Available online at http://scik.org
J. Math. Comput. Sci. 7 (2017), No. 1, 175-188
ISSN: 1927-5307

ON A BATCH ARRIVAL QUEUE WITH SECOND OPTIONAL SERVICE, RANDOM BREAKDOWNS, DELAY TIME FOR REPAIRS TO START AND RESTRICTED AVAILABILITY OF ARRIVALS DURING BREAKDOWN PERIODS

KAILASH C. MADAN* AND EBRAHIM MALALLA
Department of Mathematical Sciences, College of Arts, Science and Education,
Ahlia University, Kingdom of Bahrain

Abstract: We study a batch arrival single service channel queuing system where the server (service channel) provides two stages of general service to customers, the first essential service followed by the second optional service. It is assumed that the service channel is subject to breakdowns and on the occurrence of a breakdown, the service channel waits for the repairs to start and this waiting time (termed as the set-up time or delay time for repairs) is assumed general. Further, the repair times are also assumed general. We employ the supplementary variable technique using four supplementary variables, one each for the elapsed service time of the first essential service, the elapsed service time of the second optional service, the elapsed delay time and the elapsed repair time. In addition, we add an important assumption that during breakdown periods, the arriving batches are admitted into the system based on a policy of restricted admissibility. We derive queue size distribution for this system at a random epoch under the steady state conditions. Further, we derive some important performance measures of this system. This extends many models studied earlier by several authors. Finally, a few interesting particular cases are discussed.

Keywords: first essential service; second optional service; random breakdowns; delay time; repair time; restricted admissibility; queue size distribution at a random epoch; steady state.

2010 AMS Subject Classification: 60K25.

1 Introduction

Server breakdowns are common in many queueing situations. During the repair times of the service facility, the units or the customers have to wait until the system becomes operable again.

---

*Corresponding author

Received October 15, 2016

Consequently, such breakdowns have a definite effect on the system, particularly on the queue length and customers' waiting time in the system. Among some earlier papers on service interruptions, we refer the reader to [1], [3] and [6]. Recently, [5] and [12] have studied some queueing systems with service interruptions and the present author [11] has studied a queueing system with time-homogeneous server breakdowns and deterministic repair time. Most of these and other systems assume single (one by one) arrivals and they further assume that as soon as the service channel fails, the repairs start instantly. However, in the present paper, we deal with a bulk input queue $M^X/(G_1 \to G_2(Optional))/1$ with random breakdowns and delayed repairs, in which we assume that the service channel has to wait for the repairs to start, which is a much more realistic assumption in many real-life queueing situations. This delay in starting repairs may occur due to the non-availability of the repair people or the necessary apparatus needed for the repairs. This type of delay time was earlier introduced by one of the present authors, [7] in an M/M/1 queue with random breakdowns, general delay time and exponential repair time. Recently, [2] studied a queueing system with random breakdowns and delay times. However, in the present paper we attempt a wider generalization of the models studied by[7] and [2]. Not only that, we also generalize [8] in which he introduced the idea of second optional service but assumed single arrivals and assumed the second optional service times to be exponential. In this paper, we assume that system receives input of customers in batches of variable size and that the arriving customers are provided the first essential service (FES) followed by the second optional service (SOS). However, we assume that the service times of FES, the service times of the SOS, the delay time for the repairs to start and the repair time of the service facility, all the four random variables, follow a general arbitrary distribution. We employ the supplementary variable technique by introducing four independent supplementary variables, one each for these four variables. We further assume that service facility may only fail while it is working unlike Madan [10] who assumed that it may fail even when it is idle.

Another very important assumption in this work is the policy of restricted admissibility of arriving groups during breakdown periods. [9] Introduced restricted admissibility of arrivals in a vacation queue. In that paper, they assume that not all arriving batches are allowed to join the system as a policy to control overflow of the input into the system. Subsequently, [10] studied a vacation queue with restricted admissibility and assumed different restricted policies for the case when the server is present in the system and the case when the server is on vacation. Earlier, [4] studied a queueing system with control policies different from the ones studied by [9] and [10]. In the present paper, unlike [9] and [10], we employ the policy of restricted admissibility of batches only during the breakdown periods of the server. This is indeed a very valid and a realistic assumption, which would help alleviate the system's overall congestion.

