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APPROXIMATING SOLUTION FOR GENERALIZED QUADRATIC FUNCTIONAL INTEGRAL EQUATIONS WITH MAXIMA

DNYANESHWAR V. MULE[1,2,*], BHIMRAO R. AHIRRAO[3] AND VILAS S. PATIL[4]

[1]Department of Mathematics, North Maharashtra University, Jalgaon 425001, India

[2]Department of Mathematics, Mahatma Phule College, Kingaon, Ahmedpur-413515, India

[3]Department of Mathematics, Z.B. Patil College, Dhule 424002, India

[4]Department of Mathematics, S.I.C.E. Society's Degree College of Arts, Science and Commerce,

Ambernath(W)-421505, India

Abstract. In this paper we prove an existence as well as approximation result for a certain nonlinear generalized quadratic functional integral equation with maxima. An algorithm for the solutions is developed and it is shown that the sequence of successive approximations starting with a lower or an upper solution converges monotonically to the solution of the related quadratic functional integral equation with maxima under some suitable mixed hybrid conditions. We base our main results on the Dhage iteration principle embodied in a recent hybrid fixed point theorem of Dhage (2014) in a partially ordered normed linear space.An example illustrating the existence result is also presented.

Keywords:quadratic functional integral equation; approximate positive solution; fixed point theorem.

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1. Introduction

*Corresponding author

E-mail address: dnyaneshmule56@gmail.com

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The quadratic integral equations have been a topic of interest since long time because of their occurrence in the problems of some natural and physical processes of the universe. See Argyros [2], Deimling [6], Chandrasekher [4] and the references therein. The study gained momentum after the formulation of fixed point principles in Banach algebras due to Dhage [10],[12]. The existence results for such equations are generally proved under the mixed Lipschitz and compactness type conditions together with a certain growth condition on the nonlinearities of the quadratic integral equations. See Dhage [10],[12] and the references therein. The Lipschitz and compactness hypotheses are considered to be very strong conditions in the theory of nonlinear differential and integral equations which do not yield any algorithm to determine the numerical solutions. Therefore, it is of interest to relax or weaken these conditions in the existence and approximation theory of quadratic integral equations.However, the literature on existence results for a special class of functional differential equations, namely nonlinear quadratic differential equations with maxima under weaker partial Lipschitz and partial compactness type conditions via Dhage iteration method. is not enriched yet, recently, the first authors in [12] ,[13],[14] have studied the existence results of functional differential equations with maxima. Therefore, it is admirable to extend this method to nonlinear quadratic integral equations with maxima. This is the main motivation of the present paper.

In this paper we prove the existence as well as approximations of the positive solutions of a certain generalized quadratic integral equation with maxima via an algorithm based on successive approximations under partially Lipschitz and compactness conditions

Given a closed and bounded interval $J=[0,T]$ of the real line R for some $T>0$, we consider the quadratic functional integral equation (in short GQFIE) with maxima

$$x(t)=k(t,x(t),X(t))+[f(t,x(t),X(t))]\left(a(t)+\int_0^t v(t,s)g(s,x(s),X(s))ds\right) \tag{1}$$

for all $t\in J$, $v:J\times R\to R$ and $f,g:J\times R\times R\to R$, are continuous functions, and $X(t)=\max_{0\leq\eta\leq t}x(\eta)$.

By a *solution* of the GQFIE (1) with maxima we mean a function $x\in C(J,R)$ that satisfies the equation (1) on J, where $C(J,R)$ is the space of continuous real-valued functions defined on J.

The GQFIE (1) with maxima is new to the literature . In particular, If $f(t,x,y) = 0$ for all $t \in J$ and $x, y \in R$ the GQFIE (1) with maxima reduces to the nonlinear functional equation with maxima

$$x(t) = k(t, x(t), X(t))), \ t \in J, \tag{2}$$

and if $k(t,x,y) = 0$ and $f(t,x,y) = 1$ for all $t \in J$ and $x, y \in R$, it is reduced to nonlinear usual Volterra integral equation with maxima

$$x(t) = a(t) + \int_0^t v(t,s)g(s,x(s),X(s)))\,ds, \ t \in J. \tag{3}$$

Therefore, the QFIE (1) is general and the results of this paper include the existence and approximations results for above nonlinear functional and Volterra integral equations with maxima as special cases.

