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THE EXTENSIONS OF ISOMETRIES BETWEEN THE 2-DIMENSIONAL NORMED SPACES

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Abstract. Isometric operator is a very significant subject in the study of space structure. In this paper, we will give some results about the Tingley's problem, and give sufficient conditions for isometric operator T_0, which is between classical 2-dimensional normed sequence spaces such as $\ell_p^{(2)}$ and $\ell_\infty^{(2)}$, to have a linear extension in some way.

Keywords: 2-dimensional normed space; Tingley's problem; linearity; isometric; extension; $\ell_p^{(2)}$; $\ell_\infty^{(2)}$.

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1. Introduction

In this paper, X and Y are both normed spaces, $S(X)$ and $S(Y)$ are denoted the unit spheres of X and Y respectively, and a mapping $T : X \to Y$ is said to be an isometric operator if

$$\|Tx - Ty\| = \|x - y\|, \forall x, y \in X.$$

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The normed spaces X and Y are said to be isometric if there exists a surjective isometric operator between X and Y. If X and Y are isometric, then they are essentially identical as metric spaces. And we can obtain that Y is a strictly convex space if X is strictly convex.

As the advantageous tools in the research of normed linear spaces, isometric operator and linear operator have a vital meaning in the functional analysis. Metric geometry, space structure, equivalence theory and affine motion theory are all need the support of those two tools. In 1932, Mazur and Ulam gave the famous Mazur-Ulam theorem in [1].

Theorem 1.1 [1] (**Mazur-Ulam theorem**)[1] Every surjective isometric operator T from normed space X to normed space Y is an affine. And it is also linear if $T(0) = 0$.

This is to say that if the normed space X and Y are isometric, then they must be linearly isometric to each other.

For all metric vector spaces, it is till open that whether the Mazur-Ulam theorem can holds. In 1953, Charzynski got an important result when X and Y are both the finite dimensional and have the same dimensions in [2].

Theorem 1.2 [2] X and Y are both the finite dimensional metric vector spaces, $dim(X) = dim(Y) = n$, and $T : X \to Y$ is an isometry with $T(0) = 0$, then T is linear.

From then on, many researchers which both at home and abroad have tried to popularized the Mazur-Ulam theorem by different patterns, and have paid attention to weaken the condition of the Mazur-Ulam theorem from different patterns. Through the research of the space structure, many open problems were raised as well including the Alesandrov problem, the Aleksandrov-Rassias problem and the Tingley's problem. In the consideration of the into isometric operator, Baker extended the Mazur-Ulam theorem to strictly convex normed spaces in [3], and obtained the following conclusion :

Theorem 1.3 [3] X is a normed space and Y ia a strictly convex normed space, $T : X \to Y$ is an into isometric operator with $T(0) = 0$, then T is linear.

In [4], Mankiewicz extended the Mazur-Ulam theorem, and showing that an isometric operator which maps a connected subset of a normed space X onto an open subset of another normed space Y can be extended to an affine operator on the whole space. Applying these results to the unit balls of X and Y, we obtain that two normed spaces are linearly isometric if and only if their unit balls are isomeric.

Clearly, we have an negatively example that the subsets of Banach spaces are isometric, but not in any sense affine.

Example 1.4 A mapping $T:\mathbf{R}\to\ell_\infty^{(2)}$ ($\mathbf{R}^2$ with the max norm) given by

$$T(x)=(x,\sin x),$$

then T is an isometry, but it is not affine.

Instead of connected subset, D.Tingley raised the following problem, which called Tingley's problem in [5].

Problem 1.5 [5] Suppose that $T_0:S(X)\to S(Y)$ is a surjective isometric operator, does there exist a linear isometric operator $T:X\to Y$ such that $T|_{S(X)}=T_0$?

And Tingley has proved the following main result:

Theorem 1.6 [5] Let X and Y are both finite-dimensional normed spaces, if $T_0:S(X)\to S(Y)$ is a surjective isometry, then $T_0(-x)=-T_0(x)$, $\forall x\in S(X)$.

In this case, we call the T_0 preserves anti-polar points.

For this question, we only considerate the real normed space, because the Tingley's problem is negative when X and Y are complex normed spaces, like $X=Y=\mathbb{C}$ and $T(x)=\bar{x}$ for all $x\in\mathbb{C}$ with $|x|=1$. It's hard to make an affirmative answer because there is no linear or even metrically convex structure on this unit sphere. Even for 2-dimensional normed space, Tingley's problem is not salved until recently.

And for this question, we always considerate the surjective operator, because for the into isometric operator between unit spheres, it is not difficult to find the example that the into isometric operator between unit spheres can't be linear extended. In [15], L. Zhang obtained an counterexample which a operator from $S_1(\ell_\infty^{(2)})$ into $S_1(\ell_\infty^{(3)})$ shows that the isometric extension problem fails.