2 Description of the Model and Definitions

We consider a batch arrival queueing system, where arrivals occur according to a compound Poisson process with the batch size random variable 'I'. The server provides FES, one by one, to all customers on a first come, first served basis. On completion of the FES, a customer opts for the SOS with probability p and leaves the system with probability $1-p$. The two service time random variables S_1 and S_2 of a customer follow a general probability law with respective distribution functions (DF) $G_1(x)$ and $G_2(x)$, Laplace-Stieltjes Transforms (LST) $G_1^*(\theta)$ and $G_2^*(\theta)$ and finite moments $E(S_1^k)$ and $E(S_2^k)$, $k \geq 1$, respectively. It is further assumed that the server is subject to random breakdowns such that $\alpha\, dt$ is the first order probability that the service channel will fail during the short interval of time $(t, t+dt]$. We assume that as a result of a random breakdown, the unit whose service (FES or SOS) gets interrupted, instantly goes back to the head of the queue. As soon as the server breaks down, it has to wait for the repairs to start. We define this waiting time as the *delay time* and assume that the delay time random variable D follows a general probability law with DF $D(x)$, LST $D^*(\theta)$ and finite moments $E(D^K)$, $k \geq 1$. Next, we assume that the repair time random variable R of the service channel also follows a general probability law with DF $R(x)$, LST $R^*(\theta)$ and finite moments $E(R^K)$, $k \geq 1$. Let c $(0 \leq c \leq 1)$ be the probability that an arriving batch will be allowed to join the system during the period of time when the server is under breakdown state, either waiting for repairs to start or under repairs.

Next, we define $\lambda =$ batch arrival rate, $X =$ batch size (a random variable),

$a_k =$ Prob $[X = k]$, $X(z) = \sum_{k=1}^{\infty} z^k a_k$, the PGF of *X*, and $E[X_{[k]}] = E[X(X-1)...(X-k+1)]$, the *k*-th factorial moment of *X*.

Further, it may be noted that since $G_1(x)$, $G_2(x)$, $D(x)$ and $R(x)$ are distribution functions, we have $G_1(0) = 0$, $G_1(\infty) = 1$, $G_2(0) = 0$, $G_2(\infty) = 1$, $D(0) = 0$, $D(\infty) = 1$ and $R(0) = 0$, $R(\infty) = 1$.

Further, since $G_1(x)$, $G_2(x)$, $D(x)$ and $R(x)$ are continuous at x=0, therefore, $\mu_1(x)dx = \frac{dG_1(x)}{1-G_1(x)}$, $\mu_2(x)dx = \frac{dG_2(x)}{1-G_2(x)}$ $\beta(x)dx = \frac{dD(x)}{1-D(x)}$ and $\gamma(x)dx = \frac{dR(x)}{1-R(x)}$ are the first order differential functions (hazard rates) of $G_1(x)$, $G_2(x)$, $D(x)$ and $R(x)$, respectively.

Next, we define

$W_n^{(1)}(x;t)$ = probability that at time t, there are n (≥ 1) customers in the system, including one customer being provided FES since the elapsed service time *x*,

$W_n^{(2)}(x;t)$ = probability that at time *t*, there are n (≥ 1) customers in the system, including one customer being provided SOS since the elapsed service time *x*,

$F_n^D(x;t)$ = probability that at time *t*, there are n (≥ 1) customers in the queue, the server is in the failed state and waiting for repairs to start with elapsed waiting time *x*,

$F_n^R(x;t)$ = probability that at time *t*, there are n (≥ 1) customers in the queue and the server is under repairs with elapsed repair time x,

$Q(t)$ = probability that at time *t*, the system is empty and server is idle (but available in the system for service).

Now we shall analyze the limiting behavior of this queueing process at a random epoch with the help of Kolmogorov forward equations provided the following limits exist and are independent of the initial state:

$Q = \lim_{t\to\infty} Q(t)$, $W_n^{(1)}(x)dx = \lim_{t\to\infty} W_n^{(1)}(x,t)dx,$

$W_n^{(2)}(x)dx = \lim_{t\to\infty} W_n^{(2)}(x,t)dx,$ $F_n^D(x)dx = \lim_{t\to\infty} F_n^D(x,t)dx,$ and

$F_n^R(x)dx = \lim_{t\to\infty} F_n^R(x,t)dx,$ where $x > 0,$ and $n \geq 1$.

