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CONNECTED DOMINATION NUMBER ON CARTESIAN PRODUCT OF SIMPLE FUZZY GRAPHS

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Abstract: A dominating set D of a fuzzy graph G=(V,σ,μ) is connected dominating set if the induced subgraph <D> is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$. The Cartesian product of simple fuzzy graphs is also a fuzzy graph. In this paper we introduce the connected domination number on Cartesian product of simple fuzzy graphs and obtain the results for this parameter in fuzzy graph.

Keywords: dominating set; connected dominating set; Cartesian product of fuzzy graphs.

2010 AMS Subject Classification: 05C07, 05C38, 05C51, 05C62, 05C76.

I. Introduction

A mathematical model to describe the occurrence of uncertainty in real life situation is first suggested by L.A. Zadeh [9] in 1965. The fuzzy definition of fuzzy graphs was introduced by Kaufmann [7], from the fuzzy relation introduced by Zadeh. Although Rosenfield [6] introduced another elaborated definition, including fuzzy vertex and fuzzy edge. And introduced the notion of fuzzy graph and several fuzzy analogs of graph theoretic concepts such as paths, cycles and connectedness. The study of dominating sets in graphs was begun by Orge and Berge. The domination number was introduced by Cockayne and Hedetniemi [1]. A.Somasundaram and S.Somasundaram [8] discussed domination with reference to degree in edges. Nagoorgani and Chandrasekaran [4] discussed domination in fuzzy graph using strong arcs. A. Somasundaram

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presented the concept of connected domination number of fizzy graphs [10] [8]. In this paper, we introduce the concept of connected domination number on Cartesian product of fuzzy graphs and characterize the graphs attaining these bounds.

II. Preliminaries

In this section, basic concepts of fuzzy graph and coloring are discussed. Notation and more formal definitions which are followed as in [2], [4], [8].

Definition 2.1[4]:

A fuzzy graph G=(V,σ,μ) is a pair of functions $\sigma : V\rightarrow [0,1]$ and $\mu : V\times V\rightarrow[0,1]$, where for all u,v∈V, we have $\mu(u,v)\leq \sigma(u)\wedge\sigma(v)$.

Definition 2.2[8]:

The order p and size q of a fuzzy graph G=(V,σ,μ) are defined to be $p = \sum_{x \in V} \sigma(x)$ and $q=\sum_{xy \in V} \mu(xy)$.

Definition 2.3[2]:

Two vertices u and v in Ĝ are called adjacent if $(\frac{1}{2})[\sigma(u) \wedge \sigma(v)] \leq \mu(uv)$.

Definition 2.4[2]:

An arc (u, v) is said to be a strong arc or strong edge, if $\mu(u, v) \geq \mu^{\infty}(u, v)$ and the node v is said to be the strong neighbor of u. A node u is said to be isolated if μ(u, v) = 0 for all u ≠ v. In a fuzzy graph, every arc is a strong arc then the graph is called the strong arc fuzzy graph.

Definition 2.5[4]:

u is a node in fuzzy graph G then N(u) = {v: (u, v) is a strong arc} is called the neighborhood of u and N[u] = N(u) ∪{u} is called closed neighborhood of u.

Definition 2.6[2]:

A path in which every arc is a strong arc then the path is called strong path and the path contains n strong arcs is denoted by P_n.

Definition 2.7[2]:

A cycle in G is said to be fuzzy cycle if it contains more than one weakest arc.

Definition 2.8[8]:

A fuzzy graph $G=(V,\sigma,\mu)$ is called complete fuzzy graph if $\mu(u,v)=\sigma(u)\wedge\sigma(v)$, for all $u,v\in V$ and is denoted by K_σ.

Definition 2.9[4]:

The fuzzy subgraph $H(V_1,\tau,\rho,)$ is said to be a spanning fuzzy subgraph of $G(V,\sigma,\mu)$ if $\tau(u)= \sigma(u)$ for all $u\in V_1$ and $\rho(u,v) \leq \mu(u,v)$ for all $u,v\in V$. let $G(V,\sigma,\mu)$ be a fuzzy graph and τ be any fuzzy subset of σ, ie., $\tau(u) \leq \sigma(u)$ for all u.