The paper is organized as follows. In the following section we give the preliminaries and auxiliary results needed in the subsequent part of the paper.

2. Preliminary Notes / Materials and Methods

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$ in which the addition and the scalar multiplication by positive real numbers are preserved by $\preceq$. A few details of a partially ordered normed linear space appear in Dhage [8], Heikkilä and Lakshmikantham [15] and the references therein.

We need the following notion and results.

Definition 2.1 : A mapping $\mathscr{T} : E \to E$ is called isotone or monotone nondecreasing if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathscr{T}x \preceq \mathscr{T}y$ for all $x, y \in E$. Similarly, $\mathscr{T}$ is called monotone nonincreasing if $x \preceq y$ implies $\mathscr{T}x \succeq \mathscr{T}y$ for all $x, y \in E$. Finally, $\mathscr{T}$ is called monotonic or simply monotone if it is either monotone nondecreasing or monotone nonincreasing on E.

Definition 2.2 : A mapping $\mathscr{T} : E \to E$ is called partially continuous at a point $a \in E$ if for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\mathscr{T}x - \mathscr{T}a\| < \varepsilon$ whenever x is comparable to a and $\|x - a\| < \delta$.

$\mathscr{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathscr{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 2.3 : A non-empty subset S of the partially ordered Banach space E is called partially bounded if every chain C in S is bounded. An operator $\mathscr{T}$ on a partially normed linear space E into itself is called partially bounded if $\mathscr{T}(E)$ is a partially bounded subset of E. $\mathscr{T}$ is called uniformly partially bounded if all chains C in $\mathscr{T}(E)$ are bounded by a unique constant.

Definition 2.4 : A non-empty subset S of the partially ordered Banach space E is called partially compact if every chain C in S is a relatively compact subset of E. A mapping $\mathscr{T}: E \to E$ is called partially compact if $\mathscr{T}(E)$ is a partially relatively compact subset of E. $\mathscr{T}$ is called uniformly partially compact if $\mathscr{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathscr{T}$ is called partially totally bounded if for any bounded subset S of E, $\mathscr{T}(S)$ is a partially relatively compact subset of E. If $\mathscr{T}$ is partially continuous and partially totally bounded, then it is called partially completely continuous on E.

Remark 2.1: Suppose that $\mathscr{T}$ is a nondecreasing operator on E into itself. Then $\mathscr{T}$ is a partially bounded or partially compact if $\mathscr{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.

Definition 2.5 : The order relation $\preceq$ and the metric d on a non-empty set E are said to be compatible if $\{x_n\}_{n\in N}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in N}$ of $\{x_n\}_{n\in N}$ converges to x^* implies that the original sequence $\{x_n\}_{n\in N}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.

Clearly, the set R of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space R^n with usual componentwise order relation and the standard norm possesses the compatibility property.

Definition 2.6 : A upper semi-continuous and monotone nondecreasing function $\psi: R_+ \to R_+$ is called a $\mathscr{D}$-function provided $\psi(r) = 0$ iff $r = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathscr{T}: E \to E$ is called partially nonlinear $\mathscr{D}$-Lipschitz if there exists a $\mathscr{D}$-function $\psi : R_+ \to R_+$ such that

$$\|\mathscr{T}x - \mathscr{T}y\| \leq \psi(\|x-y\|) \tag{4}$$

for all comparable elements $x,y \in E$. If $\psi(r) = kr$, $k > 0$, then $\mathscr{T}$ is called a partially Lipschitz with a Lipschitz constant k.

