Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 6, 1671-1684
ISSN: 1927-5307

A MATHEMATICAL MODEL FOR THE TRANSMISSION DYNAMICS OF HIV/AIDS IN A TWO-SEX POPULATION CONSIDERING COUNSELING AND ANTIRETROVIRAL THERAPY (ART)

RICHARD A. KIMBIR[1], MARTIN J. I. UDOO[2], AND TERHEMEN ABOIYAR[1,*]

[1]Department of Mathematics, Statistics/Computer Science,University of Agriculture, PMB 2373, Makurdi, Nigeria

[2]Department of Mathematics and Computer Science, Modibbo Adama University of Technology, Yola, Nigeria

Abstract. An extended version of a one-sex mathematical model of HIV/AIDS transmission dynamics considering Counseling and Antiretroviral Therapy (ART) has been carried out. We proved that the disease-free equilibrium states (DFES) of the sub-model without ART and the sub-models with only infected males or females receiving ART are locally and asymptotically stable under prescribed conditions on the given model parameters. Threshold conditions are therefore derived, in terms of the given model parameters, for stability of DFES of the sub-models, as well as the proportion of infected people to receive ART. This means that HIV/AIDS can be eradicated under such conditions. Furthermore, results from the numerical experiments show the interplay of the model parameters in the control or eradication of HIV/AIDS. From these results, we see that the control or eradication of HIV/AIDS in heterosexual populations is dependent on the net transmission rates of the infection, the effectiveness of counseling and ART, and proportion of infected people receiving ART for each sex.

Keywords: two-sex model, HIV/AIDS, ART, counseling, disease free equilibrium state

2000 AMS Subject Classification: 92B05; 93A30

*Corresponding author

Received March 28, 2012

1. Introduction

The much dreaded Acquired Immunodeficiency Syndrome (AIDS), which is one of the greatest health problems of this millennium, is caused by a virus called Human Immunedeficiency Virus (HIV). AIDS quickly developed into a worldwide epidemic, affecting virtually every nation. Presently there is no permanent cure for HIV/AIDS. However, a major method used in the fight against HIV/AIDS, apart from behavioral change (which includes: abstinence from sex, sexual intercourse with one mutually faithful uninfected partner and having protected sex using the condom), is Counseling and Antiretroviral Therapy (ART). By this approach, HIV positives are detected, counseled, and placed on antiretroviral drugs. Through public awareness campaigns, the general public is encouraged to go for tests in order to determine their HIV/AIDS status so as to benefit from ART. The giving of antiretroviral drugs in the correct way, with adherence support, is called Anti-Retro-Viral Therapy, ART. ART does not cure HIV infection; it only boosts the immune system of infected people against secondary infections, thereby prolonging their life span. People who are HIV positive are also detected through random screening and contact tracing [7].

Several mathematical models on HIV/AIDS have been formulated and analyzed. Examples can be found in Swanson [1], Gumel *et al* [5], Corbett *et al* [3] and Kimbir & Oduwole [8]. In this paper, we formulate a two-sex mathematical model for the transmission dynamics of HIV/AIDS in a two-sex population considering counselling and antiretroviral therapy. This model extends the one-sex mathematical model for the transmission dynamics of HIV/AIDS considering Counseling and ART by Kimbir & Oduwole [8].

2. The Mathematical Model

In this section, we will provide the two-sex model which is the basis of this work. This model is based on the following assumptions:

(1) The population is homogeneous;
(2) Vertical transmission and age-structure are ignored;

(3) Infected persons in addition to dying naturally; irrespective of whether they use the control measure; ART, die due to the infection;
(4) The control measure is of varying efficacy;
(5) The population is heterosexual (that is males and females);
(6) The control measure may not be totally efficacious, since infected individuals that are on ART may transmit the infection;
(7) Transmission is by heterosexual contact only, excluding all other means of transmission;
(8) Natural birth and death rates are the same for all sexes;
(9) Both infected males and females use the control measure-ART;
(10) Age structure is ignored.