3 Steady State Equations Governing the System

Then following the usual probability reasoning, we have, for $x > 0,$ and $n \geq 1$, the following set of Kolmogorov forward equations under the steady state conditions:

$$\frac{d}{dx}W_n^{(1)}(x) + [\lambda + \mu_1(x) + \alpha]W_n^{(1)}(x) = \lambda\sum_{k=1}^{n} a_k W_{n-k}^{(1)}(x), \tag{1}$$

$$\frac{d}{dx}W_n^{(2)}(x) + [\lambda + \mu_2(x) + \alpha]W_n^{(2)}(x) = \lambda\sum_{k=1}^{n} a_k W_{n-k}^{(2)}(x), \tag{2}$$

$$\frac{d}{dx}F_n^D(x)+\left[\lambda+\beta(x)\right]F_n^D(x)=\lambda(1-c)F_n^D(x)+c\lambda\sum_{k=1}^{n}a_k\,F_{n-k}^D(x), \tag{3}$$

$$\frac{d}{dx}F_n^R(x)+\left[\lambda+\gamma(x)\right]F_n^R(x)=\lambda(1-c)F_n^R(x)+c\lambda\sum_{k=1}^{n}a_k\,F_{n-k}^R(x), \tag{4}$$

$$\lambda Q=(1-p)\int_0^\infty W_1^{(1)}(x)\mu_1(x)dx+\int_0^\infty W_1^{(2)}\mu_2(x)dx. \tag{5}$$

The above set of equations is to be solved under the following boundary conditions at $x = 0$ and for $n \ge 1$:

$$W_n^{(1)}(0)=\lambda a_n Q+(1-p)\int_0^\infty W_{n+1}^{(1)}(x)\mu_1(x)dx+\int_0^\infty W_{n+1}^{(2)}(x)\mu_2(x)dx+\int_0^\infty F_n^R(x)\gamma(x)dx, \tag{6}$$

$$W_n^{(2)}(0)=p\int_0^\infty W_n^{(1)}(x)\mu_1(x)dx, \tag{7}$$

$$F_n^D(0)=\alpha\left(W_n^{(1)}+W_n^{(2)}\right),\quad \text{where}\quad W_n^{(j)}=\int_0^\infty W_n^{(j)}(x)dx,\ j=1,2, \tag{8}$$

$$F_n^R(0)=\int_0^\infty F_n^D(x)\beta(x)dx, \tag{9}$$

and the normalizing condition

$$Q+\sum_{j=1}^{2}\ \sum_{n=1}^{\infty}\int_0^\infty W_n^{(j)}(x)dx+\sum_{n=1}^{\infty}\int_0^\infty F_n^D(x)dx+\sum_{n=1}^{\infty}\int_0^\infty F_n^R(x)dx=1. \tag{10}$$

4 Queue Size Distribution at a Random Epoch

Next, we define the following Probability Generating Functions for $|z|<1$:

$$W^{(1)}(x,z)=\sum_{n=1}^{\infty}z^nW_n^{(1)}(x),\ x>0;\quad W^{(1)}(0,z)=\sum_{n=1}^{\infty}z^nW_n^{(1)}(0),$$

$$W^{(1)}(z)=\int_0^\infty W^{(1)}(x,z)dx=\sum_{n=1}^{\infty}z^nW_n^{(1)}, \tag{11a}$$

$$W^{(2)}(x,z)=\sum_{n=1}^{\infty}z^nW_n^{(2)}(x),\ x>0;\quad W^{(2)}(0,z)=\sum_{n=1}^{\infty}z^nW_n^{(2)}(0),$$

$$W^{(2)}(z)=\int_0^\infty W^{(2)}(x,z)dx=\sum_{n=1}^{\infty}z^nW_n^{(2)}, \tag{11b}$$

$$F^{D}(x,z)=\sum_{n=1}^{\infty} z^{n} F_{n}^{D}(x),\ x>0; \quad F^{D}(0,z)=\sum_{n=1}^{\infty} z^{n} F_{n}^{D}(0)\ ,$$

$$F^{D}(z)=\int_{0}^{\infty} F^{D}(x,z)dx, \tag{11c}$$

$$F^{R}(x,z)=\sum_{n=1}^{\infty} z^{n} F_{n}^{R}(x),\ x>0; \quad F^{R}(0,z)=\sum_{n=1}^{\infty} z^{n} F_{n}^{R}(0)\ ,$$

$$F^{R}(z)=\int_{0}^{\infty} F^{R}(x,z)dx. \tag{11d}$$

We multiply equations (1) to (4) as well as the boundary conditions (6) to (9) by suitable powers of z, sum over all possible values of *n*, use (5) and use (11). Thus we obtain the following results:

$$W^{(1)}(z)=\frac{\lambda z\left(X(z)-1\right)\left(\dfrac{1-G_1^*(m)}{m}\right)Q}{z-(1-p)G_1^*(m)-pG_1^*(m)G_2^*(m)-\alpha\, z\,\Psi(m)\,D^*(k)\,R^*(k)}, \tag{12}$$