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\}$$

and

$$\mathscr{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u,v \in E^+\}. \tag{5}$$

The elements of $\mathscr{K}$ are called the positive vectors of the normed linear algebra E. The following lemma follows immediately from the definition of the set $\mathscr{K}$ and which is often times used in the applications of hybrid fixed point theory in Banach algebras.

Lemma 2.1 [Dhage [8]]**:** If $u_1, u_2, v_1, v_2 \in \mathscr{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.

Definition 2.7 : An operator $\mathscr{T} : E \to E$ is said to be positive if the range $R(\mathscr{T})$ of $\mathscr{T}$ is such that $R(\mathscr{T}) \subseteq \mathscr{K}$.

Theorem 2.1 [Dhage [8]]**:** Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear algebra such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain of E. Let $\mathscr{A}, \mathscr{B} : E \to \mathscr{K}$ and $\mathscr{C} : E \to E$ be three nondecreasing operators such that

(a) $\mathscr{A}$ and $\mathscr{C}$ is partially bounded and partially nonlinear $\mathscr{D}$-Lipschitz with $\mathscr{D}$-functions $\psi_{\mathscr{A}}$ and $\psi_{\mathscr{C}}$ respectively,

(b) $\mathscr{B}$ is partially continuous and uniformly partially compact, and

(c) $M\psi_{\mathscr{A}}(r)$ + $\psi_{\mathscr{C}}$,$< r$, $r > 0$, where $M = \sup\{\|\mathscr{B}(C)\| : C \text{ is a chain in } E\}$, and

(d) there exists an element $x_0 \in X$ such that $x_0 \preceq \mathscr{A}x_0\mathscr{B}x_0 + \mathscr{C}x_0$ or $x_0 \succeq \mathscr{A}x_0\mathscr{B}x_0 + \mathscr{C}x_0$.

Then the operator equation

$$\mathscr{A}x\mathscr{B}x + \mathscr{C}x = x \tag{6}$$

has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathscr{A} x_n \mathscr{B} x_n + \mathscr{C} x_n,\ n = 0, 1, \ldots,$ converges monotonically to x^*.

Remark 2.2: The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$. This simple fact has been utilized to prove the main results of this paper.

3. Results and Discussion

The QFIE (1) is considered in the function space $C(J,R)$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J,R)$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{7}$$

and

$$x \leq y \iff x(t) \leq y(t)\ \forall t \in J, \tag{8}$$

respectively.

Clearly, $C(J,R)$ is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach algebra $C(J,R)$ has some nice properties concerning the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in certain subsets of of it. The following lemma in this connection follows by an application of Arzelá-Ascoli theorem.

Lemma 3.1 : Let $\big(C(J,R), \leq, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (7) and (8) respectively. Then $\|\cdot\|$ and $\leq$ are compatible in every compact chain C in S.

Proof. Let S be a partially compact subset of $C(J,R)$ and let $\{x_n\}_{n\in N}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq \cdots, \tag{9}$$

for each $t \in J$.

Suppose that a subsequence $\{x_{n_k}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}_{k\in N}$ of the monotone real sequence $\{x_n(t)\}_{n\in N}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}_{n\in N}$ is convergent and converges to a point $x(t)$ in R for each $t\in J$. This shows that the sequence $\{x_n\}_{n\in N}$ converges to x point-wise on J. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}_{n\in N}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}_{n\in N}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}_{n\in N}$ is convergent and converges uniformly to x. As a result $\|\cdot\|$ and $\leq$ are compatible in S. This completes the proof.

We need the following definition in what follows.

Definition 3.1 : A function $p\in C(J,R)$ is said to be a lower solution of the GQFIE (1) if it satisfies

$$p(t)=k(t,p(t),P(t))+[f(t,p(t),P(t))]\left(a(t)+\int_0^t v(t,s)g(s,p(s),P(s))ds\right)$$

for all $t\in J$. Similarly, a function $q\in C(J,R)$ is said to be an upper solution of the GQFIE (1) with maxima if it satisfies the above inequalities with reverse sign.