In addition, Table 1 below gives all the parameters and variables used in the model. The flow diagram in Figure 1 will be found useful in the derivation of the model equations. From the assumptions in the stated in above and the flow diagramn in Figure 1, the

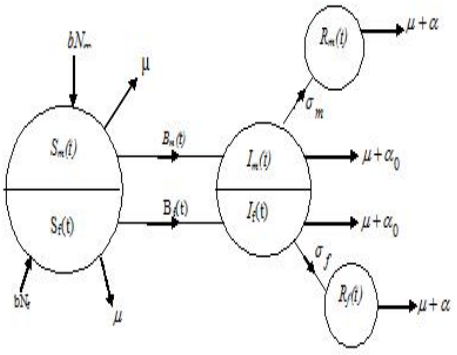

FIGURE 1. The flow diagram for the model.

TABLE 1. Model Parameters and Variables

Variable/ Parameter	Description
$S_m(t)$	Number of susceptible males at time t
$I_m(t)$	Number of infected males at time t
$R_m(t)$	Number of infected males receiving ART at time t
$c_m(t)$	Average number of sexual partners of infected males with females per unit time
$c_m^*(t)$	Average number of sexual partners of infected males receiving ART with females per unit time
β_m	Probability of transmission by an infected male
β_m^*	Probability of transmission by an infected male receiving ART
σ_m	Proportion of infected males receiving ART per unit time
$B_m(t)$	The infection rate of males at time t (that is incidence rate of males)
$S_f(t)$	Number of susceptible females at time t
$I_f(t)$	Number of infected females at time t
$R_f(t)$	Number of infected females receiving ART at time t
$c_f(t)$	Average number of sexual partners of infected females with males per unit time
$c_f^*(t)$	Average number of sexual partners of infected females receiving ART with males per unit time
β_f	Probability of transmission by an infected female
β_f^*	Probability of transmission by an infected female receiving ART
σ_f	Proportion of infected females receiving ART per unit time
$B_f(t)$	The infection rate of females at time t (that is incidence rate of females)
T	Maximum life-span of both sexes after infection
k	Efficacy of ART for both sexes per unit time
b	Population birth rate, $b > 0$
μ	Population death rate, $\mu > 0$
α	Population death rate of infected receiving ART
α_0	Population death rate of infected not receiving ART

following model equations are derived

$$\frac{dS_m}{dt} = bN_m - B_m S_m - \mu S_m \tag{1}$$

$$\frac{dI_m}{dt} = B_m S_m - (\mu + \alpha_0 + \sigma_m) I_m \tag{2}$$

$$\frac{dR_m}{dt} = \sigma_m I_m - (\mu + \alpha) R_m \tag{3}$$

$$\frac{dS_f}{dt} = bN_f - B_f S_f - \mu S_f \tag{4}$$

$$\frac{dI_f}{dt} = B_f S_f - (\mu + \alpha_0 + \sigma_f) I_f \tag{5}$$

$$\frac{dR_f}{dt} = \sigma_f I_f - (\mu + \alpha) R_f \tag{6}$$

Moreover, the total population of males and females N_m and N_f are given by

$$N_m = S_m + I_m + R_m \tag{7}$$

$$N_f = S_f + I_f + R_f. \tag{8}$$

Following Hseih [6], the incidence rates for the male and female populations denoted $B_m(t)$ and $B_f(t)$ are given as

$$B_m = \frac{c_m \beta_f I_f + c_m^* \beta_f^* R_f}{N_f} \tag{9}$$

and

$$B_f = \frac{c_f \beta_m I_m + c_f^* \beta_m^* R_m}{N_m} \tag{10}$$

Finally, we note that $\alpha = \alpha_0 e^{-kT}$.

