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SOME RESULTS ON 2-FUZZY n-NORMED LINEAR SPACES

WENWEN YANG

Department of Mathematics, Tianjin University of Technology, Tianjin 300384, China

Abstract. In this paper, standard results in fuzzy n-normed linear spaces are extended to 2-fuzzy n-normed linear spaces. Also the equivalence of α-n-norms and the Riesz theorem are proved in the real of 2-fuzzy n-normed linear spaces.

Keywords: 2-fuzzy n-norm; α-n-norm; cauchy sequence; equivalent norms; fuzzy n-compact set.

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1. Introduction

In 1989, Misiak [8,9] defined the concept of n-normed spaces and investigated the properties of these spaces. The concept of fuzzy set was introduced by Zadeh [7] in 1965. Cheng [6] and Bag and Samanta [5] introduced a concept of fuzzy norm on a linear space.

Recently, C. Park and C. Alaca [1] introduced the concept of 2-fuzzy n-normed linear spaces. The authors gave the notion of α-n-norm on a linear space corresponding to the 2-fuzzy n-norme by using some ideas of Bag and Samanta [5]. In [2], B. S. Reddy and H. Dutta investigated some properties of linear n-normed spaces. Some fundamental properties in fuzzy 2-normed space in terms of α-2-norms was discussed by Somasundaram and Beaula [10].

E-mail address: 15922083505@163.com

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In this paper, we introduce the concept of cauchy sequence, convergence and completeness in fuzzy n-normed linear space in term of α-n-norms, and then we give some fundamental properties of this space and obtain necessary and sufficient conditions for α-n-norms to be equivalent when X is a 2-fuzzy n-normed linear space or $F(X)$ is a fuzzy n-normed linear space. Finally, we generalize the Riesz theorem to 2-fuzzy n-normed linear spaces.

2. Preliminaries

Definition 2.1. [8,9] Let X be a real linear space of dimension greater than $n-1$ and let $\|\cdot,\cdots,\cdot\|$ be a real valued function on X^n satisfying the following condition:

(1) $\|x_1,x_2,\cdots,x_n\|=0$ if and only if $x_1,x_2,\cdots,x_n$ are linearly dependent;

(2) $\|x_1,x_2,\cdots,x_n\|$ is invariant under any permutation;

(3) $\|\alpha x_1,x_2,\cdots,x_n\|=|\alpha|\|x_1,x_2,\cdots,x_n\|$ for any $\alpha\in\mathbb{R}$;

(4) $\|x_0+x_1,x_2,\cdots,x_n\|\leq\|x_0,x_2,\cdots,x_n\|+\|x_1,x_2,\cdots,x_n\|$ for all $x_0,x_1,\cdots,x_n\in X$.

$\|\cdot,\cdots,\cdot\|$ is called an *n-norm* on X and the pair $(X,\|\cdot,\cdots,\cdot\|)$ is called an *n-normed linear space*.

Definition 2.2. [1] Let X be a linear space over K (field of real or complex numbers). A fuzzy subset N of $X^n\times\mathbb{R}$ is called a fuzzy n-norm on X if and only if :

(N1) For all $t\in\mathbb{R}$ with $t\leq 0$, $N(x_1,x_2,\cdots,x_n,t)=0$;

(N2) For all $t\in\mathbb{R}$ with $t>0$, $N(x_1,x_2,\cdots,x_n,t)=1$ if and only if $x_1,x_2,\cdots,x_n$ are linearly dependent;

(N3) $N(x_1,x_2,\cdots,x_n,t)$ is invariant under any permutation of $x_1,x_2,\cdots,x_n$;

(N4) For all $t\in\mathbb{R}$ with $t>0$, $N(x_1,x_2,\cdots,\lambda x_n,t)=N(x_1,x_2,\cdots,x_n,\frac{t}{|\lambda|})$, if $\lambda\neq 0$;

(N5) For all $s,t\in\mathbb{R}$, $N(x_1,x_2,\cdots,x_n+x_n^{'},s+t)\geq\min\{N(x_1,x_2,\cdots,x_n,s),N(x_1,x_2,\cdots,x_n^{'},t)\}$;

(N6) $N(x_1,x_2,\cdots,x_n,t)$ is a non-decreasing function of $t\in\mathbb{R}$ and $\lim\limits_{t\to\infty}N(x_1,x_2,\cdots,x_n,t)=1$.

Then (X,N) is called a fuzzy n-normed linear space or *f-n-NLS* in short.

Theorem 2.3. [1] *Let (X,N) be an f-n-NLS. Assume further that*

(N7) *$N(x_1,x_2,\cdots,x_n,t)>0$ for all $t>0$ implies that $x_1,x_2,\cdots,x_n$ are linearly dependent.*

Define

$\|x_1,x_2,\cdots,x_n\|_\alpha = inf\{t : N(x_1,x_2,\cdots,x_n,t) \geq \alpha,\ \alpha \in (0,1)\}$. *Then* $\{\|x_1,x_2,\cdots,x_n\|_\alpha : \alpha \in (0,1)\}$ *is an ascending family of n-norms on X (or)* α*-n-norms on X corresponding to the fuzzy n-norm on X.*

Definition 2.4. [10] Let X be a non-empty set and $F(X)$ be the set of all fuzzy sets on X. For $U,V \in F(X)$ and $k \in K$ the field of real numbers, define

$$U+V = \{(x+y, \lambda \wedge \mu) : (x,\lambda) \in U, (y,\mu) \in V\}.$$

and $kU = \{(kx,\lambda) : (x,\lambda) \in U\}$.