Example 1.7 [15] Isometry T_0 which from $S_1(\ell_\infty^{(2)})$ into $S_1(\ell_\infty^{(3)})$ is given by

$$T_0[(\xi_1,\xi_2)] = \begin{cases} (1,\frac{3}{4}\xi_2,\xi_2), & \xi_1 = 1, \xi_2 \geq 0; \\ (-1,\xi_2,\frac{3}{4}\xi_2), & \xi_1 = -1, \xi_2 \geq 0; \\ (\xi_1, 1-\frac{1}{4}\xi_1, 1), & \xi_2 = 1, \xi_1 \geq 0; \\ (\xi_1, 1, 1+\frac{1}{4}\xi_1), & \xi_2 = 1, \xi_1 < 0; \\ (\xi_1,\xi_2,\xi_2), & \xi_2 < 0. \end{cases}$$

then T_0 is an isometry from $S_1(\ell_\infty^{(2)})$ to $S_1(\ell_\infty^{(3)})$, but it can't be extended to an isometric operator or linear operator on whole space $\ell_\infty^{(2)}$.

From now on, we can say that the question of whether an isometry from $S(X)$ into $S(Y)$ can be extended an isometric and linear operator for all Banach space X and Y is negative. Some affirmative conclusions were obtained between the same type and different type of classical Banach spaces which can be found in [7-13].

We notice that, for finite-dimensional normed spaces X and Y, if $dim(X) > dim(Y)$ then there is no linear isometry from X to Y. Since when $dim(X) = 1$ is so trivial, in this paper, we consider the Tingley's problem between the 2-dimensional normed spaces, and we obtain that when $dim(Y) = 2$ the dimension of X should be less then or equal to 2, so we just consider the problem when $dim(X) = dim(Y) = 2$.

And for 2-dimensional normed spaces, Wang R D and Wang P has obtained an useful result to help extending the isometry on unit spheres in [19].

Theorem 1.8 [19] Let X and Y are both two dimensional normed space , $T_0 : S(X) \to S(Y)$ is a surjective isometric operator. If

$$\|T_0(y) - (\|T_0(x) + T_o(y)\| - 1)T_o(x)\| = \|y - (\|x+y\| - 1)x\|,$$

$$\|T_0(x)-(\|T_0(x)+T_o(y)\|-1)T_o(y)\| = \|x-(\|x+y\|-1)y\|,$$

for $\forall x,y \in S(X)$, $\|x+y\| \geq 1$, then T_0 can extended to be an linear isometric operator on X.

In this paper, we let $1 \leq p \leq \infty$, so we consider the space $\ell_p^{(2)}(\Gamma)$ and $\ell_\infty^{(2)}(\Gamma)$ together by symbol $\ell_p^{(2)}(\Gamma)$ $(1 \leq p \leq \infty)$. First, we give some metric property of the unit sphere of 2-dimensional normed space $\ell_p^{(2)}(\Gamma)$ $(1 \leq p \leq \infty)$. Then we study the Tingley's problem between $\ell_p^{(2)}(\Gamma)$ $(1 \leq p \leq \infty)$, we obtain that surjective isometric operator form $S[\ell_p^{(2)}(\Gamma)]$ to $S[\ell_p^{(2)}(\Gamma)]$ can be extended to the whole space $\ell_p^{(2)}(\Gamma)$ linearly.

2. Some Properties of the $S[\ell_p^{(2)}(\Gamma)]$ $(1 \leq p \leq \infty)$

By using the characteristics of norm of sequence spaces $\ell^p(\Gamma)$ $(1 \leq p < \infty)$ and $\ell^\infty(\Gamma)$, Ding G G obtained the representation theorem of the surjective isometric operator of unit spheres of this type spaces in [13, 14]. For Hilbert spaces, we have got a good answer about the Tingley'y problem in [18], when $p=2$, as a special case we also solved the extension of isometries between two $\ell_2^{(2)}$, so we no longer discuss this problem when $p=2$.

Theorem 2.1 [13,14] Suppose that

$$T_0 : S[\ell^p(\Gamma)] \to S[\ell^p(\Delta)] \qquad (1 \leq p \leq \infty,\ p \neq 2)$$

is an surjective isometric operator. Then there exist a 1-1 permutation mapping $\pi : \Delta \to \Gamma$ and the number set $\{\theta_\sigma\}_{\sigma\in\Delta}$ with $|\ \theta_\sigma\ |=1$ for all $\sigma \in \Delta$, such that

$$T_0(x) = \sum_{\sigma\in\Delta} \theta_\sigma \cdot \xi_{\pi(\sigma)} d_\sigma, \qquad \forall x = \sum_{\gamma\in\Gamma} \xi_\gamma e_\gamma \in S[\ell^p(\Gamma)].$$

With the representation theorem of the surjective isometric operator, we can easily get the affirmative answer to the Tingley's question in the ℓ_p type space, but we can get more explicit representation theorem of the surjective isometric operator on $\ell_p^{(2)}$ by letting the index sets $\Gamma = \Delta = \{1,2\}$ in the third chapter.