We consider the following set of assumptions in what follows:

(H_1) f defines a nonnegative function $f: J\times R\times R\to R$

(H_2) There exists a constant $M_f>0$ such that $0\leq f(t,x,y)\leq M_f$ for all $t\in J$ and $x,y\in R$.

(H_3) There exists a $\mathscr{D}$-function ψ_f such that

$$0\leq f(t,x_1,x_2)-f(t,y_1,y_2)\leq \psi_f(\max\{x_1-y_1,x_1-y_1\})$$

for all $t\in J$ and $x_1,x_2,y_1,y_2\in R$, $x_1\geq y_1$, $x_2\geq y_2$.

(H_4) a defines a continuous function $a: J\to R_+$

(H_5) v defines a continuous and nonnegative function on $J\times J$

(H_6) $g(t,x,y)$ is nondecreasing in x and y for all $t\in J$.

(H_7) There exists a constant $M_g>0$ such that $g(t,x,y)\leq M_g$ for all $t\in J$ and $x,y\in R$.

(H_8) There exists a constant $M_k>0$ such that $k(t,x,y)\leq M_k$ for all $t\in J$ and $x,y\in R$.

(H_9) There exists a $\mathscr{D}$-function ψ_k such that

$$0 \le k(t,x_1,x_2) - k(t,y_1,y_2) \le \psi_k(\max\{x_1-y_1, x_1-y_1\})$$

for all $t \in J$ and $x_1,x_2,y_1,y_2 \in R$, $x_1 \ge y_1$, $x_2 \ge y_2$.

(H_{10}) The GQFIE (1) with maxima has a lower solution $p \in C(J,R)$.

Theorem 3.1 : Assume that hypotheses (H_1)-(H_{10}) hold. Furthermore, assume that

$$(\|a\| + VT\,M_g)\,\psi_f(r) + \psi_k(r) < r,\ r > 0, \tag{10}$$

then the GQFIE (1) with maxima has a solution x^* defined on J and the sequence $\{x_n\}_{n\in N}$ of successive approximations defined by

$$x_{n+1}(t) = k(t,x_n(t),X_n(t)) + [f(t,x_n(t),X_n(t))]\left(a(t) + \int_0^t v(t,s)g(s,x_n(s),X_n(s))\right)ds,\ t \in J. \tag{11}$$

for all $t \in J$, where $x_0 = p$ and $X_n(t) = \max_{0\le\eta\le t} x_n(\eta)$, converges monotonically to x^*.

Proof. Set $E = C(J,R)$. Then, from Lemma 3.1, it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ in E.

Define two operators $\mathscr{A}$ and $\mathscr{B}$ on E by

$$\mathscr{A}x(t) = f(t,x(t),X(t)),\ t \in J, \tag{12}$$

and

$$Bx(t) = a(t) + \int_{t_0}^t v(t,s)g(s,x(s),X(s))\,ds,\ t \in J, \tag{13}$$

and

$$\mathscr{C}x(t) = k(t,x(t),X(t)),\ t \in J, \tag{14}$$

From the continuity of the integral and the hypotheses (H_1)-(H_{10}), it follows that $\mathscr{A}$,$\mathscr{B}$ and $\mathscr{C}$ define the maps $\mathscr{A},\mathscr{B} : E \to \mathscr{K}$ and $\mathscr{C} : E \to E$. Now by definitions of the operators $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ the GQFIE (1) is equivalent to the operator equation

$$\mathscr{A}x(t)\mathscr{B}x(t) + \mathscr{C}x(t) = x(t),\ t \in J. \tag{15}$$

We shall show that the operators $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ satisfy all the conditions of Theorem 2.1. This is achieved in the series of following steps.