Definition 2.5. [10] A fuzzy linear space $\widetilde{X} = X \times (0,1]$ over the number field K where the addition and scalar multiplication operation on X are defined by $(x,\lambda)+(y,\mu)=(x+y,\lambda \wedge \mu)$, $k(x,\lambda) = (kx,\lambda)$ is a fuzzy normed space if for every $(x,\lambda) \in \widetilde{X}$ there is associated a non-negative real number, $\|(x,\lambda)\|$, called the fuzzy norm of (x,λ) such that

(1) $\|(x,\lambda)\|=0$ iff $x=0$ the zero element of $X, \lambda \in (0,1]$;

(2) $\|k(x,\lambda)\|=|k|\|(x,\lambda)\|$ for all $(x,\lambda) \in \widetilde{X}$ and all $k \in K$;

(3) $\|(x,\lambda)+(y,\mu)\| \leq \|(x,\lambda \wedge \mu)\| + \|(y,\lambda \wedge \mu)\|$ for all $(x,\lambda),(y,\mu) \in \widetilde{X}$;

(4) $\|(x, \vee_t \lambda_t)\| = \wedge_t \|(x,\lambda_{t)}\|$ for all $\lambda_t \in (0,1]$.

Definition 2.6. [10] Let X be a non-empty set and $F(X)$ be the set of all fuzzy sets on X. If $f \in F(X)$ then $f = \{(x,\mu) : x \in \text{X and } \mu \in (0,1]\}$. Clearly f is a bounded function with $|f(x)| \leq 1$. Let K be the space of real numbers. Then $F(X)$ is a linear space over the field K where the addition and scalar multiplication are defined by

$$f+g = \{(x,\mu)+(y,\eta)\} = \{(x+y,\mu \wedge \eta) : (x,\mu) \in f, (y,\eta) \in g\}.$$

and

$$kf = \{(kf,\mu) : (x,\mu) \in f\}.$$

where $k \in K$.

The linear space $F(X)$ is said to be a normed space if for every $f \in F(X)$, there is associated a non-negative real number $\|f\|$ called the norm of f such that

(1)$\|f\| = 0$ if and only if $f = 0$. For

$$\|f\| = 0 \Longleftrightarrow \{\|(x,\mu)\| : (x,\mu) \in f\} = 0 \Longleftrightarrow x = 0, \mu \in (0,1] \Longleftrightarrow f = 0.$$

(2) $\|kf\| = |k|\|f\|, k \in K$. For

$$\|kf\| = \{\|k(x,\mu)\| : (x,\mu) \in f, k \in K\} = \{|k|\|x,\mu\| : (x,\mu) \in f\} = |k|\|f\|.$$

(3)$\|f+g\| \leq \|f\| + \|g\|$ for every $f, g \in F(X)$. For

$$\begin{aligned} \|f+g\| &= \{\|(x,\mu)+(y,\eta)\| : x,y \in X, \mu,\eta \in (0,1]\} \\ &= \{\|(x,y),(\mu \wedge \eta)\| : x,y \in X, \mu,\eta \in (0,1]\} \\ &\leq \{\|(x,\mu \wedge \eta)\| + \|(y,\mu \wedge \eta)\| : (x,\mu) \in f, (y,\eta) \in g\} \\ &= \|f+g\|. \end{aligned}$$

Then $(F(X), \|.\|)$ is a normed linear space.

Definition 2.7. [10] A 2-fuzzy set on X is a fuzzy on $F(X)$.

Definition 2.8. [1] Let X be a real vector space of dimension $d \geq n (n \in N)$ and $F(X)$ be the set of all fuzzy sets in X. Here we allow d to be infinite. Assume that a $[0,1]$ valued function $\|\cdot,\cdots,\cdot\|$ on $F(X)^n$ satisfies the following properties

(1) $\|f_1, f_2, \cdots, f_n\| = 0$ if and only if $f_1, f_2, \cdots, f_n$ are linearly dependent;

(2) $\|f_1, f_2, \cdots, f_n\|$ is invariant under any permutation;

(3) $\|f_1, f_2, \cdots, \lambda f_n\|$=$|\lambda|\|f_1, f_2, \cdots, f_n\|$ for any $\lambda \in K$;

(4) $\|f_1, f_2, \cdots, f_{n-1}, y+z\| \leq \|f_1, f_2, \cdots, f_{n-1}, y\| + \|f_1, f_2, \cdots, f_{n-1}, z\|$.

Then $(F(X), \|\cdot,\cdots,\cdot\|)$ is an n-normed linear space.

