convex subsets with respect to C($C \notin spanK$), having intersection property. If E has normal structure, then a nonexpansive selfmap $T : K \to K$ has a fixed point.*

Proof. By *Lemma* 3.1 we can obtain a minimal element D of F_C. If $\delta(D) = 0$, the problem is solved since in this case $D = \{x_0\}$, and thus $T(x_0) = x_0$. Then we are ready to prove D has only one point. We assume D has more than one point, and since E has normal structure, there exists $u \in D$, such that

$$\alpha = sup_{x \in D} \parallel x - u, c_{i_2}, ..., c_{i_n} \parallel < \delta(D). \tag{3.1}$$

For every $x \in D$

$$\parallel T(u) - T(x), c_{i_2}, ..., c_{i_n} \parallel \leq \parallel u - x, c_{i_2}, ..., c_{i_n} \parallel \leq \alpha.$$

Since x is arbitrary, we obtain that

$$T(D) \subseteq B_C(T(u), \alpha) = \{y \in X : \parallel T(u) - y, c_{i_2}, ..., c_{i_n} \parallel \leq \alpha\}. \tag{3.2}$$

Let $G = D\bigcap B_C(T(u),\alpha)$, then G is a $T-$ invariant, nonempty ($T(u) \in G$) bounded closed and convex subset with respect to C *i.e.*, $G \in F_C$. Since D is the minimal element, then $D = G$. We know $D \subseteq B_C(T(u),\alpha)$, then

$$sup_{y\in D} \parallel T(u)-y,c_{i_2},...,c_{i_n} \parallel \leq \alpha. \tag{3.3}$$

Let

$$D^{'} = \{z \in D, sup_{y\in D} \parallel z-y,c_{i_2},...,c_{i_n} \parallel \leq \alpha\}, \tag{3.4}$$

so, it is easy to get $D^{'} \subseteq D$ is a nonempty($T(u) \in D^{'}$) bounded closed and convex set with respect to C. And from (3.1), (3.4)

$$\delta(D^{'}) \leq \alpha < \delta(D),$$

then $D^{'} \subsetneqq D$. Next, we will prove $D^{'}$ is $T-$ invariant.

Actually, for every $w \in D^{'} \subsetneqq D$, because D is the minimal element in F_C, by *Lemma* 3.2, we know $D = \overline{Co\, TD}_C$. Then for every $y \in D$, and arbitrary $\varepsilon > 0$, existing $x_i \in D$, $\lambda_i \geq 0$, $\sum_{i=1}^{n}\lambda_i = 1$, such that

$$\parallel y-\sum_{i=1}^{n}\lambda_i T(x_i),c_{i_2},...,c_{i_n} \parallel < \varepsilon. \tag{3.5}$$

We can see that

$$\begin{aligned}
& \parallel T(w)-y,c_{i_2},...,c_{i_n} \parallel \\
\leq\ & \parallel T(w)-\sum_{i=1}^{n}\lambda_i T(x_i),c_{i_2},...,c_{i_n} \parallel + \parallel \sum_{i=1}^{n}\lambda_i T(x_i)-y,c_{i_2},...,c_{i_n} \parallel \\
\leq\ & \sum_{i=1}^{n}\lambda_i \parallel T(w)-T(x_i),c_{i_2},...,c_{i_n} \parallel + \varepsilon \\
\leq\ & \sum_{i=1}^{n}\lambda_i \parallel w-x_i,c_{i_2},...,c_{i_n} \parallel + \varepsilon \\
\leq\ & \alpha+\varepsilon.
\end{aligned}$$

Then $Tw \in D^{'}$, thus $D^{'} \in F$, which contradicts the minimality of D. We know that D has only one point, which is fixed by T, so T has at least a fixed point in $D \subseteq K$. Thus we complete the proof.