Definition 2.2 For $\forall x,y \in \ell_p^{(2)}(1 \leq p \leq \infty, p \neq 2)$, we say that x is orthogonal to y (denoted by $x \perp y$) if

$$\|x\| \leq \|x+\lambda y\|$$

hold for all scalars λ.

Definition 2.3 For $\forall x, y \in \ell_p^{(2)} (1 \le p \le \infty, p \neq 2)$, we say that $\{e_1, e_2\}$ is a normalized orthogonal basis of $\ell_p^{(2)}$ if

$$\|e_1\| = \|e_2\| = 1 \quad and \quad e_1 \perp e_2.$$

Let $e_1 = (1,0), e_2 = (0,1) \in S[\ell_p^{(2)}]$ $(1 \le p \le \infty, p \neq 2)$, it is easy to prove that $\{e_1, e_2\}$ is a normalized orthogonal basis, and for $\forall x = (\xi_1, \xi_2) \in \ell_p^{(2)}$, we can let $x = \xi_1 e_1 + \xi_2 e_2$.

Definition 2.4 For $\forall x, y \in \ell_p^{(2)} (1 \le p \le \infty, p \neq 2)$, $\{e_1, e_2\}$ is a normalized orthogonal basis of $\ell_p^{(2)}$, for $\forall x = (\xi_1, \xi_2) = \xi_1 e_1 + \xi_2 e_2 \in \ell_p^{(2)}$, $|\theta_1| = |\theta_2| = 1$, we remember $\theta_1 \xi_1 e_1 + \theta_2 \xi_2 e_2$ as $\tilde{x}$, remember $\theta_2 \xi_2 e_1 + \theta_1 \xi_1 e_2$ as $\bar{x}$.

Proposition 2.5 For $\ell_p^{(2)} (1 \le p \le \infty, p \neq 2)$, $\tilde{x}, \bar{x} \in S[\ell_p^{(2)}]$ if and only if $x \in S[\ell_p^{(2)}]$, and

$$\|x\| = \|\tilde{x}\| = \|\bar{x}\|.$$

Proof. For $\ell_p^{(2)} (1 \le p \le \infty, p \neq 2)$, $\forall x = (\xi_1, \xi_2)$,

When $1 \le p < \infty$,

$$x \in S[\ell_p^{(2)}] \;\Leftrightarrow\; \|x\| = 1 \;\Leftrightarrow\; |\xi_1|^p + |\xi_2|^p = 1 \;\Leftrightarrow\; |\theta_1 \xi_1|^p + |\theta_2 \xi_2|^p = 1$$

$$\Leftrightarrow\; \|\widetilde{x}\| = \|\overline{x}\| = 1 \;\Leftrightarrow\; \tilde{x}, \bar{x} \in S[\ell_p^{(2)}];$$

When $p = \infty$,

$$x \in S[\ell_p^{(2)}] \;\Leftrightarrow\; \|x\| = 1 \;\Leftrightarrow\; \max\{|\xi_1|, |\xi_2|\} = 1 \;\Leftrightarrow\; \max\{|\theta_1 \xi_1|, |\theta_2 \xi_2|\} = 1$$

$$\Leftrightarrow\; \|\widetilde{x}\| = \|\overline{x}\| = 1 \;\Leftrightarrow\; \tilde{x}, \bar{x} \in S[\ell_p^{(2)}].$$

□

3. The Extensions of Isometries between the 2-dimensional Normed Spaces

In this section, we considerate the extension of isometries between the 2-dimensional normed sequence spaces $\ell_p^{(2)}$ $(1 \le p \le \infty, p \neq 2)$.

Proposition 3.1 For $\ell_p^{(2)}$, when $1 \le p < \infty, p \neq 2$, $\forall x,y \in S[\ell_p^{(2)}]$, then

$$supp(x) \cap supp(y) = \varnothing \quad \Leftrightarrow \quad \|x \pm y\|^p = \|x\|^p + \|y\|^p.$$

Proof. For $\forall x,y \in S[\ell_p^{(2)}]$, let $x = (\xi_1, \xi_2)$ $y = (\eta_1, \eta_2)$.

$' \Rightarrow '$

If $supp(x) \cap supp(y) = \varnothing$, without loss of generality, let $supp(x) = \{1\}$, and $supp(y) = \{2\}$. So $x = (\xi_1, 0)$ $y = (0, \eta_2)$, then

$$\|x \pm y\|^p = \|(\xi_1, \pm\eta_2)\|^p = \mid \xi_1 \mid^p + \mid \eta_2 \mid^p = \|x\|^p + \|y\|^p.$$

$' \Leftarrow '$

If $\|x \pm y\|^p = \|x\|^p + \|y\|^p$, then $\|x+y\|^p + \|x-y\|^p = 2(\|x\|^p + \|y\|^p)$,

$$|\xi_1 + \xi_2|^p + |\eta_1 + \eta_2|^p + |\xi_1 - \xi_2|^p + |\eta_1 - \eta_2|^p = 2(|\xi_1|^p + |\xi_2|^p + |\eta_1|^p + |\eta_2|^p).(*)$$

Following the famous complex inequalities $(for \quad \forall \xi \neq 0, \eta \neq 0)$

$$\mid \xi + \eta \mid^p + \mid \xi - \eta \mid^p < 2(\mid \xi \mid^p + \mid \eta \mid^p), (1 \le p < 2)$$

and

$$\mid \xi + \eta \mid^p + \mid \xi - \eta \mid^p > 2(\mid \xi \mid^p + \mid \eta \mid^p), (p > 2)$$

with the $(*)$, we can obtain that

$$\xi_1 \cdot \eta_1 = \xi_2 \cdot \eta_2 = 0,$$

then $supp(x) \cap supp(y) = \varnothing$.