Definition 2.9. [1] Let $F(X)$ be a linear space over K. A fuzzy subset N of $F(X)^n \times \mathbb{R}$ is called a 2-fuzzy n-norm on X (fuzzy n-norm on $F(X)$) if and only if:

(N1) For all $t \in \mathbb{R}$ with $t \leq 0$, $N(f_1, f_2, \cdots, f_n, t) = 0$;

(N2) For all $t \in \mathbb{R}$ with $t > 0$, $N(f_1, f_2, \cdots, f_n, t) = 1$ if and only if $f_1, f_2, \cdots, f_n$ are linearly dependent;

(N3) $N(f_1, f_2, \cdots, f_n, t)$ is invariant under any permutation of $f_1, f_2, \cdots, f_n$;

(N4) For all $t \in \mathbb{R}$ with $t > 0$, $N(f_1, f_2, \cdots, \lambda f_n, t) \doteq N(f_1, f_2, \cdots, f_n, \frac{t}{|\lambda|})$ for $\lambda \neq 0$ and $\lambda \in K$;

(N5) For all $s, t \in \mathbb{R}$, $N(f_1, f_2, \cdots, f_n + f_n', s+t) \geq$ min $\{N(f_1, f_2, \cdots, f_n, s), N(f_1, f_2, \cdots, f_n', t)$ };

(N6) $N(f_1, f_2, \cdots, f_n, .)$: $(0, +\infty) \to (0, 1)$ is continuous;

(N7) $\lim_{t\to\infty} N(f_1, f_2, \cdots, f_n, t) = 1$.

Then $(F(X), N)$ is a fuzzy n-normed linear spaces or (X, N) is a 2-fuzzy n-normed linear space.

Theorem 2.10. [1]*Let $(F(X), N)$ be a fuzzy n-normed linear space. Assume that*

(N8) $N(f_1, f_2, \cdots, f_n, t) > 0$ *for all* $t > 0$ *implies that* $f_1, f_2, \cdots, f_n$ *are linearly dependent.*

Define

$\|f_1, f_2, \cdots, f_n\|_\alpha = inf\{t : N(f_1, f_2, \cdots, f_n, t) \geq \alpha, \alpha \in (0, 1)\}$.

Then $\{\|\cdot, \cdots, \cdot\|_\alpha : \alpha \in (0, 1)\}$ *is an ascending family of n-norms on* $F(X)$. *The n-norms are called* α*-n-norms on* $F(X)$ *corresponding to the fuzzy n-norms.*

Hereafter we use the notion fuzzy n-norm on $F(X)$ instead of 2-fuzzy n-normed linear space on X.

3. Convergence and completeness for 2-fuzzy n-normed linear space

In this section, we shall discuss some elementary properties for sequence in a fuzzy n-normed linear space with respect to α-n-norms on $F(X)$.

Definition 3.1. A sequence $\{f_k\}$ in a fuzzy n-normed linear space $(F(X), N)$ is called a cauchy sequence if $\lim_{k,m\to\infty} \|f_k - f_m, \omega_2, \omega_3, \cdots, \omega_n\|_\alpha = 0$ with respect to α-n-norm for all $\omega_2, \cdots, \omega_n \in F(X)$.

Theorem 3.2. *Let $(F(X), N)$ be a fuzzy n-normed linear space.*

(a) *If $\{f_k\}$ is a cauchy sequence in $(F(X), N)$ with respect to α-n-norm, then $\{\|f_k, \omega_2, , \cdots, \omega_n\|_\alpha : \omega_2, , \cdots, \omega_n \in F(X)\}$ is a cauchy sequence of non-negative reals.*

(b) *If $\{f_k\}$ and $\{g_k\}$ are cauchy sequence in $(F(X), N)$ with respect to α-n-norm and $\{\alpha_k\}$ is a real cauchy sequence, then $\{f_k + g_k\}$ and $\{\alpha_k f_k\}$ are cauchy sequences in $(F(X), N)$ with respect to α-n-norm where $\alpha_k \in [0, 1]$.*

Proof. (a)

$$\begin{aligned} \|f_k, \omega_2, \cdots, \omega_n\|_\alpha &= \|f_k - f_m + f_m, \omega_2, \cdots, \omega_n\|_\alpha \\ &\leq \|f_k - f_m, \omega_2, \cdots, \omega_n\|_\alpha + \|f_m, \omega_2, \cdots, \omega_n\|_\alpha. \end{aligned}$$

Also

$$\|f_k,\omega_2,\cdots,\omega_n\|_\alpha-\|f_m,\omega_2,\cdots,\omega_n\|_\alpha\le\|f_k-f_m,\omega_2,\cdots,\omega_n\|_\alpha.$$

Similarly, we have

$$\|f_m,\omega_2,\cdots,\omega_n\|_\alpha-\|f_k,\omega_2,\cdots,\omega_n\|_\alpha\le\|f_k-f_m,\omega_2,\cdots,\omega_n\|_\alpha,$$

that is

$$|\|f_k,\omega_2,\cdots,\omega_n\|_\alpha-\|f_m,\omega_2,\cdots,\omega_n\|_\alpha|\le\|f_k-f_m,\omega_2,\cdots,\omega_n\|_\alpha.$$

Therefore, $\{\|f_k,\omega_2,,\cdots,\omega_n\|_\alpha\}$ is a real cauchy sequence, since we have $\lim\limits_{k,m\to\infty}\|f_k-f_m,\omega_2,\cdots,\omega_n\|_\alpha=0$.