$$W^{(2)}(z)=\frac{p\ \lambda z\left(X(z)-1\right)G_1^*(m)\left(\dfrac{1-G_2^*(m)}{m}\right)Q}{z-(1-p)G_1^*(m)-pG_1^*(m)G_2^*(m)-\alpha\, z\,\Psi(m)\,D^*(k)\,R^*(k)}, \tag{13}$$

$$F^{D}(z)=\frac{\alpha\lambda z\left(X(z)-1\right)\Psi(m)\left(\dfrac{1-D^*(k)}{k}\right)Q}{z-(1-p)G_1^*(m)-pG_1^*(m)G_2^*(m)-\alpha\, z\,\Psi(m)\,D^*(k)\,R^*(k)}, \tag{14}$$

$$F^{R}(z)=\frac{\alpha\ \lambda z\left(X(z)-1\right)\Psi(m)\,D^*(k)\left(\dfrac{1-R^*(k)}{k}\right)Q}{z-(1-p)G_1^*(m)-pG_1^*(m)G_2^*(m)-\alpha\, z\,\Psi(m)\,D^*(k)\,R^*(k)}. \tag{15}$$

Where $m=\lambda\left(1-X(z)\right)+\alpha\,,\ k=\lambda\, c\left(1-X(z)\right)$,

$G_1^*\left[\lambda\,\left(1-X(z)\right)+\alpha\right]=\int_0^{\infty} e^{-\left[\lambda\,(1-X(z))+\alpha\right]x} dG_1(x)$ is the Laplace-Steiltjes transform of the first

essential service time, $G_2^*\left[\lambda\,\left(1-X(z)\right)+\alpha\right]=\int_0^{\infty} e^{-\left[\lambda\,(1-X(z))+\alpha\right]x} dG_2(x)$ is the Laplace-Steiltjes transform of the second optional service time, $D^{*}\left[\lambda c\left(1-X(z)\right)\right]=\int_{0}^{\infty} e^{-\left[\lambda c\left(1-X(z)\right)\right]x} dD(x)$ is the Laplace-Steiltjes transform of the waiting time before repairs to start, and $R^{*}\left[\lambda c\left(1-X(z)\right)\right]=\int_{0}^{\infty} e^{-\left[\lambda c\left(1-X(z)\right)\right]x} dR(x)$ is the Laplace-Steiltjes transform of the repair time.

Further, it is easy to see that at z=1 the right hand side expressions in equations (12) to (15) are all of zero/zero form. Therefore, applying L'Hopital's rule, we get

$$W^{(1)}(1)=\lim_{z\to 1} W^{(1)}(z)=\frac{\lambda E(X)\left(\frac{1-G_1^{*}(\alpha)}{\alpha}\right) Q}{1-\Psi(\alpha)\left[\alpha+\lambda E(X)+\alpha\lambda E(X)\left(E(D)+E(R)\right)\right]}, \tag{16}$$

This is the steady state probability that at any random epoch, the server is busy providing first essential service,

$$W^{(2)}(1)=\lim_{z\to 1} W^{(2)}(z)=\frac{p\,\lambda E(X)\, G_1^{*}(\alpha)\left(\frac{1-G_2^{*}(\alpha)}{\alpha}\right) Q}{1-\Psi(\alpha)\left[\alpha+\lambda E(X)+\alpha\lambda E(X)\left(E(D)+E(R)\right)\right]}, \tag{17}$$

This is the steady state probability that at any random epoch, the server is busy providing second optional service,

$$F^{D}(1)=\lim_{z\to 1} F^{D}(z)=\frac{\alpha\,\lambda E(X)\, E(D)\Psi(\alpha)Q}{1-\Psi(\alpha)\left[\alpha+\lambda E(X)+\alpha\lambda E(X)\left(E(D)+E(R)\right)\right]}, \tag{18}$$

This is the steady state probability that at any random epoch, the server is in the failed state and waiting for repairs to start,

$$F^{R}(1)=\lim_{z\to 1} F^{R}(z)=\frac{\alpha\,\lambda E(X) E(R)\Psi(\alpha)Q}{1-\Psi(\alpha)\left[\alpha+\lambda E(X)+\alpha\lambda E(X)\left(E(D)+E(R)\right)\right]}, \tag{19}$$

This is the steady state probability that at any random epoch, the server is in the failed state and under repairs.