□

Proposition 3.2 For $\ell_p^{(2)}$, when $p = \infty$, $\forall x,y \in S[\ell_\infty^{(2)}]$, then

$$supp(x) \cap supp(y) = \varnothing \quad \Leftrightarrow \quad \|x \pm y\| = \max\{\|x\|, \|y\|\}.$$

Proof. For $\forall x,y \in S[\ell_\infty^{(2)}]$, let $x = (\xi_1, \xi_2)$ $y = (\eta_1, \eta_2)$.

$' \Rightarrow '$

If $supp(x) \cap supp(y) = \varnothing$, without loss of generality, let $supp(x) = \{1\}$, and $supp(y) = \{2\}$, that is $x = (\xi_1, 0)$, $y = (0, \eta_2)$, then

$$\|x \pm y\| = \|(\xi_1, \pm\eta_2)\| = \max\{|\ \xi_1\ |, |\ \pm\eta_2\ |\} = \max\{\|x\|, \|y\|\}.$$

$' \Leftarrow '$

For $\forall x, y \in S[\ell_\infty^{(2)}]$, if $\|x \pm y\| = \max\{\|x\|, \|y\|\}$, that is

$$\begin{aligned}\|x \pm y\| &= \|(\xi_1 \pm \eta_1, \xi_2 \pm \eta_2)\| \\ &= \max\{\|x\|, \|y\|\} \\ &= \max\{\max\{|\xi_1|, |\xi_2|\}, \max\{|\eta_1|, |\eta_2|\}\} \\ &= \max\{|\xi_1|, |\xi_2|, |\eta_1|, |\eta_2|\} \\ &= 1,\end{aligned}$$

$$\max\{|\xi_1 + \eta_1|, |\xi_2 + \eta_2|\} = \max\{\|x\|, \|y\|\} = \max\{|\xi_1|, |\xi_2|, |\eta_1|, |\eta_2|\}, \qquad (**)$$

and

$$\max\{|\xi_1 - \eta_1|, |\xi_2 - \eta_2|\} = \max\{\|x\|, \|y\|\} = \max\{|\xi_1|, |\xi_2|, |\eta_1|, |\eta_2|\}, \qquad (***)$$

If $supp(x) \cap supp(y) \neq \varnothing$.

Case I $\quad supp(x) \cap supp(y)$ has only one point.

Without loss of generality,let $\eta_2 = 0$, that is $supp(x) \cap supp(y) = \{1\}$, $x = (\xi_1, \xi_2)$, $y = (\eta_1, 0)$.

$\|y\| = 1$ because $y \in S[\ell_\infty^{(2)}]$, that is

$$\max\{|\eta_1|, |0|\} = |\eta_1| = 1.$$

It is clearly that

$$\max\{|\xi_1 + \eta_1|, |\xi_1 - \eta_1|\} > |\eta_1| = 1,$$

but following the $(**)$ and $(***)$, we obtain that

$$\max\{|\xi_1 + \eta_1|, |\xi_2 + \eta_2|\} = \max\{|\xi_1 - \eta_1|, |\xi_2 - \eta_2|\} = \max\{|\xi_1|, |\xi_2|, |\eta_1|, |\eta_2|\} = 1,$$

it is contradictory.

Case II $supp(x) \cap supp(y)$ has two points, namely that $supp(x) \cap supp(y) = \{1, 2\}$, $\xi_1, \xi_2, \eta_1, \eta_2$ are all not equal to 0.

In this case, it is clearly that

$$\max\{|\xi_1 + \eta_1|, |\xi_1 - \eta_1|\} > |\xi_1|, \quad \max\{|\xi_1 + \eta_1|, |\xi_1 - \eta_1|\} > |\eta_1|,$$

$$\max\{|\xi_2 + \eta_2|, |\xi_1 - \eta_1|\} > |\xi_2|, \quad \max\{|\xi_2 + \eta_2|, |\xi_1 - \eta_1|\} > |\eta_2|,$$

but this is contradictory with the $(**)$ and $(***)$.