(b)

$$\begin{aligned}\|(f_k+g_k)-(f_m+g_m),\omega_2,\cdots,\omega_n\|_\alpha &= \|(f_k-f_m)+(g_k-g_m),\omega_2,\cdots,\omega_n\|_\alpha\\ &\le \|f_k-f_m,\omega_2,\cdots,\omega_n\|_\alpha+\|g_k-g_m,\omega_2,\cdots,\omega_n\|_\alpha\\ &\to 0 \quad (k,m\to\infty).\end{aligned}$$

Therefore $\{f_k+g_k\}$ is a cauchy sequence on $(F(X),N)$ with respect to α-n-norm.

$$\begin{aligned}\|\alpha_kf_k-\alpha_mf_m,\omega_2,\cdots,\omega_n\|_\alpha &= \|\alpha_kf_k-\alpha_kf_m+\alpha_kf_m-\alpha_mf_m,\omega_2,\cdots,\omega_n\|_\alpha\\ &\le \|\alpha_kf_k-\alpha_kf_m,\omega_2,\cdots,\omega_n\|_\alpha+\|\alpha_kf_m-\alpha_mf_m,\omega_2,\cdots,\omega_n\|_\alpha\\ &= |\alpha_k|\|f_k-f_m,\omega_2,\cdots,\omega_n\|_\alpha+|\alpha_k-\alpha_m|\|f_m,\omega_2,\cdots,\omega_n\|_\alpha\\ &\to 0\end{aligned}$$

Since $\{\alpha_k\}$ and $\{\|f_k,\omega_2,\cdots,\omega_n\|_\alpha\}$ are real cauchy sequences. Therefore, $\{\alpha_kf_k\}$ is a cauchy sequence in $(F(X),N)$ with respect to α-n-norm.

Definition 3.3. A sequence $\{f_k\}$ in a fuzzy n-normed linear space $(F(X),N)$ is said to converge to f if $\|f_k-f,\omega_2,\omega_3,\cdots,\omega_n\|_\alpha\to 0$ as $k\to\infty$ with respect to α-n-norm for all $\omega_2,\cdots,\omega_n\in F(X)$.

Theorem 3.4. *In the fuzzy n-normed linear space* $(F(X),N)$.

(a) *if* $f_k \to f$ *and* $g_k \to g$*, then* $f_k + g_k \to f + g$ *as* $k \to \infty$.

(b) *if* $f_k \to f$ *and* $\alpha_k \to \alpha$*, then* $\alpha_k f_k \to \alpha f$.

(c) *if* $dim(F(X),N) \geq n$, $f_k \to f$ *and* $f_k \to g$ *then* $f = g$ *convergence is with respect to* α*-n-norm.*

Proof.

(a)

$$\begin{aligned}\|(f_k+g_k)-(f+g),\omega_2,\cdots,\omega_n\|_\alpha &= \|(f_k-f)+(g_k-g),\omega_2,\cdots,\omega_n\|_\alpha \\ &\leq \|f_k-f,\omega_2,\cdots,\omega_n\|_\alpha+\|g_k-g,\omega_2,\cdots,\omega_n\|_\alpha \to 0\end{aligned}$$

Therefore,$f_k + g_k \to f + g$.

(b) If $\omega_2,\cdots,\omega_n \in F(X)$

$$\begin{aligned}\|\alpha_k f_k-\alpha f,\omega_2,\cdots,\omega_n\|_\alpha &= \|\alpha_k f_k-\alpha_k f+\alpha_k f-\alpha f,\omega_2,\cdots,\omega_n\|_\alpha \\ &\leq \|\alpha_k f_k-\alpha_k f,\omega_2,\cdots,\omega_n\|_\alpha+\|\alpha_k f-\alpha f,\omega_2,\cdots,\omega_n\|_\alpha \\ &= |\alpha_k|\|f_k-f,\omega_2,\cdots,\omega_n\|_\alpha+|\alpha_k-\alpha|\|f,\omega_2,\cdots,\omega_n\|_\alpha \to 0.\end{aligned}$$

Since $\|f_k - f,\omega_2,\cdots,\omega_n\|_\alpha \to 0$ and $|\alpha_k - \alpha| \to 0$, it follows that $\alpha_k f \to \alpha f$.

(c) For any $\omega_2,\cdots,\omega_n \in F(X)$

$$\begin{aligned}\|f-g,\omega_2,\cdots,\omega_n\|_\alpha &= \|f_k-f_k+f-g,\omega_2,\cdots,\omega_n\|_\alpha \\ &\leq \|f_k-g,\omega_2,\cdots,\omega_n\|_\alpha+\|-(f_k-f),\omega_2,\cdots,\omega_n\|_\alpha \\ &= \|f_k-g,\omega_2,\cdots,\omega_n\|_\alpha+\|f_k-f,\omega_2,\cdots,\omega_n\|_\alpha \to 0.\end{aligned}$$

Since $f_k \to f$ and $f_k \to g$. Hence $f - g,\omega_2,\cdots,\omega_n$ are linearly dependent for all $\omega_2,\cdots,\omega_n \in F(X)$. Since $\dim(F(X),N) \geq n$, the possibility if $f - g$ can be linearly dependent for all $\omega_2,\cdots,\omega_n \in F(X)$ implies that $f - g = 0$ which implies that $f = g$.