Transforming the model equations into proportions reduces the number of equations and the proportions of infected individuals define prevalence of infection, which has biological meaning. To achieve this, we let

$$s_m = \frac{S_m}{N_m}, \quad i_m = \frac{I_m}{N_m} \quad \text{and} \quad r_m = \frac{R_m}{N_m}$$

so that

$$s_m + i_m + r_m = 1.$$

Likewise,

$$s_f = \frac{S_f}{N_f}, \quad i_f = \frac{I_f}{N_f} \quad \text{and} \quad r_f = \frac{R_f}{N_f}$$

so that

$$s_f + i_f + r_f = 1.$$

To this end, the reduced system of governing equations is given as

$$\frac{di_m}{dt} = (c_m\beta_f i_f + c_m^*\beta_f^* r_f)(1 - i_m - r_m) - (b_0 + \sigma_m)i_m + \alpha i_m r_m + \alpha i_m^2 \tag{11}$$

$$\frac{di_f}{dt} = (c_f\beta_m i_m + c_f^*\beta_m^* r_m)(1 - i_f - r_f) - (b_0 + \sigma_f)i_f + \alpha i_f r_f + \alpha i_f^2 \tag{12}$$

$$\frac{dr_m}{dt} = \sigma_m i_m - (b + \alpha)r_m + \alpha_0 i_m r_m + \alpha r_m^2 \tag{13}$$

$$\frac{dr_f}{dt} = \sigma_f i_f - (b + \alpha)r_f + \alpha_0 i_f r_f + \alpha r_f^2 \tag{14}$$

where $b_0 = b + \alpha_0$.

3. Stability Analysis of Disease Free Equilibrium States of the Model

In this section, we shall be analyzing the stability of the disease free equilibrium states (DFES) of some sub-models so as to derive conditions under which HIV/AIDS can be controlled or eradicated.

Case I: The DFES of the sub-model without ART ($\sigma_m = \sigma_f = 0$)

In this case, $c_m^* = c_f^* = \beta_m^* = \beta_f^* = r_m = r_f = 0$, and the general model (11)–(14) reduces to

$$\frac{di_m}{dt} = c_m\beta_f i_f(1 - i_m) - b_0 i_m + \alpha_0 i_m^2 \tag{15}$$

$$\frac{di_f}{dt} = c_f\beta_m i_m(1 - i_f) - b_0 i_f + \alpha_0 i_f^2 \tag{16}$$

The DFES is obtained by setting the left hand sides of (15) and (16) to zero and then solving simultaneously. This gives the DFES as $(i_m^*, i_f^*) = (0, 0)$.

The Jacobian matrix, J_0, evaluated at the DFES is given by

$$J_0 = \begin{pmatrix} -b_0 & c_m\beta_f \\ c_f\beta_m & -b_0 \end{pmatrix}.$$

Now,

$$\mathrm{tr}J_0 = -2b_0 < 0,$$

and

$$\det J_0 = b_0^2 - c_m\beta_f c_f\beta_m > 0$$

provided

$$c_m\beta_f c_f c\beta_m < b_0^2.$$

Therefore the DFES is locally and asymptotically stable (LAS) (see for example, Benyah [1]).

Thus we have proved the following theorem.

Theorem 3.1. *Given* $b, \alpha_0, c_m, \beta_f, c_f, \beta_m > 0$. *If*

$$c_m\beta_f c_f c\beta_m < b_0^2,$$

then the DFES for this sub-model is LAS.

Case II: The sub-model with only infected males receiving ART ($\sigma_m > 0, \sigma_f = 0$)

In this case, $c_f^* = \beta_f^* = r_f = 0$ and the general model (11)–(14) reduces to

$$\frac{di_m}{dt} = c_m\beta_f i_f(1 - i_m - r_m) - (b_0 + \sigma_m)i_m + \alpha i_m r_m + \alpha_0 i_m^2, \tag{17}$$

$$\frac{di_f}{dt} = c_f\beta_m i_m(1 - i_f) - b_0 i_f + \alpha_0 i_f^2, \tag{18}$$

$$\frac{dr_m}{dt} = \sigma_m i_m - (b + \alpha)r_m + \alpha_0 i_m r_m + \alpha r_m^2. \tag{19}$$

The DFES for this sub-model is obtained as $(i_m^*, i_f^*, r_m^*) = (0, 0, 0)$.