□

Corollary 3.3 For $\ell_p^{(2)}$ $(1 \le p \le \infty, p \ne 2)$, $T_0 : S[\ell_p^{(2)}] \to S[\ell_p^{(2)}]$ is a surjective isometry, $\forall x, y \in S[\ell_p^{(2)}]$, then

$$supp(x) \cap supp(y) = \varnothing \quad \Leftrightarrow \quad supp[T_0(x)] \cap supp[T_0(y)] = \varnothing.$$

Proof. For $\forall x, y \in S[\ell_p^{(2)}]$, $T_0(x), T_0(y) \in S[\ell_p^{(2)}]$, let $x = (\xi_1, \xi_2)$ $y = (\eta_1, \eta_2)$.

$' \Rightarrow '$ When $1 \le p < \infty$, and $p \ne 2$, if $supp(x) \cap supp(y) = \varnothing$, then by the proposition 3.1, we have

$$\|x \pm y\|^p = \|x\|^p + \|y\|^p = 2,$$

and

$$\|T_0(x) \pm T_0(y)\|^p = \|x \pm y\|^p = \|x\|^p + \|y\|^p = 2 = \|T_0(x)\|^p + \|T_0(y)\|^p,$$

then

$$supp[T_0(x)] \cap supp[T_0(y)] = \varnothing.$$

When $p = \infty$, for $\ell_\infty^{(2)}$, if $supp(x) \cap supp(y) = \varnothing$, then following the proposition 3.2, we have

$$\|x \pm y\| = \max\{\|x\|, \|y\|\} = 1,$$

and

$$\|T_0(x) \pm T_0(y)\| = \|x \pm y\| = \max\{\|x\|, \|y\|\} = 1 = \max\{\|T_0(x)\|, \|T_0(y)\|\},$$

then

$$supp[T_0(x)] \cap supp[T_0(y)] = \varnothing.$$

$'\Leftarrow'$ When $1 \leq p < \infty$, and $p \neq 2$, if $supp[T_0(x)] \cap supp[T_0(y)] = \varnothing$, then following the proposition 3.1, we have

$$\|T_0(x) \pm T_0(y)\|^p = \|T_0(x)\|^p + \|T_0(y)\|^p = 2,$$

and

$$\|x \pm y\|^p = \|T_0(x) \pm T_0(y)\|^p = \|T_0(x)\|^p + \|T_0(y)\|^p = 2 = \|x\|^p + \|y\|^p,$$

then

$$supp(x) \cap supp(y) = \varnothing.$$

When $p = \infty$, for $\ell_\infty^{(2)}$, if $supp[T_0(x)] \cap supp[T_0(y)] = \varnothing$, then following the proposition 3.2, we have

$$\|T_0(x) \pm T_0(y)\| = \max\{\|T_0(x)\|, \|T_0(y)\|\} = 1,$$

and

$$\|x \pm y\| = \|T_0(x) \pm T_0(y)\| = \max\{\|T_0(x)\|, \|T_0(y)\|\} = 1 = \max\{\|x\|, \|y\|\},$$

then

$$supp(x) \cap supp(y) = \varnothing.$$

□

Theorem 3.4 For $\ell_p^{(2)} (1 \leq p \leq \infty, p \neq 2)$, $T_0 : S[\ell_p^{(2)}] \to S[\ell_p^{(2)}]$ is a surjective isometry, $e_1 = (1,0)$, $e_2 = (0,1)$, $\{e_1, e_2\}$ is a normalized orthogonal basis of $\ell_p^{(2)}$, then

$$T_0(e_1) = \tilde{e_1}, \quad T_0(e_2) = \tilde{e_2}$$

or

$$T_0(e_1) = \bar{e_1}, \quad T_0(e_2) = \bar{e_2}.$$

Proof. For $e_1, e_2 \in S[\ell_p^{(2)}]$, T_0 is a surjective isometry on $S[\ell_p^{(2)}]$, so

$$T_0(e_1), \quad T_2(e_2) \in S[\ell_p^{(2)}].$$

Following the proposition 3.1, 3.2 and 3.3, we obtain that

$$supp(e_1) \cap supp(e_2) = \varnothing \quad \Rightarrow supp[T_0(e_1)] \cap supp[T_0(e_2)] = \varnothing,$$

and $\|T_0(e_1)\| = \|T_0(e_2)\| = 1$, so $\{|T_0(e_1)|, |T_0(e_2)|\} = \{|e_1|, |e_2|\}$, then there are $|\theta_1| = |\theta_2| = 1$ and

$$T_0(e_1) = \theta_1 e_1 = \tilde{e_1}, \quad T_0(e_2) = \theta_2 e_2 = \tilde{e_2}$$

or

$$T_0(e_1) = \theta_1 e_2 = \bar{e_1}, \quad T_0(e_2) = \theta_2 e_1 = \bar{e_2}.$$

□

Corollary 3.5 For $\ell_p^{(2)}(1 \leq p \leq \infty, p \neq 2)$, $T_0 : S[\ell_p^{(2)}] \to S[\ell_p^{(2)}]$ is a surjective isometry, $\{e_1, e_2\}$ is a normalized orthogonal basis of $\ell_p^{(2)}$, then

$$T_0(x) = \tilde{x} \quad or \quad T_0(x) = \bar{x} \quad for \forall x \in S[\ell_p^{(2)}].$$

Proof. For $\forall x = (\xi_1, \xi_2) \in S[\ell_p^{(2)}]$, let $x \neq \pm e$, $T_0(x) = x' = (\xi_1', \xi_2') \in S[\ell_p^{(2)}]$, so $T_0(x) \neq \pm e$, by the theorem 3.4, there are only two cases.