Theorem 3.5. *Let $(F(X),N)$ be fuzzy n-normed linear space. If $\{f_k\}$ is a cauchy sequence in $F(X)$. Then $\{\|f_k - f, \omega_2, \cdots, \omega_n\|_\alpha\}$ is a cauchy sequence of non-negative reals for each $f \in F(X)$.*

Proof.

$$\begin{aligned} \|f_k - f, \omega_2, \cdots, \omega_n\|_\alpha &= \|f_k - f_m + f_m - f, \omega_2, \cdots, \omega_n\|_\alpha \\ &\leq \|f_k - f_m, \omega_2, \cdots, \omega_n\|_\alpha + \|f_m - f, \omega_2, \cdots, \omega_n\|_\alpha \end{aligned}$$

Therefore

$$\|f_k - f, \omega_2, \cdots, \omega_n\|_\alpha - \|f_m - f, \omega_2, \cdots, \omega_n\|_\alpha \leq \|f_k - f_m, \omega_2, \cdots, \omega_n\|_\alpha$$

Also

$$\big|\|f_k - f, \omega_2, \cdots, \omega_n\|_\alpha - \|f_m - f, \omega_2, \cdots, \omega_n\|_\alpha\big| \leq \|f_k - f_m, \omega_2, \cdots, \omega_n\|_\alpha.$$

Hence, $\|f_k - f, \omega_2, \cdots, \omega_n\|_\alpha$ is a cauchy sequence of non-negative reals for each $f \in F(X)$ since $\lim_{k,m\to\infty} \|f_k - f_m, \omega_2, \omega_3, \cdots, \omega_n\|_\alpha = 0$.

Theorem 3.6. *Let $(F(X),N)$ be a fuzzy n-normed linear space. If for all $\omega_2, \cdots, \omega_n \in F(X)$, $\lim_{k\to\infty} \|f_k - f, \omega_2, \omega_3, \cdots, \omega_n\|_\alpha = 0$, then $\lim_{k\to\infty} \|f_k, \omega_2, \omega_3, \cdots, \omega_n\|_\alpha = \|f, \omega_2, \cdots, \omega_n\|_\alpha$.*

Proof. Since

$$\big|\|f_k, \omega_2, \cdots, \omega_n\|_\alpha - \|f, \omega_2, \cdots, \omega_n\|_\alpha\big| \leq \|f_k - f, \omega_2, \cdots, \omega_n\|_\alpha,$$

it follows that

$$\big|\|f_k, \omega_2, \cdots, \omega_n\|_\alpha - \|f, \omega_2, \cdots, \omega_n\|_\alpha\big| \to 0 \quad (k \to \infty).$$

Hence

$$\lim_{k\to\infty} \|f_k, \omega_2, \omega_3, \cdots, \omega_n\|_\alpha = \|f, \omega_2, \cdots, \omega_n\|_\alpha.$$

Definition 3.7. The fuzzy n-normed linear space $(F(X),N)$ in which every cauchy sequence converges is called a complete fuzzy n-normed linear space. The fuzzy n-normed linear space

$(F(X),N)$ is a fuzzy n-Banach space with respect to α-n-norm for it is a complete fuzzy n-normed linear space with respect to α-n-norm.

4. The equivalence of α-n-norms in fuzzy n-normed linear spaces

In this section, we prove necessary and sufficient conditions for α-n-norms to be equivalent on a fuzzy n-normed linear space $F(X)$. Let X be a real vector space of dimension $d \geq n \, (n \in N)$, and let $F(X)$ be the set of all fuzzy sets in X.

Definition 4.1. Two α-n-norms $\|\cdot,\cdots,\cdot\|_\alpha^1$ and $\|\cdot,\cdots,\cdot\|_\alpha^2$ on a fuzzy n-normed linear space $F(X)$ are said to be equivalent if there exist constants $\beta > 0, \gamma > 0$ such that

$$\beta\|\omega_1,\omega_2,\cdots,\omega_n\|_\alpha^1 \leq \|\omega_1,\omega_2,\cdots,\omega_n\|_\alpha^2 \leq \gamma\|\omega_1,\omega_2,\cdots,\omega_n\|_\alpha^1 \quad \forall \omega_1,\omega_2,\cdots,\omega_n \in F(X).$$

Theorem 4.2. *Two α-n-norms $\|\cdot,\cdots,\cdot\|_\alpha^1$ and $\|\cdot,\cdots,\cdot\|_\alpha^2$ are equivalent on a fuzzy n-normed linear space $F(X)$ if and only if every cauchy sequence with respect to one of the α-n-norms is a cauchy sequence with respect to other α-n-norm.*