Definition 2.10[8]:

G is a fuzzy graph on V and $S \subseteq V$, then the fuzzy cardinality of S is defined to be Σ $v\in S$ $\sigma(v)$.

Definition 2.11[4]:

The Cartesian product $G=G_1\times G_2=(V,X)$ of graphs G_1 and G_2. Then $V=V_1\times V_2$, and $X=\{(u,u_2),(u,v_2)|\ u\in V_1,\ (u_2,v_2)\in X_2\}\cup\{(u_1,w),(v_1,w)|\ w\in V_2,\ (u_1,v_1)\in X_1\}$.

Let σ_i be a fuzzy subset of V_i and let μ_i be a fuzzy subset of X_i, i=1,2. Define the fuzzy subsets $\sigma_1\times\sigma_2$ of V and $\mu_1\times\mu_2$ of X as follows:

$(\sigma_1\times\sigma_2)(u_1,u_2)= \min\{\sigma_1(u_1),\sigma_2(u_2)\}$ for all $(u_1,u_2)\in V$

$(\mu_1\times\mu_2)((u,u_2),(u,v_2)) = \min\{\sigma_1(u_1),\mu_2(u_2,v_2)\}$ for all $u\in V_1$ and $(u_2,v_2)\in X_2$

$(\mu_1\times\mu_2)((u_1,w),(v_1,w)) = \min\{\sigma_2(w),\mu_1(u_1,v_1)\}$ for all $w\in V_2$ and $(u_1,v_1)\in X_1$,

Then the fuzzy graph $G=(\sigma_1\times\sigma_2,\ \mu_1\times\mu_2)1$ is said to be the cartesian product of $G_1=(\sigma_1,\mu_1)$ and $G_2 = (\sigma_2,\mu_2)$.

Definition 2.12[4]:

A dominating set D of fuzzy cardinality $|D| = \Sigma\, \sigma(u)$, for all $u \in D = \gamma(G)$ is called minimum dominating set or γ-set. A dominating set D of a fuzzy graph G is called the minimal dominating set if and only if for each vertex $v \in V$, $D-\{u\}$ is not a dominating set of G.

The minimum fuzzy cardinality taken over all minimal dominating sets in G is called the domination number of G and is denoted by $\gamma(G)$.

Definition 2.13[8]:

A dominating set D of a fuzzy graph $G=(V,\sigma,\mu)$ is connected dominating set if the induced subgraph <D> is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$.

III. Results on connected domination number of Cartesian product of fuzzy graphs

In this paper, the Cartesian product on same type of two fuzzy graphs (say G and H) such as fuzzy path, fuzzy cycle and complete fuzzy graph. The order of H should be greater than or equal to the order of G. Let $G=(V_1, \sigma_1, \mu_1)$ be a fuzzy graph. and $H==(V_2, \sigma_2, \mu_2)$ be a fuzzy graph. The Cartesian product $G\times H=(V, X)$ of fuzzy graph G and H. where $V=V_1\times V_2$, $X=\{ ((u,u_2), (u,v_2))|\ u \in V_1, (u_2,v_2) \in X_2 \} \cup \{((u_1,w), (v_1,w)) \mid w \in V_2, (u_1,v_1) \in X_1\}$. Using these notations, the following theorems are defined. To derive the theorem we use independent fuzzy path and fuzzy cycles which does not have unique strong neighbors in fuzzy path and fuzzy cycle of $G\times H$.