Case I $T_0(e_1) = \tilde{e_1}$, $T_0(e_2) = \tilde{e_2}$.

When $1 \leq p < \infty$, $p \neq 2$, $x, x' \in S[\ell_p^{(2)}]$, that is

$$\|x\|^p = 1 = |\xi_1|^p + |\xi_2|^p \quad \Rightarrow \quad 0 < |\xi_1| < 1, \quad 0 < |\xi_2| < 1,$$

$$\|x'\|^p = 1 = |\xi_1'|^p + |\xi_2'|^p \quad \Rightarrow \quad 0 < |\xi_1'| < 1, \quad 0 < |\xi_2'| < 1.$$

Then

$$\|T_0(x) \pm T_0(e_1)\|^p = \|x \pm e_1\|^p = \|x' \pm \tilde{e_1}\|^p = \|x' \pm \theta_1 e_1\|^p, \quad (3)$$

$$\|T_0(x) \pm T_0(e_2)\|^p = \|x \pm e_2\|^p = \|x' \pm \tilde{e_2}\|^p = \|x' \pm \theta_2 e_2\|^p, \quad (4)$$

that is

$$\|(\xi_1 \pm 1, \xi_2)\|^p = \|(\xi_1' \pm \theta_1, \xi_2')\|^p,$$

$$\|(\xi_1, \xi_2 \pm 1)\|^p = \|(\xi_1', \xi_2' \pm \theta_2)\|^p.$$

$$\begin{aligned} \|x \pm e_1\|^p &= \|(\xi_1 \pm 1, \xi_2)\|^p \\ &= |\xi_1 \pm 1|^p + |\xi_2|^p \\ &= |\xi_1 \pm 1|^p + 1 - |\xi_1|^p, \end{aligned}$$

$$\begin{aligned}\|x' \pm \tilde{e_1}\|^p &= \|(\xi_1' \pm \theta_1, \xi_2')\|^p \\ &= |\xi_1' \pm \theta_1|^p + |\xi_2'|^p \\ &= |\xi_1' \pm \theta_1|^p + 1 - |\xi_1'|^p,\end{aligned}$$

so by the (3) we get

$$|\xi_1 + 1|^p - |\xi_1|^p = |\xi_1' + \theta_1|^p - |\xi_1'|^p, \quad (5)$$

$$|\xi_1 - 1|^p - |\xi_1|^p = |\xi_1' - \theta_1|^p - |\xi_1'|^p. \quad (6)$$

When $\theta_1 = 1$, by (5) and $f(t) = |t+1|^p - |t|^p$ is a strictly increasing mapping when $-1 < t < 1$, we obtain that $\xi_1' = \xi_1 = \theta_1\xi_1$, and similarly we obtain that $\xi_2' = \xi_2 = \theta_2\xi_2$ when $\theta_2 = 1$.

When $\theta_1 = -1$, by (5) we get

$$|\xi_1 + 1|^p - |\xi_1|^p = |\xi_1' - 1|^p - |\xi_1'|^p = |1 - \xi_1'|^p - |\xi_1'|^p,$$

by $f(t) = |t+1|^p - |t|^p$ we obtain that $\xi_1' < 0$ and $\xi_1' = -\xi_1 = \theta_1\xi_1$, and similarly we obtain that $\xi_2' = -\xi_2 = \theta_2\xi_2$ when $\theta_2 = -1$, that is

$$T_0(x) = \tilde{x}.$$

When $p = \infty$, for $\ell_\infty^{(2)}$, $x, x' \in S[\ell_\infty^{(2)}]$, that is

$$\|x\| = 1 = \max\{|\xi_1, |\xi_2|\} \quad \Rightarrow \quad 0 < |\xi_1| \leq 1, \quad 0 < |\xi_2| \leq 1,$$

$$\|x'\| = 1 = \max\{|\xi_1'|, |\xi_2'|\} \quad \Rightarrow \quad 0 < |\xi_1'| \leq 1, \quad 0 < |\xi_2'| \leq 1.$$

$$\|T_0(x) \pm T_0(e_1)\| = \|x \pm e_1\| = \|x' \pm \tilde{e_1}\|,$$

$$\|T_0(x) \pm T_0(e_2)\| = \|x \pm e_2\| = \|x' \pm \tilde{e_2}\|,$$

that is

$$\|(\xi_1 + 1, \xi_2)\| = \max\{|\xi_1 + 1|, |\xi_2|\} = \|(\xi_1' + \theta_1, \xi_2')\| = \max\{|\xi_1' + \theta_1|, |\xi_2'|\},$$

$$\|(\xi_1 - 1, \xi_2)\| = \max\{|\xi_1 - 1|, |\xi_2|\} = \|(\xi_1' - \theta_1, \xi_2')\| = \max\{|\xi_1' - \theta_1|, |\xi_2'|\},$$

$$\|(\xi_1, \xi_2 + 1)\| = \max\{|\xi_1|, |\xi_2 + 1|\} = \|(\xi_1', \xi_2' + \theta_2)\| = \max\{|\xi_1'|, |\xi_2' + \theta_2|\},$$

and

$$\|(\xi_1,\xi_2-1)\| = \max\{|\xi_1|,|\xi_2-1|\} = \|(\xi_1',\xi_2'-\theta_2)\| = \max\{|\xi_1'|,|\xi_2'-\theta_2|\},$$