Proof. Suppose that two α-n-norms $\|\cdot,\cdots,\cdot\|_\alpha^1$ and $\|\cdot,\cdots,\cdot\|_\alpha^2$ are equivalent on a fuzzy n-normed linear space $F(X)$. Then there exists constants $\beta > 0$, $\gamma > 0$ such that

$$\beta\|\omega_1,\omega_2,\cdots,\omega_n\|_\alpha^1 \leq \|\omega_1,\omega_2,\cdots,\omega_n\|_\alpha^2 \leq \gamma\|\omega_1,\omega_2,\cdots,\omega_n\|_\alpha^1 \quad \forall \omega_1,\omega_2,\cdots,\omega_n \in F(X)$$

For a sequence $\{f_k\}$ in $F(X)$, we have

$$\beta\|f_k - f_m,\omega_2,\cdots,\omega_n\|_\alpha^1 \leq \|f_k - f_m,\omega_2,\cdots,\omega_n\|_\alpha^2 \leq \gamma\|f_k - f_m,\omega_2,\cdots,\omega_n\|_\alpha^1 \tag{1}$$

for all $\omega_2,\cdots,\omega_n \in F(X)$ and $k,m \in N$. The second inequality shows that if $\{f_k\}$ is a cauchy sequence with respect to $\|\cdot,\cdots,\cdot\|_\alpha^1$ if and only if it is a cauchy sequence with respect to $\|\cdot,\cdots,\cdot\|_\alpha^2$. For the converse part, suppose that the α-n-norms are not equivalent. Then without loss of generality we can assume the following two cases:

(a)There does not exist β such that

$$\beta\|\omega_1,\omega_2,\cdots,\omega_n\|_\alpha^1 \leq \|\omega_1,\omega_2,\cdots,\omega_n\|_\alpha^2 \quad \forall \omega_1,\omega_2,\cdots,\omega_n \in F(X).$$

(b)There does not exist γ such that

$$\|\omega_1,\omega_2,\cdots,\omega_n\|_\alpha^2 \leq \gamma\|\omega_1,\omega_2,\cdots,\omega_n\|_\alpha^1 \quad \forall\omega_1,\omega_2,\cdots,\omega_n \in F(X)$$

In case (a) for $k = 1,2,\cdots$, there exist $\{f_k\}$ in $F(X)$ such that

$$\frac{1}{k}\|f_k,\omega_2,\cdots,\omega_n\|_\alpha^1 > \|f_k,\omega_2,\cdots,\omega_n\|_\alpha^2 \tag{2}$$

Let $g_k = \frac{1}{\sqrt{k}}\frac{1}{\|f_k,\omega_2,\cdots,\omega_n\|_\alpha^2}f_k$, for each $k \in N$

Then $\|g_k,\omega_2,\cdots,\omega_n\|_\alpha^2 = \frac{1}{k} \to 0$ as $k \to \infty$.

and using (b)we get

$$\|g_k,\omega_2,\cdots,\omega_n\|_\alpha^1 = \frac{1}{\sqrt{k}}\frac{1}{\|f_k,\omega_2,\cdots,\omega_n\|_\alpha^2}\|f_k,\omega_2,\cdots,\omega_n\|_\alpha^1 > \frac{k}{\sqrt{k}} = \sqrt{k} \to \infty(k \to \infty).$$

Note that every convergent sequence in a fuzzy n-normed linear space $F(X)$ is a cauchy sequence. $\{g_k\}$ is a cauchy sequence with sespect to $\|\cdot,\cdots,\cdot\|_\alpha^2$ but not with respect to $\|\cdot,\cdots,\cdot\|_\alpha^1$. Similarly, we can prove the case (b). Hence the proof of the theorem is complete.

Corollary 4.3. *Let $\|\cdot,\cdots,\cdot\|_\alpha^1$ and $\|\cdot,\cdots,\cdot\|_\alpha^2$ be two equivalent α-n-norms on a fuzzy n-normed linear space $F(X)$. Then $f_k \to f$ with respect to $\|\cdot,\cdots,\cdot\|_\alpha^1$ if and only if $f_k \to f$ with respect to $\|\cdot,\cdots,\cdot\|_\alpha^2$.*

Proof. By replacing $'f_k \to f_m'$ with $'f_k \to f'$ in (a) of Theorem 4.2, we get the result.

5. The Riesz theorem in fuzzy n-normed linear spaces

Definition 5.1. A subset Y of $F(X)$ is said to be a fuzzy n-compact subset with respect to α-n-norm if for every sequence $\{y_k\}$ in Y, there exists a subsequence $\{y_{n_k}\}$ of $\{y_k\}$ which converges to an element $y \in Y$.