Step I: *$\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are nondecreasing on E.*

Let $x,y \in E$ be such that $x \geq y$. Then $x(t) \geq y(t)$ for all $t \in J$. Since y is continuous on $[a,t]$, there exists a $\eta^* \in [a,t]$ such that $y(\eta^*) = \max_{a\leq\eta\leq t} y(\eta)$. By definition of $\leq$, one has $x(\eta^*) \geq y(\eta^*)$. Consequently, we obtain

$$X(t) = \max_{a\leq\eta\leq t} x(\eta) = x(\eta^*) \geq y(\eta^*) = \max_{a\leq\eta\leq t} y(\eta) = Y(t)$$

for each $t \in J$. Then by hypothesis (H_2), we obtain

$$\mathscr{A}x(t) = f(t,x(t),X(t)) \geq f(t,y(t),Y(t))) = \mathscr{A}y(t),$$

and

$$\mathscr{C}x(t) = k(t,x(t),X(t)) \geq k(t,y(t),Y(t))) = \mathscr{C}y(t),$$

for all $t \in J$. This shows that $\mathscr{A}$ and $\mathscr{C}$ are nondecreasing operators on E into E. Similarly, using hypothesis (H_5),

$$\begin{aligned} \mathscr{B}x(t) &= a(t) + \int_{t_0}^{t} v(t,s)g(s,x(s),X(s))\,ds \\ &\geq a(t) + \int_{t_0}^{t} v(t,s)g(s,y(s),Y(s))\,ds \\ &= \mathscr{B}y(t) \end{aligned}$$

for all $t \in J$. Hence, it is follows that the operator $\mathscr{B}$ is also a nondecreasing operator on E into itself. Thus, $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are nondecreasing positive operators on E into itself.

Step II: *$\mathscr{A}$ and $\mathscr{C}$ are partially bounded and partially $\mathscr{D}$-Lipschitz on E.*

Let $x \in E$ be arbitrary. Then by (H_2),

$$|\mathscr{A}x(t)| \leq \left|f(t,x(t),X(t))\right| \leq M_f,$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathscr{A}x\| \leq M_f$ and so, $\mathscr{A}$ is bounded . This further implies that $\mathscr{A}$ is partially bounded on E.

Next, let $x,y \in E$ be such that $x \geq y$. Then, we have

$$|x(t)-y(t)| \leq |X(t)-Y(t)|$$

and that

$$\begin{aligned}
|X(t)-Y(t)| &= X(t)-Y(t) \\
&= \max_{t_0\leq\eta\leq t} x(\eta) - \max_{t_0\leq\eta\leq t} y(\eta) \\
&\leq \max_{t_0\leq\eta\leq t} [x(\eta)-y(\eta)] \\
&= \max_{t_0\leq\eta\leq t} |x(\eta)-y(\eta)| \\
&\leq \|x-y\|
\end{aligned}$$

for each $t \in J$. As a result, by hypothesis (A_3),

$$\begin{aligned}
|\mathscr{A}x(t)-\mathscr{A}y(t)| &= \big|f(t,x(t),X(t))-f(t,y(t),Y(t))\big| \\
&\leq \psi_f(\max\{|x(t)-y(t)|\ |X(t)-Y(t)|\}) \\
&\leq \psi_f(\|x-y\|),
\end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|\mathscr{A}x-\mathscr{A}y\| \leq \psi_f(\|x-y\|)$$

for all $x,y \in E$ with $x \geq y$. Similarly, by hypothesis (A_3)

$$\|\mathscr{C}x-\mathscr{C}y\| \leq \psi_k(\|x-y\|)$$

for all $x,y \in E$ with $x \geq y$. Hence $\mathscr{A}$ and $\mathscr{C}$ are partially nonlinear $\mathscr{D}$-Lipschitz operators on E which further implies that $\mathscr{A}$ and $\mathscr{C}$ are partially continuous on E .