Next, the normalizing condition in (10) is equivalent to

$$Q+W^{(1)}(1)+W^{(2)}(1)+F^{D}(1)+F^{R}(1)=1. \tag{20}$$

Utilizing (4.6) to (4.9) in (4.10), we obtain on simplifying

$$Q = \frac{1-[\lambda E(X) + \alpha\,\lambda\, E(X)\left(E(D)+E(R)\right)]\Psi(\alpha)}{1-\alpha\,\Psi(\alpha)}. \tag{21}$$

We note that (21) yields the following stability condition under which the steady state exists:

$$\rho_0 := [\alpha + \lambda E(X) + \alpha\,\lambda\, E(X)\left(E(D)+E(R)\right)]\Psi(\alpha) < 1. \tag{22}$$

Finally, replacing the value of Q found in (21) in the numerators of equations (12) to (15), we have explicitly determined all the probability generating functions. Similarly, equations (16) to (19) can be simplified as follows.

$$W^{(1)}(1) = \lim_{z\to 1} W^{(1)}(z) = \frac{\lambda\, E(X)\left(\frac{1-G_1^*(\alpha)}{\alpha}\right)}{1-\alpha\,\Psi(\alpha)}, \tag{23}$$

$$W^{(2)}(1) = \lim_{z\to 1} W^{(2)}(z) = \frac{p\,\lambda E(X)\, G_1^*(\alpha)\left(\frac{1-G_2^*(\alpha)}{\alpha}\right)}{1-\alpha\,\Psi(\alpha)}, \tag{24}$$

$$F^{D}(1) = \lim_{z\to 1} F^{D}(z) = \frac{\alpha\,\lambda E(X)\, E(D)\Psi(\alpha)}{1-\alpha\,\Psi(\alpha)}, \tag{25}$$

$$F^{R}(1) = \lim_{z\to 1} F^{R}(z) = \frac{\alpha\,\lambda E(X)\, E(R)\Psi(\alpha)}{1-\alpha\,\Psi(\alpha)}, \tag{26}$$

We may note that ρ, the utilization factor of the system is the proportion of time the system is busy providing the first essential service or the second optional service. Therefore, by adding (4.6) and (4.7) and using the value of Q from (4.11), we obtain

$$\rho = \frac{\lambda\, E(X)\Psi(\alpha)}{1-\alpha\,\Psi(\alpha)}. \tag{27}$$

Now, we define the probability generating function of the queue size distribution at a random epoch irrespective of the state of the system as follows:

$$P(z) = Q + W^{(1)}(z) + W^{(2)}(z) + F^{D}(z) + F^{R}(z). \tag{28}$$

This can be obtained by adding equations (12) to (15) and (21) and simplifying.

5 The Average System Size

Let L denote the mean system size at a random epoch. Then using the PGF $P(z)$ in equation (28) and after somewhat heavy algebra and simplification, we obtain

$$L = \frac{d}{dz} P(z)\Big|_{z=1}$$

$$= \rho_0 + \frac{\lambda^2 (E(X))^2}{(1-\rho_0)\alpha^2} \left[1 - G_1^*(\alpha) - \alpha E\left(s_1 e^{-s_1 \alpha}\right) + p\left(1 - G_2^*(\alpha) - \alpha E\left(s_2 e^{-s_2 \alpha}\right)\right) \right]$$

$$+ \frac{\alpha \lambda^2 (E(X))^2}{2(1-\rho_0)} \left[E(D^2) + E(R^2) + 2E(D)\,E(R) \right] + \frac{\rho_0\, E\,(X(X-1))}{2E(X)\,(1-\rho_0)}, \qquad (29)$$

where $\rho_0 = [\alpha + \lambda E(X) + \alpha\, \lambda\, E(X)(E(D) + E(R))]\Psi(\alpha)$ and $E(X(X-1))$ is the second factorial moment of batch size of arrivals.

6 Some Particular Cases

Case 1: We assume single Poisson arrivals with no restricted admissibility and that FES, SOS, Delay Time and Repair Time all have exponential distributions.

In this case we have, c=1, $E(X)=1$, $E(S_1) = \frac{1}{\mu_1}$, $E(S_2) = \frac{1}{\mu_2}$, $E(D) = \frac{1}{\beta}$, $E(D^2) = \frac{2}{\beta^2}$,

$E(R) = \frac{1}{\gamma}$, $E(R^2) = \frac{2}{\gamma^2}$, $E(X(X-1))=0$, $G_1^*(m) = \left(\frac{\mu_1}{\mu_1 + \lambda(1 - X(z)) + \alpha} \right)$,

$G_1^*(\alpha) = \left(\frac{\mu_1}{\mu_1 + \alpha} \right)$, $G_2^*(m) = \left(\frac{\mu_2}{\mu_2 + \lambda(1 - X(z)) + \alpha} \right)$, $G_2^*(\alpha) = \left(\frac{\mu_2}{\mu_2 + \alpha} \right)$,

$D^*(k) = \left(\frac{\beta}{\beta + \lambda(1 - X(z))} \right)$, and $R^*(k) = \left(\frac{\gamma}{\gamma + \lambda(1 - X(z))} \right)$.