The Jacobian matrix, evaluated at the DFES is given by

$$J_0 = \begin{pmatrix} -(b_0 + \sigma_m) & c_m\beta_f & 0 \\ c_f\beta_m & -(b_0) & 0 \\ \sigma_m & 0 & -(b + \alpha) \end{pmatrix}$$

The characteristic polynomial associated with J_0 is given by

$$p(\lambda) = -\lambda^3 + A\lambda^2 - B\lambda + C$$

where

$$A = -[2b_0 + b + \alpha + \sigma_m] \tag{20}$$

$$-B = -[(b_0 + \sigma_m)b_0 + (b_0 + \sigma_m)(b + \alpha) + b_0(b + \alpha) - c_m\beta_f c_f \beta_m] \tag{21}$$

$$C = -[(b_0 + \sigma_m)b_0(b + \alpha) - c_m\beta_f c_f \beta_m(b + \alpha)]. \tag{22}$$

Moreover, $-p(\lambda)$ has the same roots as $p(\lambda)$. The Hurwitz matrix corresponding to $-p(\lambda)$ is

$$\begin{pmatrix} -A & -C & 0 \\ 1 & B & 0 \\ 0 & -A & -C \end{pmatrix}.$$

The three Hurwitz determinants are given as

$$D_1 = -A = 2b_0 + b + \alpha + \sigma_m > 0 \qquad (\text{ since } b, \alpha_0, \alpha, \sigma_m > 0),$$

$$D_2 = \begin{vmatrix} -A & -C \\ 1 & B \end{vmatrix} = -AB + C.$$

Now, $-C > 0$ if and only if $c_m\beta_f c_f \beta_m < (b_0 + \sigma_m)b_0$ and

$$B = [(b_0 + \sigma_m)b_0 + (b_0 + \sigma_m)(b + \alpha) + b_0(b + \alpha) - c_m\beta_f c_f \beta_m] > 0$$

provided that $c_m\beta_f c_f \beta_m < (b_0+\sigma_m)b_0$. Moreover, $-AB+C > 0$ provided that $c_m\beta_f c_f \beta_m < (b_0 + \sigma_m)b_0$. Therefore, $D_2 > 0$. Finally,

$$D_3 = -C \cdot \begin{vmatrix} -A & -C \\ 1 & B \end{vmatrix} = -C \cdot D_2 > 0,$$

Thus the Routh-Hurwitz stability conditions (a) $-A, B, C > 0$ and (b) $-AB > -C$ are satisfied. For further details see Deo & Raghavendra [4]. Therefore, the DFES for this sub-model is LAS.

Note that the condition $c_m\beta_f c_f \beta_m < (b_0 + \sigma_m)b_0$ is equivalent to the condition

$$\sigma_m > \frac{c_m\beta_f c_f \beta_m - b_0^2}{b_0}.$$

Theorem 3.2. *Given that* $b, \alpha_0, \alpha, c_m, \beta_f, c_f, \beta_m > 0$. *If*

$$\sigma_m > \frac{c_m\beta_f c_f\beta_m - b_0^2}{b_0},$$

then the DFES for the sub-model (17)– (19) is LAS.

Case III: The sub-model with only infected females receiving ART ($\sigma_m = 0, \sigma_f > 0$)

In this case, $c_m^* = \beta_m^* = r_m = 0$ and the general model (11)–(14) reduces to

$$\frac{di_m}{dt} = c_m\beta_f i_f(1 - i_m) - b_0 i_m + \alpha_0 i_m^2 \tag{23}$$

$$\frac{di_f}{dt} = c_f\beta_m i_m(1 - i_f - r_f) - (b_0 + \sigma_f)i_f + \alpha i_f r_f + \alpha_0 i_f^2 \tag{24}$$

$$\frac{dr_f}{dt} = \sigma_f i_f - (b + \alpha)r_f + \alpha_0 i_f r_f + \alpha i_f \tag{25}$$