Theorem 3.1:

A connected dominating set exist for a fuzzy graph $G\times H$ if and only if G & H are connected

Theorem 3.2:

Any connected dominating set D of $G\times H$ is a minimal connected dominating set of $G\times H$ if and only if for each $v_1 \in D$ one of the following two conditions hold v_1 is adjacent to any vertex in D and $v_2 \in V \setminus D$

i) for some $v_3 \in D$, $\sigma(v_1) < \sigma(v_3)$

ii) there exists a vertex $v_2 \in V \setminus D$ ∋: $N(v_2) \cap D = \{v_1\}$

Proof:

Let us consider that G×H be a Cartesian product of two simple fuzzy graphs with mn vertices and D be a minimal connected dominating set of G×H.

If $v_1 \in D$ then $D \setminus v_1$ is a connected dominating set which implies that some vertex v_2 in $V \setminus D \cup \{v_1\}$ is not connected dominate by any vertex in $D \setminus \{v_1\}$. This deals with the following cases.

Suppose $v_1 = v_2$. That is, v_1 is adjacent to any vertex v_3 in D such that $\sigma(v_1) < \sigma(v_3)$. Otherwise $v_2 \in V \setminus D$ then v_2 is not dominated by any vertex in $D \setminus \{v_1\}$. $D \setminus \{v_1\}$ is connected dominating set but not minimally connected dominating set. Hence v_2 is the only vertex dominated by v_1. Thus $N(v_2) \cap D = \{v_1\}$.

Conversely suppose that D is a connected dominating set and for each $v_1 \in D$.

We prove that D is minimal connected dominating set. Suppose that D is not a minimal connected dominating set. There exists a vertex $v_1 \in D$ such that $D \setminus \{v_1\}$ is a connected dominating set.

Hence v_1 is adjacent to one vertex in $D \setminus \{v_1\}$ which is contradicting condition (i). Therefore $D \setminus \{v_1\}$ is connected dominating set. Every vertex of $V \setminus D$ is adjacent to atleast one vertex in $D \setminus \{v_1\}$. That is, condition (i) & (ii) does not hold. This is contradicting to our assumption that atleast one of the conditions hold.

Theorem 3.3:

Any connected dominating set of a Cartesian product of fuzzy graph G×H is a dominating set of a fuzzy graph

Proof:

Let D be a connected dominating set of a Cartesian product of fuzzy graph G×H for every $(u_1,v_1) \in V(G \times H) \setminus D$ there exist a vertex $(u_2,v_1) \in D$ such that $(u_1,v_1)(u_2,v_1) \in E(G \times H)$ and $|deg(u_1,v_1) - deg(u_2,v_1)| \leq 1$ and the subgraph $\langle D \rangle$ is connected. Thus D is a dominating set of a Cartesian product of fuzzy graph.

Note:

The converse of the above theorem need not be true.

Theorem 3.4:

If G & H are fuzzy path with m & n vertices and G×H is Cartesian product of fuzzy paths with mn vertices then for every minimal connected dominating set D in G×H, V(G×H) \ D is also a connected dominating set with at most m-1 components.

Example 3.5:

G:

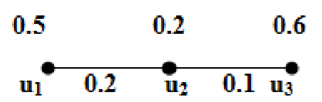


Fig.1 Fuzzy path on 3 vertices

H:

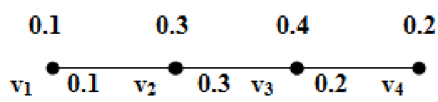


Fig.2 Fuzzy path on 4 vertices

G×H:

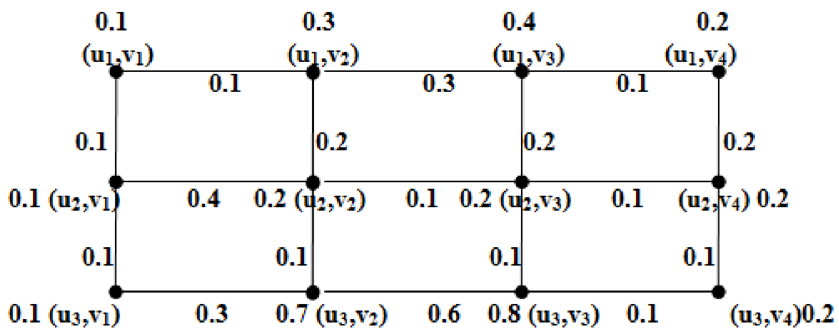


Fig.3 Cartesian product of Fuzzy paths on 3×4 vertices

D = { (u_1,v_2), (u_2,v_2), (u_3,v_2), (u_1,v_3), (u_2,v_3), (u_3,v_3) }

V(G×H) \ D = { (u_1,v_1), (u_2,v_1), (u_3,v_1), (u_1,v_4), (u_2,v_4), (u_3,v_4) }

m -1 = 3-1 = 2 components exists

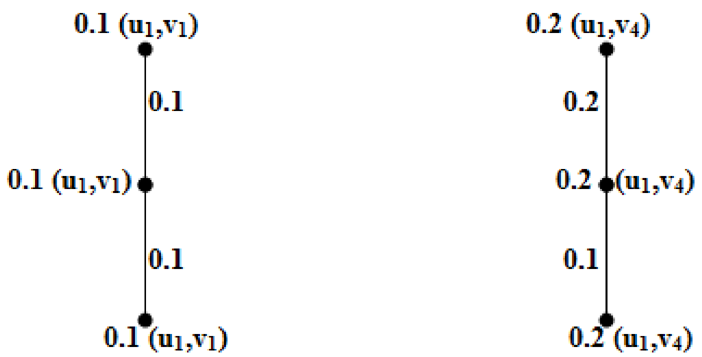


Fig. 4 Two Components of V(G×H) \ D

Theorem 3.6:

For any connected fuzzy graph of G×H, $\gamma(G\times H) \le \gamma_{cd}(G\times H)$.

Proof:

From the definition of connected dominating set of a fuzzy graph of G×H, it is clear that for any Cartesian product of fuzzy graphs G×H, any connected dominating set D is also a dominating set. $\therefore \gamma(G\times H) \le \gamma_{cd}(G\times H)$.

Theorem 3.7:

If G×H is a Cartesian product of any same types of fuzzy graph, S is a connected spanning fuzzy subgraph of G×H then every connected dominating set of S is also a connected dominating set of G×H and $\gamma_{cd}(G\times H) \le \gamma_{cd}(S)$.

Proof:

To prove this theorem, we consider a simple fuzzy paths G and H. Applying Cartesian product on fuzzy graphs G and H we get G×H.

Let S be a connected spanning subgraph of G×H. let us consider, $V(G\times H) =\{ v_n, \sigma(v_n) > 0, n= 1,2,..\}$ and $S=\{v_i\}$, $1\le i \le n$ such that $S \subseteq V(G\times H)$ in such a way that every vertex of $V \setminus S$ are at a distance of S in G×H which gives a minimal connected dominating set in G×H. let (u_i,v_j) (u_i,v_{j+1}) be any edge of G×H such that i<j and for all i,j = 1,2,3,..n.

Suppose S is minimal connected spanning subgraph of G×H and $E(S) = E(G\times H) \setminus (u_i,v_j)\ (u_i,v_{j+1})$ and we know that V(G×H)=V(S), since S is a spanning subgraph of G×H and the vertices in S have the weak connected with their neighbors. So, construct a connected dominating set in S which cover all vertices in S. therefore, connected dominating set of S contains atmost n vertices. Hence $\gamma_{cd}(G\times H) \le \gamma_{cd}(S)$.

Theorem 3.8:

If G×H are fuzzy paths which has m & n vertices and $\gamma_{cd}(G\times H)=m$ then every vertex in G×H has

i) $\Delta(G\times H) = 4$ or 3 which based on |G| and

ii) $\delta(G\times H)=2$

Theorem 3.9:

For any cartesian product of fuzzy paths G & H with m & n vertices,

$\left\lceil \frac{2mn}{1+\Delta_s(G)} \right\rceil \le \gamma_{cd}(G\times H) \le \left\lceil \frac{mn}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor$. Where p is the number of vertices in $G\times H$.