Consequently, $\left(\frac{1 - G_1^*(m)}{m} \right) = \left(\frac{1}{\mu_1 + \lambda(1 - X(z)) + \alpha} \right)$, $\left(\frac{1 - G_1^*(\alpha)}{\alpha} \right) = \left(\frac{1}{\mu_1 + \alpha} \right)$,

$\left(\frac{1 - G_2^*(m)}{m} \right) = \left(\frac{1}{\mu_2 + \lambda(1 - X(z)) + \alpha} \right)$, $\left(\frac{1 - G_2^*(\alpha)}{\alpha} \right) = \left(\frac{1}{\mu_2 + \alpha} \right)$,

$\left(\frac{1 - D^*(k)}{k} \right) = \left(\frac{1}{\beta + \lambda(1 - X(z))} \right)$, $\left(\frac{1 - R^*(k)}{k} \right) = \left(\frac{1}{\gamma + \lambda(1 - X(z))} \right)$, and $\Psi(\alpha) = \frac{p\mu_1 + \mu_2 + \alpha}{(\mu_1 + \alpha)(\mu_2 + \alpha)}$.

With these substitutions in the main results we can obtain the corresponding to this case. In addition, we find the following probabilities:

$$W^{(1)}(1) = \frac{\lambda\left(\frac{1}{\mu_1+\alpha}\right)}{1-\alpha\left[\frac{p\mu_1+\mu_2+\alpha}{(\mu_1+\alpha)(\mu_2+\alpha)}\right]}, \tag{30}$$

This is the steady state probability that at any random epoch the server is busy providing first essential service,

$$W^{(2)}(1) = \frac{p\,\lambda\left(\frac{\mu_1}{\mu_1+\alpha}\right)\left(\frac{1}{\mu_2+\alpha}\right)}{1-\alpha\left[\frac{p\mu_1+\mu_2+\alpha}{(\mu_1+\alpha)(\mu_2+\alpha)}\right]}, \tag{31}$$

This is the steady state probability that at any random epoch the server is busy providing second optional service,

$$F^{D}(1) = \frac{\alpha\,\lambda\left(\frac{1}{\beta}\right)\left[\frac{p\mu_1+\mu_2+\alpha}{(\mu_1+\alpha)(\mu_2+\alpha)}\right]}{1-\alpha\left[\frac{p\mu_1+\mu_2+\alpha}{(\mu_1+\alpha)(\mu_2+\alpha)}\right]}, \tag{32}$$

This is the steady state probability that at any random epoch the server is in the failed state and waiting for repairs to start,

$$F^{R}(1) = \frac{\alpha\,\lambda\left(\frac{1}{\gamma}\right)\left[\frac{p\mu_1+\mu_2+\alpha}{(\mu_1+\alpha)(\mu_2+\alpha)}\right]}{1-\alpha\left[\frac{p\mu_1+\mu_2+\alpha}{(\mu_1+\alpha)(\mu_2+\alpha)}\right]}, \tag{33}$$

This is the steady state probability that at any random epoch the server is in the failed state and under repairs.

Further, we have

$$Q = \frac{1-\left[\alpha+\lambda+\alpha\,\lambda\left(\frac{1}{\beta}+\frac{1}{\gamma}\right)\right]\left[\frac{p\mu_1+\mu_2+\alpha}{(\mu_1+\alpha)(\mu_2+\alpha)}\right]}{1-\alpha\left[\frac{p\mu_1+\mu_2+\alpha}{(\mu_1+\alpha)(\mu_2+\alpha)}\right]}, \tag{34}$$

$$\rho = \frac{\lambda\left[\frac{p\mu_1+\mu_2+\alpha}{(\mu_1+\alpha)(\mu_2+\alpha)}\right]}{1-\alpha\left[\frac{p\mu_1+\mu_2+\alpha}{(\mu_1+\alpha)(\mu_2+\alpha)}\right]}, \tag{35}$$