Lemma 5.2. *Let $F(X)$ be a fuzzy n-normed linear space, and let Y be a fuzzy n-compact subspace of $F(X)$. For $f_1,\cdots,f_n \in F(X)$, $inf_{y\in Y}\|f_1-y,\cdots,f_n-y\|_\alpha = 0$ then there exists an element $y_0 \in Y$ such that $\|f_1-y_0,\cdots,f_n-y_0\|_\alpha = 0$.*

Proof. For each positive integer k, there exists an element $y_k \in Y$ such that

$$\|f_1-y_k,\cdots,f_n-y_k\|_\alpha \leq \frac{1}{k}$$

Since $\{y_k\}$ is a sequence in a fuzzy n-compact space Y, we can consider that $\{y_k\}$ is a convergent sequence in Y without loss of generality. Let $y_k \to y_0$ as $k \to \infty$ for some $y_0 \in Y$. For every $\varepsilon > 0$, there exists a positive integer K such that if $k \geq K$, then $\frac{1}{k} < \frac{\varepsilon}{n+1}$ and $\|y_k - y_0, \omega_2, \cdots, \omega_n\|_\alpha < \frac{\varepsilon}{n+1}$ for all $\omega_i \in F(X)(i = 2, \cdots, n)$. For $k > K$, we have

$$\begin{aligned}
\|f_1 - y_0, f_2 - y_0, \cdots, f_n - y_0\|_\alpha &= \|y_k - y_0 + f_1 - y_k, f_2 - y_0 \cdots, f_n - y_0\|_\alpha \\
&\leq \|y_k - y_0, f_2 - y_0, \cdots, f_n - y_0\|_\alpha \\
&+ \|f_1 - y_k, f_2 - y_0, \cdots, f_n - y_0\|_\alpha \\
&\leq \|y_k - y_0, f_2 - y_0, \cdots, f_n - y_0\|_\alpha \\
&+ \|f_1 - y_k, y_k - y_0, f_3 - y_0 \cdots, f_n - y_0\|_\alpha \\
&+ \|f_1 - y_k, f_2 - y_k, f_3 - y_0 \cdots, f_n - y_0\|_\alpha \\
&\leq \|y_k - y_0, f_2 - y_0, \cdots, f_n - y_0\|_\alpha \\
&+ \|f_1 - y_k, y_k - y_0, f_3 - y_0, \cdots, f_n - y_0\|_\alpha \\
&+ \|f_1 - y_k, f_2 - y_k, y_k - y_0, \cdots, f_n - y_0\|_\alpha \\
&+ \|f_1 - y_k, f_2 - y_k, f_3 - y_k, \cdots, f_n - y_0\|_\alpha \\
&\;\;\vdots \\
&\leq \|y_k - y_0, f_2 - y_0, \cdots, f_n - y_0\|_\alpha \\
&+ \|f_1 - y_k, y_k - y_0, f_3 - y_0, \cdots, f_n - y_0\|_\alpha \\
&+ \|f_1 - y_k, f_2 - y_k, y_k - y_0, \cdots, f_n - y_0\|_\alpha \\
&+ \cdots \\
&+ \|f_1 - y_k, f_2 - y_k, f_3 - y_k, \cdots, y_k - y_0, f_n - y_0\|_\alpha \\
&+ \|f_1 - y_k, f_2 - y_k, f_3 - y_k, \cdots, f_{n-1} - y_k, f_n - y_0\|_\alpha \\
&\leq \|y_k - y_0, f_2 - y_0, \cdots, f_n - y_0\|_\alpha \\
&+ \|f_1 - y_k, y_k - y_0, f_3 - y_0, \cdots, f_n - y_0\|_\alpha
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \|f_1-y_k, f_2-y_k, y_k-y_0, \cdots, f_n-y_0\|_\alpha \\
&+ \quad \cdots \\
&+ \quad \|f_1-y_k, f_2-y_k, f_3-y_k, \cdots, y_k-y_0, f_n-y_0\|_\alpha \\
&+ \quad \|f_1-y_k, f_2-y_k, f_3-y_k, \cdots, f_{n-1}-y_k, y_k-y_0\|_\alpha \\
&+ \quad \|f_1-y_k, f_2-y_k, f_3-y_k, \cdots, f_{n-1}-y_k, f_n-y_k\|_\alpha \\
&= \quad \|y_k-y_0, f_2-y_0, \cdots, f_n-y_0\|_\alpha \\
&+ \quad \|f_1-y_0, y_k-y_0, f_3-y_0, \cdots, f_n-y_0\|_\alpha \\
&+ \quad \|f_1-y_0, f_2-y_0, y_k-y_0, \cdots, f_n-y_0\|_\alpha \\
&+ \quad \cdots \\
&+ \quad \|f_1-y_0, f_2-y_0, f_3-y_0, \cdots, y_k-y_0, f_n-y_0\|_\alpha \\
&+ \quad \|f_1-y_0, f_2-y_0, f_3-y_0, \cdots, f_{n-1}-y_0, y_k-y_0\|_\alpha \\
&+ \quad \|f_1-y_k, f_2-y_k, f_3-y_k, \cdots, f_{n-1}-y_k, f_n-y_k\|_\alpha \\
&< \quad n\frac{\varepsilon}{n+1}+\frac{1}{k} < n\frac{\varepsilon}{n+1}+\frac{1}{K} < n\frac{\varepsilon}{n+1}+\frac{\varepsilon}{n+1} = \varepsilon.
\end{aligned}
$$

Since ε is arbitrary, $\|f_1-y_0,\cdots,f_n-y_0\|_\alpha = 0$.