Lemma 3.4 *Let E be an n-normed space, $z \in S_C(E)$, if $\delta_C^E(z,1) > 0$, then E has normal structure.*

Proof. Let $K \subseteq E$ is any more than one point, bounded closed and convex subset with respect to C, then there exist $x, y \in K$ and $\{i_2, ..., i_n\} \subseteq \{1, 2, ..., n\}$, such that

$$sup_{x,y\in K} \parallel x-y, c_{i_2}, ..., c_{i_n} \parallel = \delta(K).$$

This implies that for arbitrary given $\varepsilon > 0$, we have

$$\parallel x-y, c_{i_2}, ..., c_{i_n} \parallel \geq \delta(K) - \varepsilon.$$

Clearly, for any $u \in K$,

$$\parallel u-x, c_{i_2}, ..., c_{i_n} \parallel \leq \delta(A),$$

$$\parallel u-y, c_{i_2}, ..., c_{i_n} \parallel \leq \delta(A),$$

and let $z = \frac{x-y}{\|x-y, c_{i_2}, ..., c_{i_n}\|}$, we have

$$u-x-(u-y) = \parallel x-y, c_{i_2}, ..., c_{i_n} \parallel \cdot z.$$

Then

$$\parallel u - \tfrac{x+y}{2}, c_{i_2}, ..., c_{i_n} \parallel \leq \delta(K)[1 - \delta_C^E(z, \tfrac{\delta(K)-\varepsilon}{\delta(K)})].$$

According the condition $\delta_C^E(z, 1) > 0$, we can deduce that for any $u \in K$, there exists $\frac{x+y}{2} \in K$, such that

$$\parallel u - \tfrac{x+y}{2}, c_{i_2}, ..., c_{i_n} \parallel < \delta(K).$$

Thus we complete the proof.

Corollary 3.5 *Let E be an n-UCED and $K \subseteq E$ be nonempty bounded closed and convex subsets with respect to C ($C \notin spanK$), having intersection property. $T : K \to K$ is a nonexpansive mapping, then T has at least a fixed point.*

Theorem 3.6 *Let E be an n-UCED and $K \subseteq E$ be nonempty bounded closed and convex subsets with respect to C($C \notin spanK$), having intersection property. For all $x, y \in K$ and $x_2, ..., x_n \in E$, $T : K \to K$ satisfies*

$$\parallel Tx - Ty, x_2, ..., x_n \parallel \leq a \parallel x-y, x_2, ..., x_n \parallel + b \parallel x - Tx, x_2, ..., x_n \parallel + c \parallel x - Ty, x_2, ..., x_n \parallel,$$

$a+b+c = 1$, then T has at least a fixed point.

Proof. By *Lemma* 3.1 we can obtain a minimal element D of F_C, then we are ready to prove D has only one point.

We assume D has more than one point, then there exists $u \in D$, such that

$$\alpha = sup_{x\in D} \| x-u, c_{i_2}, ..., c_{i_n} \| < \delta(D).$$

For every $x \in D$

$$\begin{aligned} & \| T(u)-T(x), c_{i_2}, ..., c_{i_n} \| \\ \leq \; & a \| u-x, c_{i_2}, ..., c_{i_n} \| + b \| u-T(u), c_{i_2}, ..., c_{i_n} \| + c \| u-T(x), c_{i_2}, ..., c_{i_n} \| \\ \leq \; & (a+b+c) \sup_{x\in D} \| u-x, c_{i_2}, ..., c_{i_n} \| \\ \leq \; & \alpha. \end{aligned}$$

Using the same technique as *theorem* 3.3, we can get condition (3.2)-(3.4). And we continue the proof to prove D' is $T-$ invariant in this case. Actually, for every $w \in D' \subsetneqq D$, because D is the minimal element in F, we have $D = \overline{Co\ TD}_C$ by *Lemma* 3.2. For each $y \in D$, and arbitrary $\varepsilon > 0$, there exists $x_i \in D$, $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$, such that

$$\| y - \sum_{i=1}^{n} \lambda_i T(x_i), c_{i_2}, ..., c_{i_n} \| < \varepsilon.$$