The DFES for this sub-model is $(i_m^*, i_f^*, r_m^*) = (0, 0, 0)$. The Jacobian matrix evaluated at the DFES is given by

$$J_0 = \begin{pmatrix} -b_0 & c_m\beta_f & 0 \\ c_f\beta_m & -(b_0 + \sigma_f) & 0 \\ 0 & \sigma_f & -(b + \alpha) \end{pmatrix}$$

The characteristic polynomial associated with J_0 is given by

$$p(\lambda) = -\lambda^3 + A\lambda^2 - B\lambda + C$$

where

$$A = -[2b_0 + b + \alpha + \sigma_f] \tag{26}$$

$$-B = -[(b_0 + \sigma_f)b_0 + (b_0 + \sigma_f)(b + \alpha) + b_0(b + \alpha) - c_m\beta_f c_f\beta_m] \text{ and} \tag{27}$$

$$C = -[(b_0 + \sigma_m)b_0(b + \alpha) - c_m\beta_f c_f\beta_m(b + \alpha)]. \tag{28}$$

Once again, $-p(\lambda)$ has the same roots as $p(\lambda)$. Thus, the Hurwitz matrix corresponding to $-p(\lambda)$ is

$$\begin{pmatrix} -A & -C & 0 \\ 1 & B & 0 \\ 0 & -A & -C \end{pmatrix}.$$

We observe that this is the same matrix as the one in the submodel with $\sigma_m > 0$ and $\sigma_f = 0$. To this end, the Routh-Hurwitz conditions are satisfied under the same conditions. All this leads us to the following result:

Theorem 3.3. *Given that* $b, \alpha_0, \alpha, c_m, \beta_f, c_f, \beta_m > 0$. *If*

$$\sigma_f > \frac{c_m \beta_f c_f \beta_m - b_0^2}{b_0},$$

then the DFES for the sub-model (23)– (25) is LAS.

4. Numerical Experiments

In this section we will use a fourth-order Runge-Kutta method, see [2], to solve the general model (11)–(14). The simulation will run for a period of $t = 48$ years. We will study three distinct situations:

(i) Prevalence of infection in the absence of counseling and antiretroviral therapy;
(ii) Prevalence of infection in the absence of effective counseling and ART and;
(iii) Prevalence of infection in the presence of very effective counseling and ART.

The plots in all three cases are for $i_m(t)$ and $i_f(t)$.

Experiment One

In this experiment, we study the prevalence of infection in the absence of Counseling and Antiretroviral Therapy (ART). We use the following values for the parameters: $b = 0.5$, $\alpha_0 = 0.2$, $k = 0$, $T = 10$, $c_m = 4$, $c_m^* = 0$, $c_f = 4$, $c_f^* = 0$, $\beta_m = 0.35$, $\beta_m^* = 0$, $\beta_f = 0.35$, $\beta_f^* = 0$, $\sigma_m = 0$, $\sigma_f = 0$. In addition, we let $i_m(0) = 0.35$, $i_f(0) = 0.35$, $r_m(0) = 0$ and $r_f(0)$.

The result, displayed in Figure 2, reflect a high prevalence of infection. The figure appears to have only one graph because the two plots coincide.

Experiment Two

In this case, we study the dynamics of the infection in the absence of effective Counseling and ART. We use the following values for the parameters: $b = 0.5$, $\alpha_0 = 0.2$, $k = 0.1$,