Step III: *$\mathscr{B}$ is a partially continuous operator on E.*

Let $\{x_n\}_{n\in N}$ be a sequence in a chain C of E such that $x_n \to x$ for all $n \in N$. Then, by dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathscr{B}x_n(t) &= a(t) + \int_{t_0}^{t} v(t,s)g(s,x_n(s),X_n(s))\,ds \\
&\geq a(t) + \int_{t_0}^{t} [\lim_{n\to\infty} v(t,s)g(s,x_n(s),X_n(s))]\,ds \\
&= a(t) + \int_{t_0}^{t} v(t,s)g(s,x(s),X(s))\,ds \\
&= \mathscr{B}x(t),
\end{aligned}$$

for all $t \in J$. This shows that $\mathscr{B}x_n$ converges monotonically to $\mathscr{B}x$ pointwise on J.

Next, we will show that $\{\mathscr{B}x_n\}_{n\in N}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned}
\Big|Bx_n(t_2) - Bx_n(t_1)\Big| &\leq |a(t_1) - a(t_2)| \\
&+ \Big|\int_0^{t_2} v(t_1,s)g(s,x_n(s),X_n(s))ds - \int_0^{t_1} v(t_1,s)g(s,x_n(s),X_n(s))ds\Big| \\
&\leq |a(t_1) - a(t_2)| + \Big|\int_0^{t_2} v(t_2,s)g(s,x_n(s),X_n(s))ds \\
&- \int_0^{t_1} v(t_2,s)g(s,x_n(s),X_n(s))ds\Big| \\
&+ \Big|\int_0^{t_2} v(t_1,s)g(s,x_n(s),X_n(s))ds - \int_0^{t_1} v(t_1,s)g(s,x_n(s),X_n(s))ds\Big| \\
&\leq |a(t_1) - a(t_2)| + \Big|\int_0^{t_2} |v(t_2,s) - v(t_1,s)||g(s,x_n(s),X_n(s))|ds\Big| \\
&+ \Big|\int_{t_1}^{t_2} |v(t_1,s)||g(s,x_n(s),X_n(s))|ds\Big| \\
&\leq \Big|a(t_1) - a(t_2)| + \Big|\int_0^{T} |v(t_2,s) - v(t_1,s)|M_g ds\Big| \\
&+ VM_g|t_2 - t_1| \qquad (16)
\end{aligned}$$

uniformly for all $n \in N$. This shows that the convergence $\mathscr{B}x_n \to \mathscr{B}x$ is uniform and hence $\mathscr{B}$ is partially continuous on E.

Step IV: *$\mathscr{B}$ is uniformly partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathscr{B}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathscr{B}(C)$ is uniformly bounded. Let $y \in \mathscr{B}(C)$ be any element. Then there is an element $x \in C$ be such that $y = \mathscr{B}x$. Now, by hypothesis (A_5),

$$\begin{aligned}
|y(t)| &\leq a(t) + \int_0^t v(t,s)|g(s,x(s),X(s))|\,ds \\
&\leq \|a\| + VM_gT \\
&\leq r
\end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|y\| \leq \|\mathscr{B}x\| \leq r$ for all $y \in \mathscr{B}(C)$. Hence, $\mathscr{B}(C)$ is a uniformly bounded subset of E. Moreover, $\|\mathscr{B}(C)\| \leq r$ for all chains C in E. Hence, $\mathscr{B}$ is a uniformly partially bounded operator on E. Next, we will show that $\mathscr{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$.Then, for any $y \in \mathscr{B}(C)$, one has,