Further, since both service times are exponential, we have

$$1-G_1^*(\alpha)=\frac{\alpha}{\mu_1+\alpha},\ 1-G_2^*(\alpha)=\frac{\alpha}{\mu_2+\alpha},\ E(s_1e^{-s_1\alpha})=\frac{\mu_1}{(\mu_1+\alpha)^2}\text{ and }E(s_2e^{-s_2\alpha})=\frac{\mu_2}{(\mu_2+\alpha)^2}.$$

Therefore, (5.1) reduces to

$$L=\rho_0+\frac{\lambda^2}{(1-\rho_0)}\left[\frac{1}{(\mu_1+\alpha)^2}+\frac{p}{(\mu_2+\alpha)^2}\right]+\frac{\alpha\lambda^2}{(1-\rho_0)}\left[\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\beta\gamma}\right], \tag{36}$$

where

$$\rho_0=\left[\alpha+\lambda+\alpha\lambda\left(\frac{1}{\beta}+\frac{1}{\gamma}\right)\right]\left[\frac{p\mu_1+\mu_2+\alpha}{(\mu_1+\alpha)(\mu_2+\alpha)}\right]. \tag{37}$$

Case 2: No Second Optional Service

In this case, we put $p=0$ in the main results.

Case 3: No Delay

In this case, we have $D^*(k)=1$ and $E(D)=0$.

Case 4: No Breakdowns

In this case, we let $\alpha=0$ and consequently, $G_i^*(\alpha)=1$, for $i=1,2$, and

$$\lim_{\alpha\to 0}\left(\frac{1-G_1^*(\alpha)}{\alpha}\right)=E(S_1),\ \lim_{\alpha\to 0}\left(\frac{1-G_2^*(\alpha)}{\alpha}\right)=E(S_2),\ \lim_{\alpha\to 0}\Psi(\alpha)=E(S_1)+pE(S_2),$$

$$\lim_{\alpha\to 0}\left\{\frac{1-G_1^*(\alpha)-\alpha E(S_1e^{-s_1\alpha})}{\alpha^2}\right\}=\frac{E(S_1^{\ 2})}{2},$$

and $$\lim_{\alpha\to 0}\left\{\frac{1-G_2^*(\alpha)-\alpha E(S_2e^{-s_2\alpha})}{\alpha^2}\right\}=\frac{E(S_2^{\ 2})}{2}.$$

With these substitutions in the main results we can derive results of this particular case.

Case 5: No Second Optional Service, No Breakdowns

In this case we put $p=0$ and $E(S_2)=0$ in the results of case 4 or we put $\alpha=0$ in the results of case 2.

7 Numerical Examples

We provide some numerical examples to check the validity of our results obtained in Case 1 and to see the effect of various parameters involved in our model (namely, the breakdown rate α, the delay parameter β and the completion of repair parameter γ) on the utilization factor ρ and on the probabilities of various steady states of the system, namely the probabilities of the idle state, the working state and the breakdown state waiting for repair to start and under repair. We assume the fixed values of the arrival rate $\lambda = 1$, the service rates $\mu_1 = 2$ and $\mu_2 = 4$, and arbitrarily choose values of the other various parameters such that the stability condition (6.8) of the particular case 1 is not violated. We obtain the following numerical values which depict results as expected.

Table 1: Effect of β on the utilization factor and on the probabilities of steady states.

λ	μ_1	μ_2	p	α	γ	ρ
1	2	4	0.5	0.5	5	0.647

β	L_q	L	Q	$W^{(1)}(1)$	$W^{(2)}(1)$	$F^D(1)$	$F^R(1)$
6	1.33	2.15	0.23	0.53	0.12	0.05	0.06
8	1.20	2.01	0.25	0.53	0.12	0.04	0.06
10	1.14	1.94	0.26	0.53	0.12	0.03	0.06
12	1.10	1.90	0.26	0.53	0.12	0.03	0.06
14	1.07	1.87	0.27	0.53	0.12	0.02	0.06
16	1.05	1.85	0.27	0.53	0.12	0.02	0.06

Table 2: Effect of α on the utilization factor and on the probabilities of steady states.

λ	μ_1	μ_2	p	β	γ
1	2	4	0.5	4	5

α	L_q	L	Q	$W^{(1)}(1)$	$W^{(2)}(1)$	$F^D(1)$	$F^R(1)$	ρ
0	0.75	1.38	0.38	0.50	0.13	0.00	0.00	0.625
0.1	0.84	1.52	0.34	0.51	0.12	0.02	0.01	0.630
0.2	0.97	1.69	0.31	0.51	0.12	0.03	0.03	0.634
0.3	1.13	1.90	0.28	0.52	0.12	0.05	0.04	0.639
0.4	1.36	2.16	0.24	0.52	0.12	0.06	0.05	0.643
0.5	1.67	2.51	0.21	0.53	0.12	0.08	0.06	0.647

Table 3: Effect of γ on the utilization factor and on the probabilities of steady states.