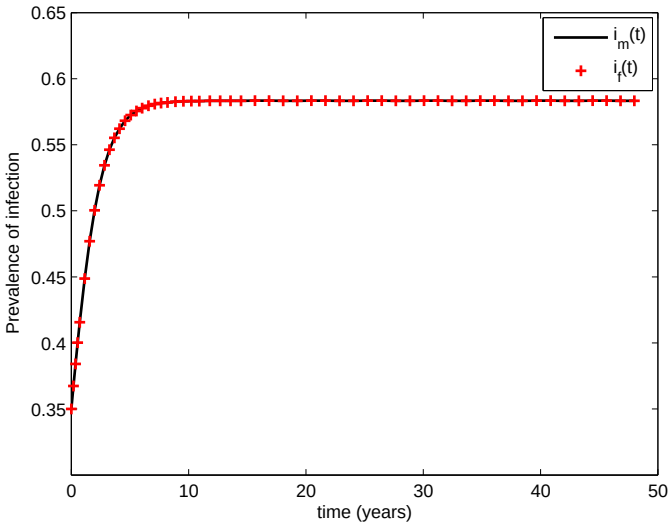

FIGURE 2. Prevalence of infection in the absence of Counseling and ART.

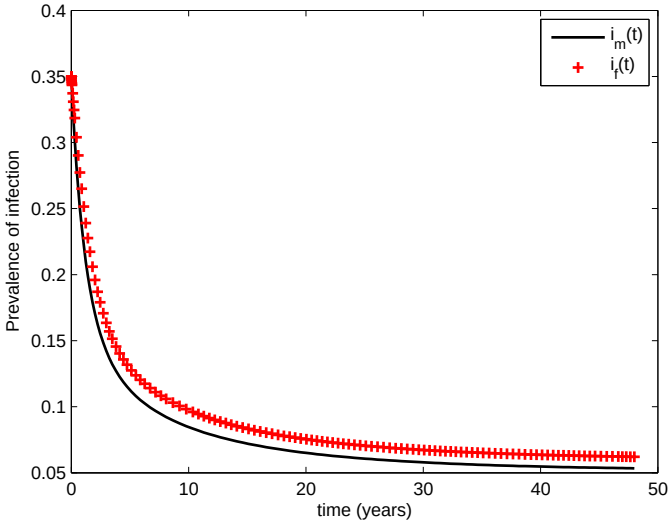

FIGURE 3. Prevalence of infection in the absence of effective Counseling and ART.

$T = 10$, $c_m = 3$, $c_m^* = 2$, $c_f = 3$, $c_f^* = 2$, $\beta_m = 0.35$, $\beta_m^* = 0.2$, $\beta_f = 0.35$, $\beta_f^* = 0.2$, $\sigma_m = 0.3$, $\sigma_f = 0.3$. Once again, we let $i_m(0) = 0.35$, $i_f(0) = 0.35$, $r_m(0) = 0$ and $r_f(0)$.

The result displayed in Figure 3, shows decrease in the prevalence of infection for both males and females. This does not, however, achieve eradication. This is due to the

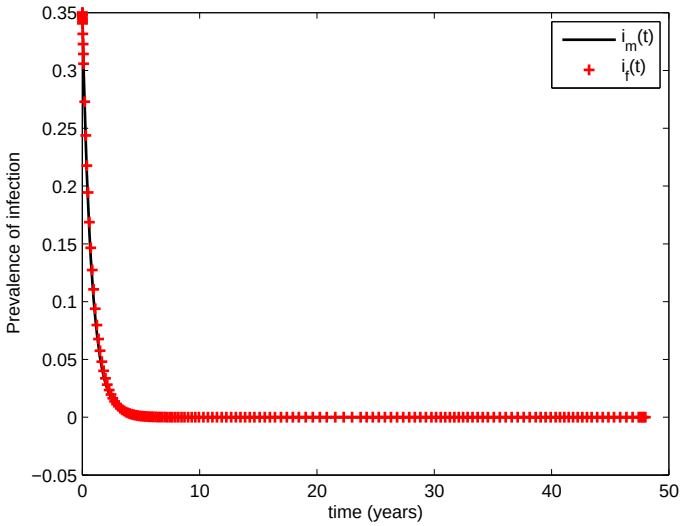

FIGURE 4. Prevalence of infection in the presence of very effective Counseling and ART.

increase in the number of sexual partners and transmission rates and low efficacy level of ART.