We can see that

$$\begin{aligned} & \| T(w)-y, c_{i_2}, ..., c_{i_n} \| \leq \| T(w) - \sum_{i=1}^{n} \lambda_i T(x_i), c_{i_2}, ..., c_{i_n} \| + \| \sum_{i=1}^{n} \lambda_i T(x_i) - y, c_{i_2}, ..., c_{i_n} \| \\ \leq \; & \sum_{i=1}^{n} \lambda_i \| T(w)-T(x_i), c_{i_2}, ..., c_{i_n} \| + \varepsilon \\ \leq \; & \sum_{i=1}^{n} \lambda_i [a \| w-x_i, c_{i_2}, ..., c_{i_n} \| + b \| x_i - T(x_i), c_{i_2}, ..., c_{i_n} \| + c \| x_i - T(w), c_{i_2}, ..., c_{i_n} \|] + \varepsilon \\ \leq \; & \alpha + \varepsilon. \end{aligned}$$

Then $Tw \in D'$, thus $D' \in F$, which contradicts the minimality of D. We know that D has only one point, which is fixed by T, thus we complete the proof. It is obvious that *theorem* 3.6 is the extension of *theorem* 3.5.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] S.Gähler, Lineare 2-normierte Räume, Math. Nachr. 28 (1964), 1-43.

[2] S.Gähler, Untersuchungen uber verallgemeinerte m-metrische Räume I, Math. Nachr. 40 (1969), 165-189.

[3] S.Gähler, Untersuchungen uber verallgemeinerte m-metrische Räume II, Math. Nachr. 40 (1969), 229-264.

[4] S.Gähler, Untersuchungen uber verallgemeinerte m-metrische Räume III, Math. Nachr. 41 (1970), 23-26.

[5] K.Iseki, P.L.Sharma and B.K.Sharma, Contration type mapping on 2-metric space, Math. Jap. 21 (1976), 67-70.

[6] K.Iseki, Mathematics on two normed spaces, Bull. Korean. Math. Soc. 3 (1976), 127-135.

[7] H.Gunawan and M.Mashadi, On n-normed spaces, Inter. J. Math. and Math. Sci. 27 (2001), 631-639.

[8] H.Gunawan and M.Mashadi, On finite dimensional 2-normed spaces, Soochow J. Math. 27 (2001), 321-329.

[9] H.Gunawan and E.Sukaesih, Fixed point theorems on bounded sets in an n-normed space, J. Math. Anal. 3 (2015),51-58.

[10] G.X Yan, n-normed spaces, J. Harbin Normal University: Natural Science. 1 (1985), 7-14.

[11] A.G.White, 2-Banach spaces, Math. Nachr. 42 (1969), 43-60.

[12] N.Redjel and A.Dehici, Some results in fixed point theory and application to the convergence of some iterative processes, Fixed Point Theory and Applications. 173 (2015), 1-17.

[13] Z.P Fan, The strong and weak convergence of sequence in n- Banach spaces, J. Gansu Lianhe University. 5 (2008), 3-8.

[14] A.Misiak, n-inner product spaces, Mathematische Nachrichten. 140 (1989), 299-319.

[15] K.Goebel and W.A.Kirk, Topics in Metric Fixed Point Theory, Cambridge Stud. Adv. Math. Cambridge Univ. Press, Cambridge, 1990.

[16] Katsuhiko Matsuzaki, Uniform convexity, normal structure and the fixed point property of metric spaces, Topology and its Applications. 196 (2015), 684-695.

[17] Clarkson J.A, Uniformly convex spaces, Trans. Amer. Math. Soc. 40 (1936), 396-414.

[18] M.Kir and H.Kiziltunc, On fixed point theorems for contraction mappings in n-normed spaces, Appl. Math. Inf. Lett. 2 (2014),59-64.