λ	μ_1	μ_2	p	α	β	ρ
1	2	4	0.5	0.2	2	0.634

γ	L_q	L	Q	$W^{(1)}(1)$	$W^{(2)}(1)$	$F^D(1)$	$F^R(1)$
1	3.75	4.60	0.18	0.51	0.12	0.06	0.13
2	1.81	2.60	0.24	0.51	0.12	0.06	0.06
3	1.47	2.24	0.26	0.51	0.12	0.06	0.04
4	1.34	2.10	0.27	0.51	0.12	0.06	0.03
5	1.27	2.03	0.28	0.51	0.12	0.06	0.03
6	1.23	1.98	0.28	0.51	0.12	0.06	0.02

Table 4: Effect of p on the utilization factor and on the probabilities of steady states.

λ	μ_1	μ_2	α	β	γ
1	2	4	0.5	6	4

p	L_q	L	Q	$W^{(1)}(1)$	$W^{(2)}(1)$	$F^D(1)$	$F^R(1)$	ρ
0	0.71	1.40	0.40	0.50	0.00	0.04	0.06	0.500
0.2	0.92	1.67	0.33	0.51	0.05	0.05	0.07	0.557
0.4	1.26	2.06	0.26	0.52	0.09	0.05	0.08	0.616
0.6	1.90	2.77	0.18	0.54	0.14	0.06	0.08	0.679
0.8	3.60	4.53	0.10	0.55	0.20	0.06	0.09	0.744
1	21.24	22.23	0.02	0.56	0.25	0.07	0.10	0.813

One can easily notice that when β increases for fixed values of α and γ, the probability of the idle state Q increases and the average queue length L decreases. Similarly, as γ increases for fixed values of α and β. However, when α increases for fixed values of β and γ, the probability of the idle state Q decreases, the average queue length L increases and the utilization factor ρ increases. Clearly, the utilization factor ρ increases as p increases for fixed values of α, β and γ. On the other hand, it is independent of the delay parameter β and the completion of repair parameter γ.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] B. Avi-Itzhak and P. Naor, Some Queueing Problems with the Service Station Subject to Breakdowns, Oper. Res., 11 (1963), 303-320.

[2] R. Fadhil, K. C. Madan and A. C. Lukas, On M(x)/G/1 Queueing System with Random Breakdowns, Server vacations, Delay Times and a Standby Server, International Journal of Operational Research, 15 (1) (2012), 30-47.

[3] D. P. Gaver, A Waiting Line with Interrupted Service Including Priorities, J. Roy. Statist. Soc., Ser. B, 24 (1962), 73-90.

[4] H. S. Lee and M. M. Srinivasan., Control Policies for the $M^X/G/1$ Queueing Systems, Management Sciences, 35 (1989), 708-721.

[5] W. Li, D. Shi and X. Chao, Reliability Analysis of M/G/1 Queueing System with Server Breakdowns and Vacations, J. Appl. Prob., 34 (1997), 546-555.

[6] K. C. Madan, A Priority Queueing System with Service Interruption, Statistica Neerlandica, 27 (3) (1973), 115-123.

[7] K. C. Madan., A Queueing System with Random Failures and Delayed Repairs, Journal of Indian Stat. Ass., 32 (1994), 39-48.

[8] K. C. Madan, An $M/G/1$ Queue with Second Optional Service, Queueing Systems, 34 (2000), 37-46.

[9] K. C. Madan and W. Abu-Dayyeh, Restricted Admissibility of Batches into an M/G/1 Type Bulk Queue with Modified Bernoulli Schedule Server Vacations, ESAIM: Probability and Statistics, 6 (2002), 113-125.

[10] K. C. Madan, and G. Choudhury, An M(X)/G/1 Queue with a Bernoulli Vacation Schedule Under Restricted Admissibility Policy, Sankhya, 66 (1) (2004), 175-193.

[11] K. C. Madan, An $M/G/1$ Queue with Time-Homogeneous Breakdowns and Deterministic Repair Times, Soochow Journal of Mathematics, 29 (1) (2003), 103-110.

[12] T. Takine, and B. Sengupta, A Single Server Queue with Service Interruptions, Queueing Systems, 26 (1997), 285-300.