When $\xi_1 > 0$, we have $|\xi_1+1| > 1$, $|\xi_2| \le 1$ and $|\xi_2'| \le 1$, so

$$\max\{|\xi_1+1|,|\xi_2|\} = |\xi_1+1| = \max\{|\xi_1'+\theta_1|,|\xi_2'|\} = |\xi_1'+\theta_1|.$$

We can get $|\xi_1+1| = |\xi_1'+\theta_1|$, and by the mapping $g(t) = |t+1|$ is strictly increasing when $0 \le t \le 1$, we obtain that $\xi_1' = \theta_1\xi_1$.

Similarly we also obtain that $\xi_2' = \theta_2\xi_2$ when $\xi_2 > 0$.

When $\xi_1 < 0$, we have $|\xi_1-1| > 1$, $|\xi_2| \le 1$ and $|\xi_2'| \le 1$, so

$$\max\{|\xi_1-1|,|\xi_2|\} = |\xi_1-1| = \max\{|\xi_1'-\theta_1|,|\xi_2'|\} = |\xi_1'-\theta_1|.$$

We can get $|\xi_1-1| = |\xi_1'-\theta_1|$, and by the mapping $g(t) = |t-1|$ is strictly decreasing when $-1 \le t \le 0$, we obtain that $\xi_1' = \theta_1\xi_1$.

Similarly we also obtain that $\xi_2' = \theta_2\xi_2$ when $\xi_2 < 0$, that is

$$T_0(x) = \tilde{x}.$$

Case II $T_0(e_1) = \bar{e_1}$, $T_0(e_2) = \bar{e_2}$.

When $1 \le p < \infty$, $p \ne 2$,

$$\|x+e_1\|^p = |\xi_1+1|^p + 1 - |\xi_1|^p,$$

$$\|x'+\bar{e_1}\|^p = |\xi_2'+\theta_1|^p + 1 - |\xi_2'|^p,$$

$$\|x-e_1\|^p = |\xi_1-1|^p + 1 - |\xi_1|^p,$$

$$\|x'-\bar{e_1}\|^p = |\xi_2'-\theta_1|^p + 1 - |\xi_2'|^p,$$

$$\|T_0(x)+T_0(e_1)\|^p = \|x+e_1\|^p = \|x'+\theta_1 e_2\|^p,$$

$$\|T_0(x)-T_0(e_1)\|^p = \|x-e_1\|^p = \|x'-\theta_1 e_2\|^p,$$

like the above proof about case I, we obtain that $\xi_2' = \theta_1\xi_1$, and similarly we also obtain that $\xi_1' = \theta_2\xi_2$, that is

$$T_0(x) = \bar{x}.$$

When $p=\infty$, for $\ell_\infty^{(2)}$,

$$\|T_0(x)\pm T_0(e_1)\|=\|x\pm e_1\|=\|x^{'}\pm\theta_1e_2\|,$$

$$\|T_0(x)\pm T_0(e_2)\|=\|x\pm e_2\|=\|x^{'}\pm\theta_2e_1\|,$$

that is

$$\|(\xi_1+1,\xi_2)\|=\max\{|\xi_1+1|,|\xi_2|\}=\|(\xi_1^{'},\xi_2^{'}+\theta_1)\|=\max\{|\xi_1^{'}|,|\xi_2^{'}+\theta_1|\},$$

$$\|(\xi_1-1,\xi_2)\|=\max\{|\xi_1-1|,|\xi_2|\}=\|(\xi_1^{'},\xi_2^{'}-\theta_1)\|=\max\{|\xi_1^{'}|,|\xi_2^{'}-\theta_1|\},$$

$$\|(\xi_1,\xi_2+1)\|=\max\{|\xi_1|,|\xi_2+1|\}=\|(\xi_1^{'}+\theta_2,\xi_2^{'})\|=\max\{|\xi_1^{'}+\theta_2|,|\xi_2^{'}|\},$$

and

$$\|(\xi_1,\xi_2-1)\|=\max\{|\xi_1|,|\xi_2-1|\}=\|(\xi_1^{'}-\theta_2,\xi_2^{'})\|=\max\{|\xi_1^{'}-\theta_2|,|\xi_2^{'}|\},$$

like the above proof about case I, we get $\xi_1^{'}=\theta_2\xi_2$ and $\xi_2^{'}=\theta_1\xi_1$, that is

$$T_0(x)=\bar{x}.$$

□

Theorem 3.6 For $\ell_p^{(2)}$ $(1\le p\le\infty,\ p\ne 2)$, $T_0:S[\ell_p^{(2)}]\to S[\ell_p^{(2)}]$ is a surjective isometry, then T_0 can be linearly extended to an real isometry from $\ell_p^{(2)}$ to $\ell_p^{(2)}$.