Theorem 5.3. *Let Y and Z be subspaces of fuzzy n-normed linear spaces $(F(X),N)$, and let Y be a fuzzy n-compact proper subset of Z with dimension greater than $n-1$. For each $\theta \in (0,1)$, there exist an element $(f_1,\cdots,f_n) \in Z^n$ such that $\|f_1,\cdots,f_n\|_\alpha = 1$, $\|f_1-y,\cdots,f_n-y\|_\alpha \geq \theta$ for all $y \in Y$.*

Proof. Let $h_1,\cdots,h_n \in Z \cap Y^\perp$ be linearly independent. Let

$$a = inf_{y\in Y}\|h_1-y,\cdots,h_n-y\|_\alpha$$

Assume that $a = 0$. By Lemma 1, there is an element $y_0 \in Y$ such that

$$\|h_1-y_0,\cdots,h_n-y_0\|_\alpha = 0 \tag{1}$$

If $y_0 = 0$, then $\|h_1, \cdots, h_n\|_\alpha = 0$. This implies that $\inf\{t : N(h_1, h_2, \cdots, h_n, t) \geq \alpha\} = 0$. Then $h_1, \cdots, h_n$ are linearly dependent. This leads to a contradiction. So y_0 is non-zero. Hence $h_1, \cdots, h_n, y_0$ are linear independent. But from definition and from (1) that $h_1 - y_0, \cdots, h_n - y_0$ are linearly dependent. This exist real numbers $\alpha_1, \cdots, \alpha_n$ not all zero such that

$$\alpha_1(h_1 - y_0) + \cdots + \alpha_n(h_n - y_0) = 0$$

This we have

$$\alpha_1 h_1 + \cdots + \alpha_n h_n + (-1)(\alpha_1 + \cdots + \alpha_n)y_0 = 0$$

Then $h_1, \cdots, h_n, y_0$ are linear dependent, which is a contradiction. Hence $a > 0$. For each $\theta \in (0,1)$, there exists an element $y_0 \in Y$ such that

$$a \leq \|h_1 - y_0, \cdots, h_n - y_0\|_\alpha \leq \frac{a}{\theta}.$$

For each $j = 1, \cdots, n$, let

$$f_j = \frac{h_j - y_0}{\|h_1 - y_0, \cdots, h_n - y_0\|_\alpha^{\frac{1}{n}}}$$

That it is obvious that $\|f_1, \cdots, f_n\|_\alpha = 1$

$$\begin{aligned}
\|f_1 - y, \cdots, f_n - y\|_\alpha &= \left\| \frac{h_1 - y_0}{\|h_1 - y_0, \cdots, h_n - y_0\|_\alpha^{\frac{1}{n}}} - y, \cdots, \frac{h_n - y_0}{\|h_1 - y_0, \cdots, h_n - y_0\|_\alpha^{\frac{1}{n}}} - y \right\|_\alpha \\
&= \frac{1}{\|h_1 - y_0, \cdots, h_n - y_0\|_\alpha} \|h_1 - (y_0 + y\|h_1 - y_0, \cdots, h_n - y_0\|_\alpha^{\frac{1}{n}}), \cdots, \\
&\quad h_n - (y_0 + y\|h_1 - y_0, \cdots, h_n - y_0\|_\alpha^{\frac{1}{n}})\|_\alpha \\
&\geq \frac{1}{\|h_1 - y_0, \cdots, h_n - y_0\|_\alpha} a \geq \frac{a}{\frac{a}{\theta}} = \theta
\end{aligned}$$

for all $y \in Y$. This completes the proof.

Definition 5.4. A subset Y of the fuzzy n-normed linear space $(F(X), N)$ is called a fuzzy partially n-closed subset if for linear independent elements $f_1, \cdots, f_n \in F(X)$ there exists a sequence y_k in Y such that $\|f_1 - y_k, \cdots, f_n - y_k\|_\alpha \to 0$ as $k \to \infty$, then $f_j \in Y$ for some j.

Theorem 5.5. *Let Y, Z be subspaces of the fuzzy n-normed linear space $F(X)$, and let Y be a fuzzy partially n-closed subset of Z. Assume that dimZ $\geq n$. For each $\theta \in (0,1)$, there exists an* *element* $(f_1,\cdots,f_n)\in Z^n$ *such that*

$$\|f_1,\cdots,f_n\|_\alpha = 1$$

$$\|f_1-y,\cdots,f_n-y\|_\alpha \geq \theta$$

for all $y\in Y$.

Proof. Let $h_1,\cdots,h_n\in Z-Y$ be linearly independent. Let

$$a = inf_{y\in Y}\|h_1-y,\cdots,h_n-y\|_\alpha$$

Assume that $a=0$. Then there is a sequence $\{y_k\}$ in Y such that $\|h_1-y_k,\cdots,h_n-y_k\|_\alpha\to 0$ as $k\to\infty$. Since Y is fuzzy partially n-closed, $h_j\in Y$ for some j, which is a contradiction. Hence $a>0$. The rest of the proof is the same as in the proof of Theorem 5.3.

Conflict of Interests

The authors declare that there is no conflict of interests.

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