Experiment Three

In this experiment, we study the prevalence of infection in the presence of very effective Counseling and ART. We use the following values for the parameters: $b = 0.5$, $\alpha_0 = 0.2$, $k = 1$, $T = 10$, $c_m = 3$, $c_m^* = 1$, $c_f = 3$, $c_f^* = 1$, $\beta_m = 0.15$, $\beta_m^* = 0$, $\beta_f = 0.15$, $\beta_f^* = 0$, $\sigma_m = 0.8$, $\sigma_f = 0.8$. Again, we let $i_m(0) = 0.35$, $i_f(0) = 0.35$, $r_m(0) = 0$ and $r_f(0)$.

Here we observe, from the result displayed in Figure 4, a fast decrease in the prevalence of infection. This is due to a decrease in number of sexual partners ($c_m^* = 1, c_f^* = 1$) and high efficacy level of ART ($k = 1$ and $\sigma_m = \sigma_f = 0.8$). Thus, with very effective Counseling and high proportion of people receiving ART, HIV/AIDS can be eradicated completely from heterosexual populations in finite time. We note that the graph for $i_m(t)$ and $i_f(t)$ coincide for most of the time except the slight difference between 0 and 5 years.

5. Conclusion

For the sub-model without any ART (i.e., $\sigma_m = \sigma_f = 0$) we see that the DFE is locally and asymptotically stable, whenever it exists, provided that $c_m\sigma_f c_f\sigma_m < b_0^2$. For the sub-model for which only infected males receive ART (i.e.,$\sigma_m > 0$, $\sigma_f = 0$) we observed that the DFE is LAS provided that

$$\sigma_m > \frac{c_m\beta_f c_f\beta_m - b_0^2}{b_0},$$

A similar result is obtained for the sub-model for which only infected females receive ART which is LAS provided that

$$\sigma_f > \frac{c_m\beta_f c_f\beta_m - b_0^2}{b_0}.$$

It is observed that withdrawing Counseling and ART from both the sub-models with only infected males/females receiving ART and the general model, we recover the sub-model without any ART. Numerical experiments, using hypothetical data also show that effective Counseling and ART will completely control or eradicate HIV/AIDS from heterosexual populations. It is also observed that for effective counseling and ART to lead to eradication, it is necessary that the same proportion of males and females should be involved in ART.

References

[1] F. Benyah, Epidemiological Modelling and Analysis, Paper presented at 13th Edward A. Bouchet/Abdus Salam workshop, University of Ghana, Legon, Accra, 9-13 July, 2007.

[2] R. L. Burden and J. D. Faires, Numerical Analysis, (Brooks/Cole, California, 1997).

[3] B. D. Corbett, S. M. Moghadas and A. B. Gumel, Sub-threshold domain of bistable equilibria for a model of HIV epidemiology, J. Math. Math. Sc.. (2004) pp.753–764.

[4] S. G. Deo and V. Raghavndra, Ordinary Differential Equations and Stability Theory,(Tata McGraw-Hill, New Delhi, 1975).

[5] A. B. Gumel, W. Z. Xue, P. N. Shivakumar, M. L. Garba and B. M. Sahai, A New Mathematical Model for Assessing Therapeutic Strategies for HIV infection. Journal of Theoretical Medicine. **4(2)** (2002) pp.147.

[6] H. Y. Hsieh, A two sex model for treatment of AIDS and behaviour change in a population of varying size. IMA J. Math. Appl. Bio. Med.. **13** (2003) pp.151-173.

[7] J. M. Hyman, J. Li and E. N. Stanley, Modelling the impact of random screening and contact tracing in reducing the spread of HIV. Math. Biosc.. **181** (2003) pp.17-54.

[8] A. R. Kimbir and H. K. Oduwole. A Mathematical model of HIV transmission dynamics considering counseling and Antiretroviral Therapy (ART). J. Mod. Math. Stat.. **2(5)** (2008) pp. 166-169.

[9] C. E. Swanson, B. Tindall and D. A. Cooper. Efficacy of Zidovudine treatment in homosexual men with AIDS-related complex. Factors influencing development of AIDS, sexual and drug tolerance, AIDS. **9** (1994) pp. 625-634.