Proof. For $\ell_p^{(2)}$, $e_1=(1,0)$, $e_2=(0,1)$, $\{e_1,e_2\}$ is a normalized orthogonal basis, for $\forall x\in\ell_p^{(2)}$, let

$$T(x)=\begin{cases}\|x\|T_0(\frac{x}{\|x\|}), & \text{if } x\ne\theta;\\ \theta, & \text{if } x=\theta.\end{cases}$$

For $\forall x=(\xi_1,\xi_2),y=(\eta_1,\eta_2)\in\ell_p^{(2)}$, $x\ne 0$, and $y\ne 0$, there are only two cases,

Case I $T_0(e_1)=\tilde{e_1}$, $T_0(e_2)=\tilde{e_2}$.

In this case, by theorem 3.5 and proposition 2.5, we obtain that

$$T_0(\frac{x}{\|x\|})=\widetilde{\frac{x}{\|x\|}}=\frac{\tilde{x}}{\|x\|},\quad T_0(\frac{y}{\|y\|})=\widetilde{\frac{y}{\|y\|}}=\frac{\tilde{y}}{\|y\|},$$

$$T(x)=\|x\|T_0(\frac{x}{\|x\|})=\|x\|\cdot(\frac{\tilde{x}}{\|x\|})=\tilde{x},$$

$$\begin{aligned}
\|T(x)-T(y)\| &= \|\|x\|T_0(\frac{x}{\|x\|}) - \|y\|T_0(\frac{y}{\|y\|})\| \\
&= \|\|x\|\cdot(\frac{\tilde{x}}{\|x\|}) - \|y\|\cdot(\frac{\tilde{y}}{\|y\|})\| \\
&= \|\tilde{x}-\tilde{y}\| \\
&= \|\widetilde{(x-y)}\| \\
&= \|x-y\|,
\end{aligned}$$

$$\begin{aligned}
T(\alpha x+\beta y) &= \|\alpha x+\beta y\|T_0(\frac{\alpha x+\beta y}{\|\alpha x+\beta y\|}) \\
&= \|\alpha x+\beta y\|\cdot(\frac{\widetilde{\alpha x+\beta y}}{\|\alpha x+\beta y\|}) \\
&= (\widetilde{\alpha x+\beta y}) \\
&= \alpha\tilde{x}+\beta\tilde{y} \\
&= \alpha T(x)+\beta T(y),
\end{aligned}$$

then T is the linearly isometric extension of T_0 to whole space $\ell_p^{(2)}$.

Case II $\quad T_0(e_1)=\bar{e_2}, T_0(e_2)=\bar{e_1}$.

In this case, by theorem 3.5 and proposition 2.5, we obtain that

$$T_0(\frac{x}{\|x\|}) = \overline{(\frac{x}{\|x\|})} = \frac{1}{\|x\|}\cdot\bar{x}, \quad T_0(\frac{y}{\|y\|}) = \overline{(\frac{y}{\|y\|})} = \frac{1}{\|y\|}\cdot\bar{y},$$

$$T(x) = \|x\|T_0(\frac{x}{\|x\|}) = \|x\|\cdot(\frac{1}{\|x\|})\cdot\bar{x} = \bar{x},$$

$$\begin{aligned}
\|T(x)-T(y)\| &= \|\|x\|T_0(\frac{x}{\|x\|}) - \|y\|T_0(\frac{y}{\|y\|})\| \\
&= \|\|x\|\cdot(\frac{1}{\|x\|})\cdot\bar{x} - \|y\|\cdot(\frac{1}{\|y\|})\cdot\bar{y}\| \\
&= \|\bar{x}-\bar{y}\| \\
&= \|\overline{(x-y)}\| \\
&= \|x-y\|,
\end{aligned}$$

$$
\begin{aligned}
T(\alpha x+\beta y) &= \|\alpha x+\beta y\| T_0\Big(\frac{\alpha x+\beta y}{\|\alpha x+\beta y\|}\Big) \\
&= \|\alpha x+\beta y\| \cdot \Big(\frac{1}{\|\alpha x+\beta y\|}\Big) \cdot \overline{\alpha x+\beta y} \\
&= \alpha \bar{x}+\beta \bar{y} \\
&= \alpha T(x)+\beta T(y),
\end{aligned}
$$

then T is the linearly isometric extension of T_0 to whole space $\ell_p^{(2)}$.

□

Since $T(\theta)=\theta$, the linearity about the T can also be proved by the Mazur-Ulam theorem.

Conflict of Interests

The authors declare that there is no conflict of interests.

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