Proof:

Let S be a dominating set of any connected dominating set then each vertex of S dominating at least $\Delta_s(G)$ other vertices of G.

Thus, $2mn \leq \gamma(G)\,(1+\Delta_s(G)) \leq \gamma_{cd}(G)$

$\frac{2mn}{1+\Delta_s(G)} \leq \gamma(G) \leq \gamma_{cd}(G)$

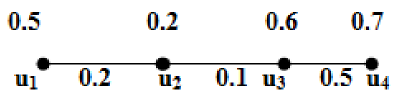
$\frac{2mn}{1+\Delta_s(G)} \leq \gamma_{cd}\,(G)$

Clearly, by the proposition, $\gamma_{cd}(G \times H) \leq \left\lceil \frac{mn}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor$

Therefore, $\frac{2mn}{1+\Delta_s(G)} \leq \gamma_{cd}(\,G \times H) \leq \left\lceil \frac{mn}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor$

Example 3.10:

G :



H :

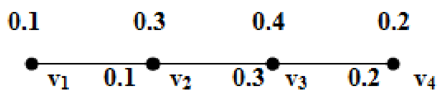


Fig.5 Fuzzy path on 4 vertices **Fig.6 Fuzzy path on 4 vertices**

G×H :

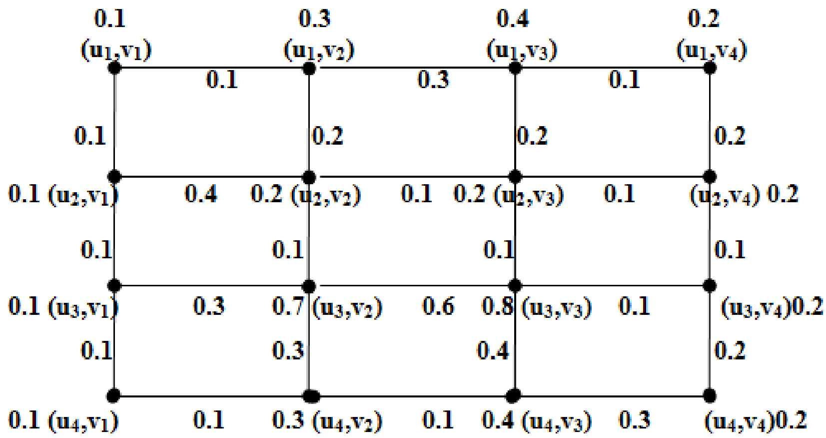


Fig.7 Cartesian product of Fuzzy paths on 4×4 vertices

$D=\{(u_1,v_2), (u_2,v_2), (u_3,v_2), (u_4,v_2), (u_1,v_3), (u_2,v_3), (u_3,v_3), (u_4,v_3)\}$. $|D| = 8$ and $\gamma_{cd}(G\times H) = 3.3$

Theorem 3.11:

For any fuzzy path G & H with m & n vertices, G×H is the Cartesian product on G and H, $n \leq \gamma_{cd}(G\times H) \leq 3n+4$

Theorem 3.12:

If G & H are fuzzy path on m & n vertices ($m \leq n$) and G×H is a cartesian product of fuzzy paths G and H then $\gamma_{cd}(G) \leq \gamma_{cd}(H) \leq \gamma_{cd}(G \times H)$

Results 3.13:

If G and H are fuzzy path on m & n vertices and $n \geq m$ then

i) $m = 2 \Rightarrow \gamma_{cd}(G \times H) = n$

ii) $m = 3 \Rightarrow \gamma_{cd}(G \times H) = n$

iii) $m = 4 \Rightarrow \gamma_{cd}(G \times H) = 2n - 1$

iv) $m = 5 \Rightarrow \gamma_{cd}(G \times H) = 2n + 1$

v) $m = 6 \Rightarrow \gamma_{cd}(G \times H) = 2n + 2$

vi) $m = 7 \Rightarrow \gamma_{cd}(G \times H) \leq 3n$

vii) $m = 8 \Rightarrow \gamma_{cd}(G \times H) \leq 3n + 3$

viii) $m = 9 \Rightarrow \gamma_{cd}(G \times H) \leq 3n + 4$

ix) $m = 10 \Rightarrow \gamma_{cd}(G \times H) \leq 4n + 2$

Theorem 3.14:

If G×H is a cartesian product of fuzzy cycles with mn vertices, D is a connected dominating set of G×H then V\D is also a connected graph and maximum length of cycle in V \ D is atmost $\gamma_{cd}(G \times H)$.

Proof:

Let G & H be a fuzzy cycles on m & n vertices. G×H be a Cartesian product of fuzzy cycles on m & n vertices. Let D be a connected dominating set of G×H. To prove that V \D is also a connected dominating set. Let $v_1 \in V \setminus D$ and it adjacents to atleast one vertex in V \ D.

Suppose $N(v_1) \cap D = \{v_2\}$ then v_1 is a pendent vertex in G×H which is contradict to our Theorem 3.2(ii). Since every vertex in G×H has maximum degree is 4. Thus V \D is also connected graph.

Suppose $|D| = n$ and $|V(G)|=mn$. ie., $\gamma_{cd}(G \times H) = n$.

Proposition 3.15:

Let G & H be a fuzzy cycle with m & n (≥ 4) vertices. G×H is a Cartesian product of fuzzy graphs with mn vertices. The connected dominating set is isomorphic to tree.

Example 3.16:

G:

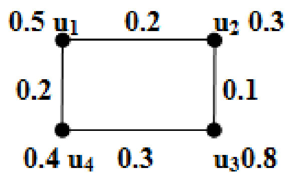


Fig. 8 Fuzzy cycle of length 4 (C_4)

H:

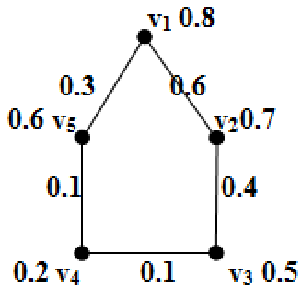


Fig.9 Fuzzy cycle of length 5 (C_5)

G × H:

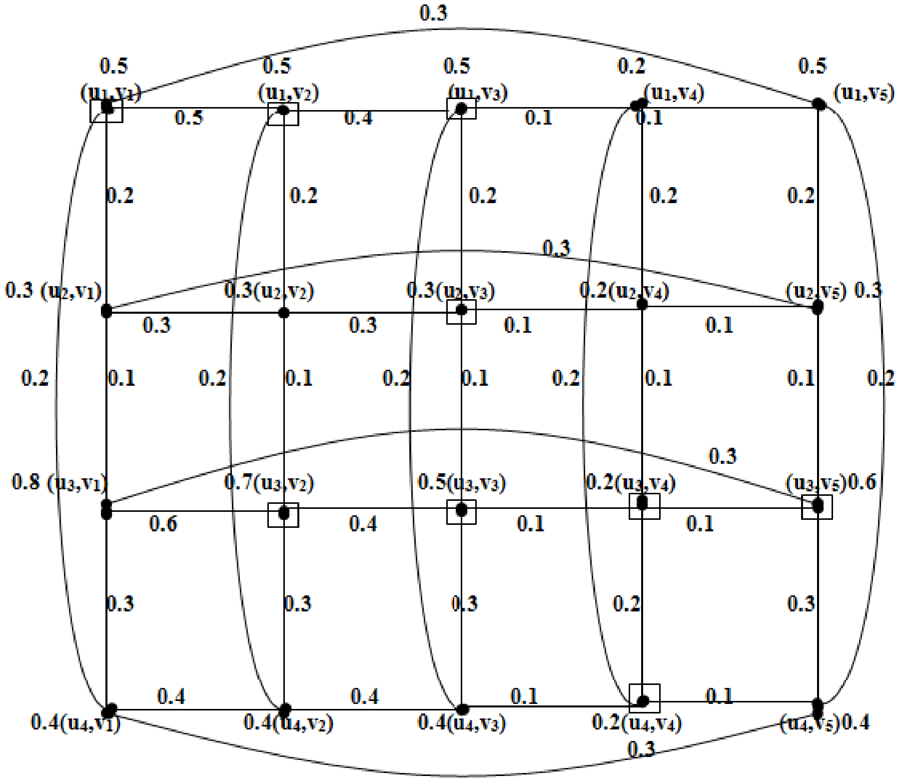


Fig. 10 Cartesian product of Fuzzy cycles on C_4 and C_5 ($C_4 \times C_5$)

CD-set, D={ u_1,v_1), (u_1,v_2), (u_1,v_3), (u_2,v_3), (u_3,v_2), (u_3,v_3), (u_3,v_4), (u_3,v_5), (u_4,v_4)}.

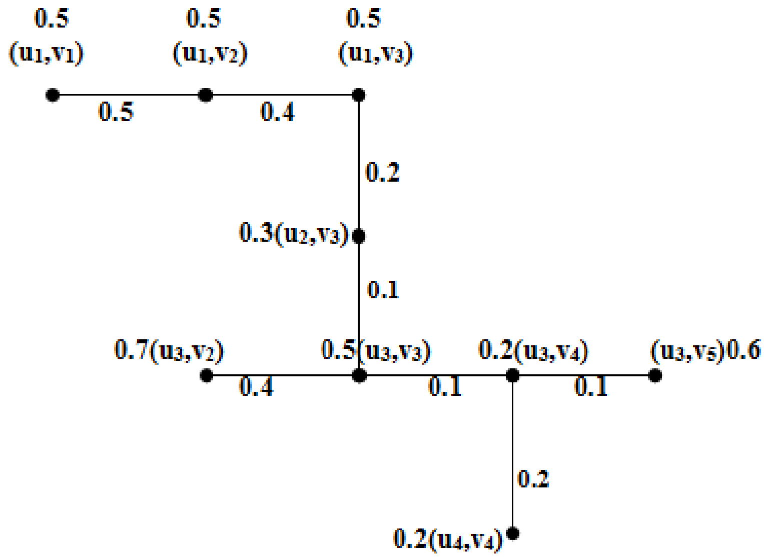


Fig. 11 Tree on 9 vertices

Results 3.17:

If G and H are fuzzy cycles on m & n vertices and $n \geq m$ and $G \times H$ is a Cartesian product of fuzzy cycles with mn vertices then

i) $m = 3 \Rightarrow \gamma_{cd}(G \times H) = n$

ii)) $m = 4 \Rightarrow \gamma_{cd}(G \times H) \leq \frac{mn}{2} - 2$

iv) $m = 5 \Rightarrow \gamma_{cd}(G \times H) \leq 2n + 1$

v) $m = 6 \Rightarrow \gamma_{cd}(G \times H) \leq m + 2n$

vi) $m = 7 \Rightarrow \gamma_{cd}(G \times H) = \begin{cases} \leq m + 2n \text{ if } m = n \\ \geq m + 2n \text{ if } n \geq m \end{cases}$

vii) $m = 8 \Rightarrow \gamma_{cd}(G \times H) \leq 3m + 4n/2$

viii) $m = 9 \Rightarrow \gamma_{cd}(G \times H) \leq 3n + m$

Proposition 3.18

Let G & H be a complete fuzzy graph with m & n vertices. $G \times H$ is a Cartesian product of complete fuzzy graphs with mn vertices. The connected dominating set is isomorphic to K_m.

Results 3.19:

If G and H are complete fuzzy graph on m & n vertices and $n \geq m$ and $G \times H$ is a Cartesian product of complete fuzzy graphs with mn vertices then $\gamma_{cd}(G \times H) = m$

Conflict of Interests

The authors declare that there is no conflict of interests.

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