$$\begin{aligned}
\left|y(t_2) - y(t_1)\right| &= \left|Bx(t_2) - Bx(t_1)\right| \\
&\leq |a(t_1) - a(t_2)| + \left|\int_0^{t_2} v(t_1,s)g(s,x(s),X(s))ds - \int_0^{t_1} v(t_1,s)g(s,x(s),X(s))ds\right| \\
&\leq |a(t_1) - a(t_2)| + \left|\int_0^{t_2} v(t_2,s)g(s,x(s),X(s))ds - \int_0^{t_2} v(t_1,s)g(s,x(s),X(s))ds\right| \\
&+ \left|\int_0^{t_2} v(t_1,s)g(s,x(s),X(s))ds - \int_0^{t_1} v(t_1,s)g(s,x(s),X(s))ds\right| \\
&\leq |a(t_1) - a(t_2)| + \left|\int_0^{t_2} |v(t_2,s) - v(t_1,s)||g(s,x(s),X(s))|ds\right| \\
&+ \left|\int_{t_1}^{t_2} |v(t_1,s)||g(s,x(s),X(s))|ds\right| \\
&\leq |a(t_1) - a(t_2)| + \left|\int_{t_1}^{t_2} |g(s,x(s),X(s))|ds\right| \\
&\leq \left|a(t_1) - a(t_2)| + \left|\int_0^{T} |v(t_2,s) - v(t_1,s)|M_g ds\right|\right. \\
&+ VM_g|t_2 - t_1| \\
&\quad \to 0 \qquad as \qquad t_2 - t_1 \to 0
\end{aligned}$$

uniformly for all $y \in \mathscr{B}(C)$.Hence $\mathscr{B}(C)$ is an equicontinuous subset of E. Now, $\mathscr{B}(C)$ is a uniformly bounded and equicontinuous set of functions in E, so it is compact. Consequently, $\mathscr{B}$ is a uniformly partially compact operator on E into itself.

Step V: *p satisfies the operator inequality $p \leq \mathscr{A} p \mathscr{B} p + \mathscr{C} p$.*

By hypothesis (H_{10}), the GQFIE (1) has a lower solution p defined on J. Then, we have

$$p(t) = k(t, p(t), P(t)) + [f(t, p(t), P(t))] \left(a(t) + \int_0^t v(t,s) g(s, p(s), P(s)) ds \right) \tag{17}$$

for all $t \in J$. From the definitions of the operators $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ it follows that $p(t) \leq \mathscr{A} p(t) \mathscr{B} p(t) + \mathscr{C} p(t)$ for all $t \in J$. Hence $p \leq \mathscr{A} p \mathscr{B} p + \mathscr{C} p$.

Step VI: *The $\mathscr{D}$-functions $\psi_{\mathscr{A}}$ and $\psi_{\mathscr{C}}$ satisfy the growth condition $M\psi_{\mathscr{A}}(r) + \psi_{\mathscr{C}} < r$, for $r > 0$.*

Finally, the $\mathscr{D}$-function $\psi_{\mathscr{A}}$ of the operator $\mathscr{A}$ satisfy the inequality given in hypothesis (d) of Theorem 2.1, viz.,

$$M\psi_{\mathscr{A}}(r) + \psi_{\mathscr{C}} \leq (\|a\| + VM_g T)\, \psi_f(r) + \psi_k(r) < r$$

for all $r > 0$.

Thus $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ satisfy all the conditions of Theorem 2.1 and we conclude that the operator equation $\mathscr{A} x \mathscr{B} x + \mathscr{C} x = x$ has a solution. Consequently the GQFIE(1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n \in N}$ of successive approximations defined by (11) converges monotonically to x^*. This completes the proof.

Example

Given a closed and bounded interval $J = [0, 1]$, consider the GQFIE,

$$\begin{aligned} x(t) = \frac{1}{4}\left[4 + \tan^{-1} x(t) + \tan^{-1} X(t)\right] \left(\frac{1}{t+1} + \int_0^t \frac{1}{t^2+1} \cdot \frac{[1 + \tanh x(s)]}{4} ds \right) \\ + \frac{1}{4}[\tan^{-1} x(t) + \tan^{-1} X(t)] \end{aligned} \tag{18}$$

for $t \in J$, where $X(t) = \max_{0 \leq \eta \leq t} x(\eta)$.

Here, $q(t) = \frac{t}{t+1}$ and $v(t,s) = \frac{1}{t^2+1}$ which are continuous and $\|q(t)\| = \frac{1}{2}$ and V=1. Similarly, the functions k ,f and g are defined by $k(t,x,y) = \frac{1}{4}[\tan^{-1}x(t) + \tan^{-1}y(t)]$

$$f(t,x,y) = \frac{1}{4}\left[4 + \tan^{-1}x(t) + \tan^{-1}y(t)\right]$$

and

$$g(t,x,y) = g(t,x) = \frac{1+\tanh x}{4}.$$

The function f satisfies the hypothesis (H_3) with $\psi_f(r) = \frac{1}{2}\frac{r}{1+\eta^2}$ for each $0 < \eta < r$. To see this, we have

$$\begin{aligned} 0 &\leq f(t,x_1,x_2) - f(t,y_1,y_2) \\ &\leq \frac{1}{4}\cdot\frac{1}{1+\eta^2}\cdot(x_1-y_1) + \frac{1}{4}\cdot\frac{1}{1+\eta^2}\cdot(x_2-y_2) \\ &\leq \frac{1}{2}\cdot\frac{1}{1+\eta^2}\cdot\max\{x_1-y_1\,,\,x_2-y_2\} \end{aligned}$$

for all $x_1,y_1,x_2,y_2 \in R$, $x_1 \geq y_1$ and $x_2 > \eta > y_2$. Moreover, the function f is nonnegative and bounded on $J \times R \times R$ with bound $M_f = 2$ and so the hypothesis (H_2) is satisfied.

Similarly,the function k satisfies the hypothesis (H_9) $\psi_k(r) = \frac{1}{2}\frac{r}{1+\eta^2}$ for each $0 < \eta < r$. To see this, we have

$$\begin{aligned} 0 &\leq k(t,x_1,x_2) - k(t,y_1,y_2) \\ &\leq \tfrac{1}{4}\cdot\tfrac{1}{1+\eta^2}\cdot(x_1-y_1) + \tfrac{1}{4}\cdot\tfrac{1}{1+\eta^2}\cdot(x_2-y_2) \\ &\leq \tfrac{1}{2}\cdot\tfrac{1}{1+\eta^2}\cdot\max\{x_1-y_1\,,\,x_2-y_2\} \end{aligned}$$

for all $x_1,y_1,x_2,y_2 \in R$, $x_1 \geq y_1$ and $x_2 > \eta > y_2$. Moreover, the function k is bounded on $J \times R \times R$ with bound $M_k = \frac{\pi}{4}$ and so the hypothesis (H_9) is satisfied. Also we have

$$(\|a\| + VT\,M_g)\,\psi_f(r) + \psi_k(r) < r,\; r > 0,$$

Again, since g is nonnegative and bounded on $J \times R \times R$ with bound $M_g = \frac{1}{2}$, the hypothesis (H_5) holds. Furthermore, $g(t,x,y) = g(t,x)$ is nondecreasing in x and y for all $t \in J$, and thus hypothesis (H_6) is satisfied.

Thus, condition (10) of Theorem 3.1 is held. Finally, the GQFIE (1) has a lower solution $p(t) = 0$ on J. Thus all the hypotheses of Theorem 3.1 are satisfied. Hence we apply Theorem 2.1 and conclude that the GQFIE (1) has a solution x^* defined on J and the sequence $\{x_n\}_{n\in N}$ defined by

$$x_{n+1}(t) = \frac{1}{4}\left[4 + \tan^{-1} x_n(t) + \tan^{-1} X_n(t)\right]\left(\frac{1}{t+1} + \int_0^t \frac{1}{t^2+1}\frac{[1+\tanh x(s)]}{4}\,ds\right) + \frac{1}{4}[\tan^{-1} x_n(t) + \tan^{-1} X_n(t)] \qquad (19)$$

for all $t \in J$, where $x_0 = 0$, converges monotonically to x^*.

Conflict of Interests

The authors declare that there is no conflict of interests.

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